Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 5140-5155
https://doi.org/10.28919/jmcs/5934
ISSN: 1927-5307

GENERALIZATION OF ZYGMUND TYPE INEQUALITIES FOR THE s^{th} DERIVATIVE OF POLYNOMIALS

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Abstract. If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ in $|z| < 1$, it was proved by Hans and Lal [Anal. Math. 40, 105-115(2014)] that for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$,

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{2} p(z)\right| \leq \frac{n_s}{2}\left\{\left(\left|1+\frac{\beta}{2^s}\right| + \left|\frac{\beta}{2^s}\right|\right)\|p\|_\infty - \left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right) m\right\},$$

where $n_s = n(n-1)...(n-s+1), \|p\|_\infty = \max_{|z|=1}|p(z)|$ and $m = \min_{|z|=1}|p(z)|$,

In this paper, we prove an inequality which gives an improved and generalized extension of the above inequality into L^γ norm.

Keywords: L^γ norm; inequality; polynomial; zero.

2010 AMS Subject Classification: 30C10, 30C15.

1. INTRODUCTION

Let $\mathbb{P}_n$ be the class of polynomials of degree n. For $p \in \mathbb{P}_n$, we denote its s^{th} derivative by $p^{(s)}(z)$.

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Received April 27, 2021

Next for $p \in \mathbb{P}_n$, we define

$$\|p\|_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \quad 0 < \gamma < \infty. \tag{1}$$

If we let $\gamma \to \infty$ in the above equality and make use of the well-known fact from analysis [12] that

$$\lim_{\gamma\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, one can define $\|p\|_0 = exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log|p(e^{i\theta})| d\theta \right\}$ and show that $\lim_{\gamma\to 0^+} \|p\|_\gamma = \|p\|_0$. It would be of further interest that by taking limits as $\lim_{\gamma\to 0^+}$ that the stated result holding for $\gamma > 0$, holds for $\gamma = 0$ as well.

A famous result due to Bernstein [9](also see [13]) states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty \le n\|p\|_\infty. \tag{2}$$

Inequality (2) can be obtained by letting $\gamma \to \infty$ in the inequality

$$\|p'\|_\gamma \le n\|p\|_\gamma, \quad \gamma > 0. \tag{3}$$

Inequality (3) for $\gamma \ge 1$ is due to Zygmund [15]. Arestov [1] proved that (3) remains valid for $0 < \gamma < 1$ as well.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (2) and (3) can be respectively improved by

$$\|p'\|_\infty \le \frac{n}{2}\|p\|_\infty \tag{4}$$

and

$$\|p'\|_\gamma \le \frac{n}{\|1+z\|_\gamma}\|p\|_\gamma, \gamma > 0. \tag{5}$$

Inequality (4) was conjectured by Erdös and later verified by Lax [8] , whereas, inequality (5) was proved by de-Bruijn [3] for $\gamma \ge 1$. Rahman and Schmeisser [11] showed that (5) remains true for $0 < \gamma < 1$.

As an extension of (4), Jain [6] proved that if $p \in \mathbb{P}_n$ and $p(z) \neq 0$ in $|z| < 1$, then

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \leq \frac{n}{2}\left(\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right)\|p\|_\infty, \tag{6}$$

for $|z| = 1$ and for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$.

Further, Jain [7] improved (6) by obtaining under the same hypothesis that

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \leq \frac{n}{2}\left\{\left(\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right)\|p\|_\infty - \left(\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)m\right\}, \tag{7}$$

for $|z| = 1$ and for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $m = \min_{|z|=1}|p(z)|$.

Further, Hans and Lal [5] generalized (6) and (7) for the s^{th} derivative of polynomials under the same hypothesis that

$$\left|z^s p^{(s)}(z) + \beta\frac{n_s}{2}p(z)\right| \leq \frac{n_s}{2}\left(\left|1+\frac{\beta}{2^s}\right| + \left|\frac{\beta}{2^s}\right|\right)\|p\|_\infty \tag{8}$$

and

$$\left|z^s p^{(s)}(z) + \beta\frac{n_s}{2}p(z)\right| \leq \frac{n_s}{2}\left\{\left(\left|1+\frac{\beta}{2^s}\right| + \left|\frac{\beta}{2^s}\right|\right)\|p\|_\infty - \left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right)m\right\} \tag{9}$$

for $|z| = 1$ and for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $1 \leq s \leq n$, $m = \min_{|z|=1}|p(z)|$ and where here and throughout this paper $n_s = n(n-1)...(n-s+1)$.

