Available online at http://scik.org

J. Math. Comput. Sci. 11 (2021), No. 4, 5031-5044

https://doi.org/10.28919/jmcs/5938

ISSN: 1927-5307

EXPLICIT SOLUTION OF TIME FRACTIONAL MODIFIED EQUAL WIDTH WAVE EQUATION BY LIE SYMMETRY ANALYSIS

HARISH KUMAR[1,*], DIMPLE SINGH[1], AMIT TOMAR[2]

[1]Amity School of Applied Sciences, Amity University, Haryana, India

[2]Amity Institute of Applied Sciences, Amity University, Noida, U.P., India

Abstract: In present article, the time fractional modified equal width wave equation has been examined by Lie symmetry reduction technique. This schemed methodology with generalized Erdelyi-Kober (E-K) integral and differential operator have been used to transformed the partial differential equations (PDEs) of generalized (non integer) order into ordinary differential equations (ODEs) of fractional order with insertion of some independent variable. At last, explicit solution obtained by power series method.

Keywords: time fractional modified equal width wave (TFMEWW) equation; power series solution; Riemann Liouville (R-L) derivative; Lie symmetry method; Erdelyi-Kober operator.

2010 AMS Subject Classification: 35R11.

1. INTRODUCTION

The generalized calculus literature is as ancient as classical calculus. The concepts of fractional differential equations (FDEs) are utilized in modeling distinct phenomenon of mechanics, dynamics and drug therapy in biological systems. It is also used to study new age advance problems in neurons network, image processing, geology and hydrology. Podlubny [1], Oldham

*Corresponding author

E-mail address: harishdhull56@gmail.com

Received April 28, 2021

[2] and Debnath [3] illustrated the content on fractional order calculus. They provided Grunwald, Caputo and Riemann-Liuovili (RL) fractional derivatives and integrals definition along with their physical and geometrical interpretations in real modeling.

The schemed study of Lie symmetry and their application has been derived by Olver [4]. Bakkayaraj and Sehdaven [5] provoked about the group formalism of geometrical transforms in this technique. Biswas et al.[6, 7] suggested the dual dispersion and non-linearity laws with the exclusive use of infinitesimals in symmetry reduction. Invariance criterion of some fractional PDEs, Hirota nonlinear, Hirota-Satsoma systems has been studied by Singla et al. [8]. Sneddon [9] used the concept of E-K operators and remarked that the system of FPDEs can be reduced to FODEs with the efficient use of these operators. The symmetry properties and exact solution of real time fractional KdV of third, fourth, fifth and generalized order compiled by Zhang [10], Wang et al. [17] and Gandhi [25, 26]. Kaur et al. [11-15] has been implemented Painlike and Lie symmetry to Einstein vacuum field equation, Complex Hirota forms in multiple real and complex solutions. Huang [16] provided the total solutions of time fractional Harry-Dym equation with R-L fractional derivatives approach. Garrido et al. [19] prompted on travelling wave generalized solution of Driffield-Sokolov system and Arora et al. [20] found solitary wave solutions of modified equal width wave equations by Lie infinitesimals. The unremarkable criterion of solitary waves of equal width and regularized long wave equation has been solved by Gardner et al. [21, 22] in late 20^{th} century. The physical phenomenon, scattering of regular long solitary wave has been studied by Morison et al. [23]. Rudin [24] attempted the implicit function theorem in principal of mathematical analysis, which has been used for convergence of power series solution by [10, 17, 25-26].

In recent times, mathematicians have devoted lot of efforts to analyze the explicit and exact solutions of linear and nonlinear PDEs. It’s difficult to obtain the exact solution of nonlinear differential equation as compared to linear differential equation. Therefore, some researchers have used numerical methods. It is always challenging task to find the exact or analytic solutions for nonlinear equations, and finding such solutions is even more difficult for fractional nonlinear equations. Hence, we are accepting the challenge and in this paper, we will try to find the explicit solution of a nonlinear time fractional equation.

The MEWWE occurring from the nonlinear media with dispersion process has been paid special concentration in the past decades. Our motivation is to generate mathematical formulation of the infinitesimals and investigate the symmetry reduction with power series solution of generalized TFMEWW equation with one parameter 'γ'

$$\partial_t^{\gamma} u + 3u^2 u_x - \mu u_{xxt} = 0 \;;\; 0 < \gamma < 1,\ \mu > 0 \tag{1}$$

and *(x,t)* is space-time coordinate and *u(x, t)* is amplitude of wave for the one dimensional wave-propagation in nonlinear media with dispersion phenomenon. Time fractional MEWW nonlinear model is based on the EW equation. Morrison [23] established this model as modified regular long wave equation and modified KdV equation in fluid mechanics.