Recently, Gulzar [4] obtained an L^γ analogue of (8) by proving the following result.

Theorem 1. *If $p \in \mathbb{P}_n$ and $p(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $1 \leq s \leq n$ and $0 \leq \gamma \leq \infty$,*

$$\begin{aligned}\left\{\int_0^{2\pi}\left|e^{is\theta}p^{(s)(e^{i\theta})} + \beta\frac{n_s}{2^s}p(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leq & n_s E_\gamma \left\{\int_0^{2\pi}\left|\left(1+\frac{\beta}{2^s}\right)e^{i\alpha} + \frac{\beta}{2^s}\right|^\gamma\right\}^{\frac{1}{\gamma}} \\ & \times \left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}},\end{aligned} \tag{10}$$

where

$$E_\gamma = \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^\gamma\right\}^{-\frac{1}{\gamma}}. \tag{11}$$

The result is best possible and equality in (10) holds for $p(z) = az^n + b$ with $|a| = |b| = 1$.

2. Lemmas

For the proofs of the theorem, we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \leq k, k \leq 1$ and $1 \leq s \leq n$, then for $|z| = 1$*

$$|z^s p^{(s)}(z)| \geq \frac{n_s}{(1+k)^s}|p(z)|, \tag{12}$$

where $n_s = n(n-1)...(n-s+1)$ and for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, the zeros of polynomial

$$z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z), \text{ lie in } |z| \leq 1. \tag{13}$$

The above lemma was obtained by Zireh [14] and (13) is a consequence of Lemma 3.

Lemma 2. *Let $F(z)$ be a polynomial of degree n, having all its zeros in the disk $|z| \leq k, k \leq 1$, and $p(z)$ a polynomial of degree not exceeding that of $F(z)$. If $|p(z)| \leq |F(z)|$ for $|z| = k, k \leq 1$, then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1, 1 \leq s \leq n$,*

$$\left| z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z) \right| \leq \left| z^s F^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} F(z) \right|. \tag{14}$$

This lemma was proved by Zireh [14].

Lemma 3. *Let $F \in \mathbb{P}_n$ and let f be a polynomial of degree at most n, such that $|f(z)| \leq |F(z)|$ for $|z| = 1$. If $F(z) \neq 0$ for $|z| < 1$ (respectively $|z| > 1$) and for every $z \in \mathbb{C}$ and every real α, $f(z) \neq e^{i\alpha} F(z)$, then*

(1) $|f(z)| < |F(z)|$ for $|z| < 1$ (respectively $|z| > 1$),

(2) $F(z) + \beta f(z) \neq 0$ for $|z| < 1$ (respectively $|z| > 1$) and $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and

(3) $f(z) + \lambda F(z) \neq 0$ for $|z| < 1$ (respectively $|z| > 1$) and $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$.

The above lemma is due to Gulzar [4].

Lemma 4. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ in $|z| < k, k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$ and for $|z| = 1$,*

$$\begin{aligned} \left| z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z) \right| &\leq \left| z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z) \right| \\ &\quad - n_s \left\{ k^{-n} \left| 1 + \frac{\beta}{(1+k)^s} \right| - \left| \frac{\beta}{(1+k)^s} \right| \right\} m, \end{aligned} \tag{15}$$

where $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\overline{z}}\right)}$ *and* $m = \min_{|z|=k} |p(z)|$.

Proof. Let $m = \min_{|z|=k} |p(z)|$, then $m \leq |p(z)|$ for $|z| \leq k$. Now for λ with $|\lambda| < 1$, we have for $|z| = k$

$$|\lambda m| < m \leq |p(z)|.$$

Hence by Rouche's theorem the polynomial $G(z) = p(z) - \lambda m$ has no zero in $|z| < k$. Therefore the polynomial