We proposed definition and terminology in section 2; in section 3, Lie symmetry approach has been discussed. The application of series solution with its convergence to MEWW model described in sections 4 and 5. Finally, the remarks and conclusions established.

2. PRELIMINARIES

In this section, we would like to present the needful definitions and terminology related to fractional calculus.

In the first two definitions, let $h(t)$ be an integrable function on $(0,t)$ and for $g > 0$ assume that $h(t)$ is g-times differentiable on $(0,t)$ except for a set of measure zero.

2.1 CAPUTO FRACTIONAL DERIVATIVE:

$$D_t^{\beta}\left(h(t)\right) = \frac{1}{\Gamma(g-\beta)} \int_0^t (t-\xi)^{g-\beta-1} h^{g}(\xi) d\xi \ for\, g-1 < \beta \le g, t > 0, g \in N \tag{2}$$

2.2 R-L FRACTIONAL DERIVATIVE:

$$D_t^{\beta}\left(h(t)\right) = \frac{1}{\Gamma(g-\beta)} \frac{d^g}{dt^g} \int_0^t (t-\xi)^{g-\beta-1} h(\xi) d\xi \ for\ g-1 < \beta \le g, t > 0, g \in N \tag{3}$$

2.3 R-L PARTIAL DERIVATIVE OF GENERALIZED ORDER 'β' FOR THE FUNCTION $v(x,t)$ W.R.T. T

This R-L definition holds for the function of two variables and β is order of fractional derivative.

$$D_t^{\beta}\left(v(x,t)\right) = \frac{1}{\Gamma(g-\beta)} \frac{\partial^g}{\partial t^g} \int_0^t (t-\xi)^{g-\beta-1} v(x,\xi) d\xi \;\; for\, g-1 < \beta < g, t > 0, g \in N \qquad (4)$$
$$\frac{\partial^g v}{\partial t^g} \;\; when\, \beta = g$$

2.4 THE LEIBNITZ RULE FOR R-L FRACTIONAL DERIVATIVES:

Leibnitz rule is defined for the product of two functions. Hence, below is the definition of Leibnitz rule for fractional derivative of the product of two functions. $v(x,t)$ and $w(x,t)$ are function of two variable such that they are differentiable and integrable. β is order of fractional derivative.

$$D_t^{\beta}\left(v(x,t).w(x,t)\right) = \sum_{g=0}^{\infty} \binom{\beta}{g} D_t^{\beta-g}\left(v(x,t)\right).D_t^{g}\left(w(x,t)\right),\ \beta > 0, \qquad (5)$$
$$where \binom{\beta}{g} = \frac{(-1)^g \beta.\Gamma(g-\beta)}{\Gamma(1-\beta).\Gamma(g+1)}$$

2.5 FRACTIONAL DERIVATIVES OF A CONSTANT IS ZERO AND FOR A FUNCTION $c.w(x,p)$

$$D_p^{\alpha}\left(c.w(x,p)\right) = c.\frac{\partial^{\alpha}}{\partial t^{\alpha}} w(x,p) \qquad (6)$$
$$also if\;\; p \in (c,d] then D_p^{\alpha}(p-c)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(p-c)^{\beta-\alpha}; \alpha \geq 0, \beta > 0$$

2.6 DEFINITION OF A E-K FRACTIONAL DIFFERENTIAL AND INTEGRAL OPERATOR

$\left(P_{\partial}^{r,\gamma} G\right)(\psi)$ and $\left(K_{\partial}^{r,\gamma} H\right)(\psi)$ are left hand sided Erdelyi-Kober Fractional differential operator and Erdelyi-Kober Fractional integral operator respectively.