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\overline{z}}\right)} = Q(z) - \overline{\lambda} m \left(\frac{z}{k}\right)^n$$

will have all its zeros in $|z| \leq k$, where $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\overline{z}}\right)}$. Also $|G(z)| = |H(z)|$ for $|z| = k$. On applying Lemma 2 to the polynomial $H(z)$ for $F(z)$ of degree n, we have for $|\beta| \leq 1$ and $|z| = 1$

$$\left|z^s G^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} G(z)\right| \leq \left|z^s H^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} H(z)\right|$$

i.e,

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} (p(z) - \lambda m)\right| \leq \left|z^s Q^{(s)}(z) - n_s \overline{\lambda} m \left(\frac{z}{k}\right)^n + \beta \frac{n_s}{(1+k)^s} \left(Q(z) - \overline{\lambda} m \left(\frac{z}{k}\right)^n\right)\right|.$$

This can be rewritten as

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z) - \beta \frac{n_s}{(1+k)^s} \lambda m\right| \leq \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z) - n_s \overline{\lambda} m \left(\frac{z}{k}\right)^n \left(1 + \frac{\beta}{(1+k)^s}\right)\right|. \tag{16}$$

Since all the zeros of $Q(z)$ lie in $|z| \leq k \leq 1$, we have $|p(z)| = |Q(z)|$ for $|z| = k$. On applying Lemma 11 to the polynomial $Q(z)$, we have for $|z| = 1$

$$|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)| \geq n_s k^n |1 + \frac{\beta}{(1+k)^s}| m,$$

where $|\beta| \leq 1$. Then for an appropriate choice of the argument of λ, we have

$$\left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z) - n_s \overline{\lambda} m \left(\frac{z}{k}\right)^n \left(1 + \frac{\beta}{(1+k)^s}\right)\right|$$

$$= \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)\right| - |\lambda| n_s k^{-n} \left|1 + \frac{\beta}{(1+k)^s}\right| m. \tag{17}$$

By combining (16) and (17), we get for $|z| = 1$ and $|\beta| \leq 1$

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right| - n_s \left|\frac{\beta}{(1+k)^s} \lambda m\right| \leq \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)\right| - n_s k^{-n} \left|1 + \frac{\beta}{(1+k)^s}\right| |\lambda| m.$$

Equivalently,

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right| \leq \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)\right| - n_s \left(k^{-n} \left|1 + \frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right) |\lambda| m.$$

As $|\lambda| \to 1$, we have

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right| \leq \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)\right| - n_s \left(k^{-n} \left|1 + \frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right) m.$$

This completes the proof of lemma 4. □

Lemma 5. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| \leq k, k \leq 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$ and $|z| \geq 1$,*

$$\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right| \leq \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)\right|, \tag{18}$$

where $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\overline{z}}\right)}$.

Proof. Since $p(z)$ has no zeros in $|z| \leq k$. Correspondingly the polynomial $Q(z) = (\frac{z}{k})^n \overline{p(\frac{k^2}{\overline{z}})}$ has all its zeros in $|z| < k$ and $|p(z)| = |Q(z)|$ for $|z| = k$. Therefore, by Lemma 2, for $|\beta| \leq 1$ and $|z| = 1$, we have the desired result. □

Lemma 6. *If $p \in \mathbb{P}_n$ and $p(z)$ does not vanish in $|z| \le k, k \le 1$ and $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\bar{z}}\right)}$, then for every $\beta \in \mathbb{C}$ with $\beta \le 1, 1 \le s \le n$ and α real,*

$$\left(z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right) e^{i\alpha} + \frac{z^n}{k_n} \overline{M\left(\frac{k^2}{\bar{z}}\right)} \neq 0$$

for $|z| < 1$ (respectively $|z| \le 1$), where $M(z) = z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)$.

Proof. By hypothesis, $p(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < k, k \le 1$. Therefore, by Lemma 5 for every $\beta \in \mathbb{C}$ with $|\beta| \le 1$ and $|z| = 1$, we have,

$$\begin{aligned}\left|z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right| &\le \left|z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)\right| \\ &= |M(z)| \\ &= \left|\frac{z^n}{k^n} \overline{M\left(\frac{k^2}{\bar{z}}\right)}\right|.\end{aligned}$$

Since $p(0) \neq 0$, then $\deg(Q(z)) = n$. Moreover, $Q(z) \neq 0$ for $|z| \ge k$ and then by (13) of Lemma 1 implies that $|M(z)| \neq 0$ for $|z| > 1$. Therefore $\frac{z^n}{k^n} \overline{M\left(\frac{k^2}{\bar{z}}\right)} \neq 0$ for $|z| < 1$. Then, by Lemma 3, for $|z| < 1$,

$$\left(z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right) e^{i\alpha} + \frac{z^n}{k^n} \overline{M\left(\frac{k^2}{\bar{z}}\right)} \neq 0.$$

If $p(z) \neq 0$ for $|z| \le 1$, then we have again the above result for $|z| < 1$.

Now, for $|z| = 1$, we observe that in this case there is some $r > 1$ such that $p(rz) \neq 0$ for $|z| < 1$. Thus, if $Q_1(z) = z^n \overline{p\left(\frac{r}{\bar{z}}\right)}$ and $M_1(z) = z^s Q_1^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q_1(z)$, then we have, for $|z| < 1$,

$$\left(z^s r^s p^{(s)}(rz) + \beta \frac{n_s}{(1+k)^s} p(rz)\right) e^{i\alpha} + z^n \overline{M_1\left(\frac{1}{\bar{z}}\right)} \neq 0.$$

For $z = \frac{e^{i\theta}}{r}$, $|z| = \frac{1}{r} < 1$, we obtain

$$\left(e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{(1+k)^s} p(e^{i\theta})\right) e^{i\alpha} + \left(\frac{e^{in\theta}}{r^n}\right) \overline{M_1(re^{i\theta})} \neq 0 \text{ for } 0 \le \theta < 2\pi$$

or

$$\left(z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z)\right) e^{i\alpha} + \left(\frac{z^n}{r^n}\right) \overline{M_1\left(\frac{r}{\bar{z}}\right)} \neq 0 \text{ for } |z| = 1.$$

Also, a short calculation shows that

$$\left(\frac{z^n}{r^n}\right)\overline{M_1\left(\frac{r}{\bar{z}}\right)} = \left(\frac{z^n}{k^n}\right)\overline{M\left(\frac{k^2}{\bar{z}}\right)} \text{ for any } z$$

and so

$$\left(z^s p^{(s)}(z) + \beta\frac{n_s}{(1+k)^s}p(z)\right)e^{i\alpha} + \left(\frac{z^n}{k^n}\right)\overline{M\left(\frac{k^2}{\bar{z}}\right)} \neq 0 \text{ for } |z| = 1.$$

This completes the proof of Lemma 6. □

Next we describe a result of Arestov [1].

For $\gamma = (\gamma_0, \gamma_1, ..., \gamma_n) \in \mathbb{C}^{n+1}$ and $p(z) = \sum_{j=0}^{n} a_j z^j$, we define

$$C_\gamma p(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:

(1) $p(z)$ has all its zeros in $z \in \mathbb{C} : |z| \leq 1$,

(2) $p(z)$ has all its zeros in $z \in \mathbb{C} : |z| \geq 1$.

Lemma 7. *Let $\phi(x) = \psi(\log x)$ where ψ is a convex non-decreasing function on $\mathbb{R}$ and $p(z)$ is a polynomial of degree n. Then, for each admissible operator C_γ,*

$$\int_0^{2\pi} \phi\left(\left|C_\gamma p(e^{i\theta})\right|\right) d\theta \leq \int_0^{2\pi} \phi\left(C_\gamma |p(e^{i\theta})|\right) d\theta,$$

where $C_\gamma = \max(|\gamma_0|, |\gamma_n|)$.

In particular, Lemma 7 applies with $\phi : x \to x^\gamma$ for every $\gamma \in (0, \infty)$ and with $\phi : x \to \log x$ as well. Therefore, we have, for $0 \leq \gamma < \infty$,

$$\left\{\int_0^{2\pi} \left|C_\gamma p(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leq C_\gamma \left\{\int_0^{2\pi} \left|p(e^{i\theta})\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}.$$

The above lemma is due to Gulzar [4].

Lemma 8. *If $p \in \mathbb{P}_n$ and $p(z)$ has no zeros in $|z| \leq k, k \leq 1$ and $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\bar{z}}\right)}$, then, for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, \alpha$ real, $1 \leq s \leq n$ and $\gamma > 0$,*

$$\int_0^{2\pi} \left| \left(e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{(1+k)^s} p(e^{i\theta}) \right) e^{i\alpha} + e^{in\theta} \overline{M(e^{i\theta})} \right|^\gamma d\theta$$

$$\leq n_s^\gamma \left| k^{-n} \left(1 + \frac{\beta}{(1+k)^s} \right) e^{i\alpha} + \frac{\overline{\beta}}{(1+k)^s} \right|^\gamma \int_0^{2\pi} \left| p(e^{i\theta}) \right|^\gamma d\theta, \tag{19}$$

where $M(z) = z^s Q^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} Q(z)$.

Proof. Since $p(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| \leq k, k \leq 1$, therefore, by Lemma 6, the polynomial

$$\begin{aligned} C_\gamma p(z) &= \left(z^s p^{(s)}(z) + \beta \frac{n_s}{(1+k)^s} p(z) \right) e^{i\alpha} + \frac{z^n}{k^n} \overline{M\left(\frac{k^2}{\bar{z}}\right)} \\ &= n_s \left\{ k^{-n} \left(1 + \frac{\beta}{(1+k)^s} \right) e^{i\alpha} + \frac{\overline{\beta}}{(1+k)^s} \right\} a_n z^n \\ &\quad + \ldots + n_s \left\{ k^{-n} \left(1 + \frac{\overline{\beta}}{(1+k)^s} \right) + \frac{\beta}{(1+k)^s} e^{i\alpha} \right\} a_0 \end{aligned}$$

does not vanish in $|z| < 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and α real. Therefore, C_γ is an admissible operator. Applying Lemma 7, the desired result follows immediately for $\gamma > 0$. This completes the proof. □

Lemma 9. *If A, B, C are non-negative real numbers such that $B + C \leq A$, then for every real number α,*

$$\left| (A - C) + e^{i\alpha}(B + C) \right| \leq \left| A + e^{i\alpha} B \right|. \tag{20}$$

The above lemma was proved by Aziz and Shah [2].

Lemma 10. *Let $a, b \in \mathbb{C}$ with $|b| \geq |a|$. Then for $\gamma > 0$ and α real, we have,*

$$\int_0^{2\pi} \left| a + e^{i\alpha} b \right|^\gamma d\alpha \geq |a|^\gamma \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma d\alpha. \tag{21}$$

The above lemma is due to Mir [10].

Lemma 11. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k, k \leq 1$, then for every real or complex number β with $|\beta| \leq 1$ and $1 \leq s \leq n$,*

$$\min_{|z|=1}\left|z^s p^{(s)}(z)+\beta\frac{n_s}{(1+k)^s}p(z)\right| \geq n_s k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right|\min_{|z|=k}|p(z)|. \tag{22}$$

The above lemma was obtained by Zireh [14].

3. Main Results

In this paper, we improve as well as generalise Theorem 1 by considering polynomials not vanishing in $|z| < k, k \leq 1$. More precisely, we prove

Theorem 2. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k, k \leq 1$, then for every real or complex number β, δ with $|\beta| \leq 1, |\delta| \leq 1, 1 \leq s \leq n$ and $0 \leq \gamma < \infty$,*

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|e^{is\theta}p^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}p(e^{i\theta})+\delta m\frac{n_s}{2}\left(k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right|\right.\right.\right.\\&\left.\left.\left.-\left|\frac{\beta}{(1+k)^s}\right|\right)\right|^{\gamma}d\theta\right\}^{\frac{1}{\gamma}}\\&\leq n_s E_\gamma\left\{\int_0^{2\pi}\left|k^{-n}\left(1+\frac{\beta}{(1+k)^s}\right)e^{i\alpha}+\frac{\beta}{(1+k)^s}\right|^{\gamma}d\alpha\right\}^{\frac{1}{\gamma}}\\&\times\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^{\gamma}d\theta\right\}^{\frac{1}{\gamma}},\end{aligned} \tag{23}$$

where $m = \min_{|z|=k}|p(z)|$ and E_γ is given by (11).

The result is best possible and equality in (23) holds for the polynomial $p(z) = az^n + bk^n$ with $|a| = |b|$ and $\beta \geq 0$.

Proof. Since $p(z) \neq 0$ in $|z| < k, k \leq 1$, therefore, by Lemma 4,

$$\begin{aligned}\left|z^s p^{(s)}(z)+\beta\frac{n_s}{(1+k)^s}p(z)\right| \leq &\left|z^s Q^{(s)}(z)+\beta\frac{n_s}{(1+k)^s}Q(z)\right|\\&-n_s m\left\{k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right|-\left|\frac{\beta}{(1+k)^s}\right|\right\}\end{aligned}$$

where $Q(z) = \left(\frac{z}{k}\right)^n\overline{p\left(\frac{k^2}{\bar{z}}\right)}$ and $n_s = n(n-1)...(n-s+1)$.

For every θ, $0 \le \theta < 2\pi$, $\beta \in \mathbb{C}$ with $|\beta| \le 1$ and $1 \le s \le n$,

$$\left|e^{is\theta}p^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right| + \frac{mn_s}{2}\left\{k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right\}$$
$$\le \left|e^{is\theta}Q^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right| - \frac{mn_s}{2}\left\{k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right\}. \tag{24}$$

Taking $A = \left|e^{is\theta}Q^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right|$,
$B = \left|e^{is\theta}p^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right|$ and
$C = \frac{mn_s}{2}\left\{k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right\}$ in Lemma 9, so that by (24)
$B + C \le A - C \le A$, we get for all real α,

$$\Bigg|\left\{\left|e^{is\theta}p^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right| + \frac{mn_s}{2}\left(k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right)\right\}e^{i\alpha}$$
$$+\left\{\left|e^{is\theta}Q^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right| - \frac{mn_s}{2}\left(k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right)\right\}\Bigg|$$
$$\le \left|\left|e^{is\theta}p^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right|e^{i\alpha} + \left|e^{is\theta}Q^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right|\right|.$$

Which implies for every $\gamma > 0$,

$$\int_0^{2\pi}\left|F(\theta) + e^{i\alpha}G(\theta)\right|^\gamma d\theta \le \int_0^{2\pi}\Bigg|\left|e^{is\theta}p^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right|e^{i\alpha}$$
$$+\left|e^{is\theta}Q^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right|\Bigg|^\gamma d\theta, \tag{25}$$

where

$$F(\theta) = \left|e^{is\theta}Q^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right|$$
$$-\frac{mn_s}{2}\left(k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right)$$

and

$$G(\theta) = \left|e^{is\theta}p^{(s)}(e^{i\theta}) + \beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right| + \frac{mn_s}{2}\left(k^{-n}\left|1+\frac{\beta}{(1+k)^s}\right| - \left|\frac{\beta}{(1+k)^s}\right|\right).$$

Integrating inequality (25) with respect to α from 0 to 2π, we get from Lemma 8, that for every $\gamma > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} \left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma} d\theta d\alpha$$

$$\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|e^{is\theta}p^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right|e^{i\alpha}+\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}Q(e^{i\theta})\right|\right|^{\gamma} d\alpha\right\} d\theta$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|e^{is\theta}p^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}p(e^{i\theta})\right|e^{i\alpha}+\left|e^{in\theta}\overline{(e^{is\theta}Q^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}Q(e^{i\theta}))}\right|\right|^{\gamma} d\alpha\right\} d\theta$$

$$= \int_0^{2\pi}\left\{\int_0^{2\pi}\left|(e^{is\theta}p^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}p(e^{i\theta}))e^{i\alpha}+e^{in\theta}\overline{(e^{is\theta}Q^{(s)}(e^{i\theta})+\beta\frac{n_s}{(1+k)^s}Q(e^{i\theta}))}\right|^{\gamma} d\theta\right\} d\alpha$$

(26)

$$\leq n_s^{\gamma}\int_0^{2\pi}\left|k^{-n}\left(1+\frac{\beta}{(1+k)^s}\right)e^{i\alpha}+\frac{\overline{\beta}}{(1+k)^s}\right|^{\gamma} d\alpha \int_0^{2\pi}\left|p(e^{i\theta})\right|^{\gamma} d\theta.$$

Since

$$\int_0^{2\pi}\left|k^{-n}\left(1+\frac{\beta}{(1+k)^s}\right)e^{i\alpha}+\frac{\overline{\beta}}{(1+k)^s}\right|^{\gamma} d\alpha = \int_0^{2\pi}\left|\left|k^{-n}\left(1+\frac{\beta}{(1+k)^s}\right)\right|e^{i\alpha}+\left|\frac{\overline{\beta}}{(1+k)^s}\right|\right|^{\gamma} d\alpha$$

$$= \int_0^{2\pi}||k^{-n}(1+\frac{\beta}{(1+k)^s})|e^{i\alpha}+|\frac{\beta}{(1+k)^s}||^{\gamma} d\alpha$$

$$= \int_0^{2\pi}\left|k^{-n}\left(1+\frac{\beta}{(1+k)^s}\right)e^{i\alpha}+\frac{\beta}{(1+k)^s}\right|^{\gamma} d\alpha.$$

Using this in inequality (26), we get for every $\gamma > 0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma} d\theta d\alpha \leq n_s^{\gamma}\int_0^{2\pi}\left|k^{-n}\left(1+\frac{\beta}{(1+k)^s}\right)e^{i\alpha}+\frac{\beta}{(1+k)^s}\right|^{\gamma} d\alpha$$

$$\times\int_0^{2\pi}\left|p(e^{i\theta})\right|^{\gamma} d\theta. \tag{27}$$

When taking

$$a = G(\theta) \quad \text{and} \quad b = F(\theta),$$

since $|b| \geq |a|$ from (24), we obtain from Lemma 10, that for every $\gamma > 0$,

$$\int_0^{2\pi} \left|F(\theta) + e^{i\alpha}G(\theta)\right|^{\gamma} d\alpha \geq |G(\theta)|^{\gamma} \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^{\gamma} d\alpha. \tag{28}$$

Integrating inequality (28) with respect to θ from 0 to 2π, we get from (27), that for every $\gamma > 0$,

$$\begin{aligned}
&\left\{ \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^{\gamma} d\alpha \int_0^{2\pi} \left[\left| e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{(1+k)^s} p(e^{i\theta}) \right| \right.\right. \\
&\left.\left. + \frac{mn_s}{2} \left(k^{-n} \left| 1 + \frac{\beta}{(1+k)^s} \right| - \left| \frac{\beta}{(1+k)^s} \right| \right) \right]^{\gamma} d\theta \right\}^{\frac{1}{\gamma}} \\
&\leq n_s \left\{ \int_0^{2\pi} \left| k^{-n} \left(1 + \frac{\beta}{(1+k)^s} \right) e^{i\alpha} + \frac{\beta}{(1+k)^s} \right|^{\gamma} d\alpha \right\}^{\frac{1}{\gamma}} \left\{ \int_0^{2\pi} \left| p(e^{i\theta}) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}.
\end{aligned} \tag{29}$$

Using $\delta \in \mathbb{C}$ with $|\delta| \leq 1$,we have

$$\begin{aligned}
&\left| e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{(1+k)^s} p(e^{i\theta}) + \frac{\delta mn_s}{2} \left(k^{-n} \left| 1 + \frac{\beta}{(1+k)^s} \right| - \left| \frac{\beta}{(1+k)^s} \right| \right) \right| \\
&\leq \left| e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{(1+k)^s} p(e^{i\theta}) \right| + \frac{mn_s}{2} \left(k^{-n} \left| 1 + \frac{\beta}{(1+k)^s} \right| - \left| \frac{\beta}{(1+k)^s} \right| \right),
\end{aligned}$$

we get from (29) that for every $\gamma > 0$,

$$\begin{aligned}
&\left\{ \int_0^{2\pi} \left| e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{(1+k)^s} p(e^{i\theta}) + \frac{\delta mn_s}{2} \left(k^{-n} \left| 1 + \frac{\beta}{(1+k)^s} \right| - \left| \frac{\beta}{(1+k)^s} \right| \right) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}} \\
&\leq n_s \left\{ \int_0^{2\pi} \left| k^{-n} \left(1 + \frac{\beta}{(1+k)^s} \right) e^{i\alpha} + \frac{\beta}{(1+k)^s} \right|^{\gamma} d\alpha \right\}^{\frac{1}{\gamma}} \frac{\left\{ \int_0^{2\pi} \left| p(e^{i\theta}) \right|^{\gamma} d\theta \right\}^{\frac{1}{\gamma}}}{\left\{ \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^{\gamma} d\alpha \right\}^{\frac{1}{\gamma}}},
\end{aligned}$$

which proves the theorem. □

Remark 1. If we take $k = 1$ in Theorem 2, then inequality (23) reduces to an inequality recently proved by Mir [10], which is again a generalization of Theorem 1.

Corollary 1. *If $p \in \mathbb{P}_n$ and $p(z) \neq 0$ in $|z| < 1$, then for any $\beta, \delta \in \mathbb{C}$ with $|\beta| \leq 1, |\delta| \leq 1$, $1 \leq s \leq n$ and $0 \leq \gamma < \infty$,*

$$\left\{\int_0^{2\pi} \left| e^{is\theta} p^{(s)}(e^{i\theta}) + \beta \frac{n_s}{2^s} p(e^{i\theta}) + \delta m \frac{n_s}{2}\left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right)\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n_s E_\gamma \left\{\int_0^{2\pi} \left|\left(1+\frac{\beta}{2^s}\right)e^{i\alpha} + \frac{\beta}{2^s}\right|^{\gamma} d\alpha\right\}^{\frac{1}{\gamma}} \left\{\int_0^{2\pi} \left|p(e^{i\theta})\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}, \tag{30}$$

where $m = \min_{|z|=k} |p(z)|$ and E_γ is given by (11).

The result is best possible and equality in (30) holds for the polynomial $p(z) = az^n + b$ with $|a| = |b| = 1$.

If we take $k = 1$ and $\delta = 0$ in Theorem 2, then inequality (23) reduces to inequality (10) of Theorem 1.

If we take $s = 1$ in (23), we get the following result.

Corollary 2. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k, k \leq 1$, then for every real or complex number β, δ with $|\beta| \leq 1, |\delta| \leq 1$, and $0 \leq \gamma < \infty$,*

$$\left\{\int_0^{2\pi} \left| e^{i\theta} p'(e^{i\theta}) + \beta \frac{n}{(1+k)} p(e^{i\theta}) + \delta m \frac{n}{2}\left(k^{-n}\left|1+\frac{\beta}{(1+k)}\right| - \left|\frac{\beta}{(1+k)}\right|\right)\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}$$

$$\leq n E_\gamma \left\{\int_0^{2\pi} \left|k^{-n}\left(1+\frac{\beta}{(1+k)}\right)e^{i\alpha} + \frac{\beta}{(1+k)}\right|^{\gamma} d\alpha\right\}^{\frac{1}{\gamma}}$$

$$\times \left\{\int_0^{2\pi} \left|p(e^{i\theta})\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}, \tag{31}$$

where $m = \min_{|z|=k} |p(z)|$ and E_γ is given by (11).

Further, if we take $\beta = 0$ in Theorem 2, we have

Corollary 3. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k, k \leq 1$, then for every real or complex number δ with $|\delta| \leq 1, 1 \leq s \leq n$ and $0 \leq \gamma < \infty$,*

$$\left\{\int_0^{2\pi} \left| e^{is\theta} p^{(s)}(e^{i\theta}) + k^{-n}\delta m \frac{n_s}{2}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \leq n_s E_\gamma k^{-n} \left\{\int_0^{2\pi} \left|p(e^{i\theta})\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}, \tag{32}$$

where $m = \min_{|z|=k} |p(z)|$ and E_γ is given by (11).

For $k=1$, $s=1$ and $\delta=0$, inequality (32) reduces to inequality (5). An important result is further implied by Corollary 2 on taking limit $\gamma \to \infty$, that

Corollary 4. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<k, k\leq 1$, then for every real or complex number δ with $|\delta|\leq 1, 1\leq s\leq n$,*

$$\max_{|z|=1}\left|z^s p^{(s)}(z)+k^{-n}\delta m\frac{n_s}{2}\right| \leq \frac{n_s}{2k^n}\max_{|z|=1}|p(z)|. \tag{33}$$

Further, in Corollary 2, if we put $s=1, \delta=0$ and taking limit $\gamma\to\infty$, we get

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}k^{-n}\max_{|z|=1}|p(z)|. \tag{34}$$

Remark 2. *Inequality* (34) *is expected to have smaller bound compared to inequality* (2) *for* $k \geq \left(\frac{1}{2}\right)^{\frac{1}{n}}$. *This inequality* (34) *gives inequality analogue to Lax* [8] *for* $\frac{1}{2^{\frac{1}{n}}} \leq k \leq 1$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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