$$\left(P_{\partial}^{r,\gamma} G\right)(\psi) = \prod_{j=0}^{m-1}\left(r + j - \frac{1}{\partial}\psi \frac{d}{d\psi}\right)\left(K^{r+\gamma, m-\gamma} G\right)(\psi) \;\; with\;\; \psi > 0, \partial > 0 \, and\, \gamma > 0; m = \begin{cases} [\gamma]+1, \gamma \notin N \\ \gamma, \gamma \in N \end{cases} \qquad (7)$$
$$\left(K_{\partial}^{r,\gamma} H\right)(z) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_1^{\infty} (w-1)^{\gamma-1} w^{-(r+\gamma)}\, hH(\psi w^{\frac{1}{\partial}})\, dw, & \gamma > 0 \\ h(\psi) & \gamma = 0 \end{cases}$$

3. LIE SYMMETRY ANALYSIS

There are several semi-analytic and analytic techniques to obtain exact and approximate solutions of FPDEs but we imposed Lie approach to address the infinitesimal symmetries of generalized differential equations; as conversion of FPDEs into FODEs is major task and it is feasible after the prolongation technique explained under:

Suppose a general fractional PDE with space-time variables and $v = v(x, p)$.

$$\partial_p^\gamma v + F(p, x, v, v_x, v_p, v_{xx}, v_{xxp}, \ldots\ldots..) = 0; \, where \;\, 0 < \gamma < 1, \tag{8}$$

Lie group of transformations with parameter ε is taken as

$$\begin{aligned} \hat{x}^* &= \hat{x} + \varepsilon\zeta(x, p\,;v) + O(\varepsilon^2), \\ \hat{p}^* &= \hat{p} + \varepsilon\tau(x, p\,;v) + O(\varepsilon^2), \\ \hat{v}^* &= \hat{v} + \varepsilon\eta(x, p\,;v) + O(\varepsilon^2), \end{aligned} \tag{9}$$

Associated lie algebra of (9) is generated by vectors fields

$$X = \zeta\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial p} + \eta\frac{\partial}{\partial v} \tag{10}$$

Apply prolongation of 3^{rd} order on (8)

$$pr^{(\gamma,3)}X(\Delta)|_{\Delta=0}, \Delta = \partial_p^\gamma v - F \tag{11}$$

By preserving the operator $pr^{(\gamma,3)}X$ takes the form below:

$$pr^{(\gamma,3)}X(\Delta) = X + \eta^{\gamma,p}\frac{\partial}{\partial\left(\partial_p^\gamma v\right)} + \eta^x\frac{\partial}{\partial v_x} + \eta^{xxp}\frac{\partial}{\partial v_{xxp}} \tag{12}$$

Here, we use essential terms only which are usable in this paper and $\tau, \zeta \; and \, \eta$ are infinitesimals and η^x, η^{xxp} are extended infinitesimals of order 1 and 3 respectively and $\eta^{\gamma,p}$ is an extended time fractional infinitesimal of order γ

$$\begin{aligned} \eta^x &= D_x(\eta) - v_p D_x(\tau) - v_x D_x(\zeta) \\ &= \eta_x + \left(\eta_v - \zeta_x\right)v_x - \tau_x v_p - \zeta_v v_x^2 - \tau_v v_x v_p \end{aligned} \tag{13}$$

$$\eta^{xxp} = D_x(\eta^{xp}) - u_{xxp}D_x(\zeta) - u_{xpp}D_x(\tau) \tag{14}$$

$$\begin{aligned} &\eta^{\gamma,p} = D_p^\gamma(\eta) + \zeta D_p^\gamma(v_x) - D_p^\gamma(\zeta v_x) + \tau D_p^\gamma(v_p) - D_p^\gamma(\tau v_p) \\ &here D_p = \partial_p + v_p\frac{\partial}{\partial v} + v_{pp}\frac{\partial}{\partial v_p} + v_{xp}\frac{\partial}{\partial v_x} + \ldots\ldots\ldots.; \\ &D_x = \partial_x + v_x\frac{\partial}{\partial v} + v_{xp}\frac{\partial}{\partial v_p} + v_{xx}\frac{\partial}{\partial v_x} + \ldots\ldots\ldots. \end{aligned} \tag{15}$$

Now generalized Leibnitz Rule is

$$D_t^{\beta}\left(h(t).k(t)\right)=\sum_{n=0}^{\infty}\binom{\beta}{n}D_t^{n}(h).D_t^{\beta-n}(k) \qquad (16)$$
$$where D_t^{0}\left(h(t)\right)=h(t), D_t^{n+1}\left(h(t)\right)=D_t\left(D_t^{n}(h(t))\right)$$

We obtain expressions

$$\zeta\, D_p^{\gamma}(v_x)-D_p^{\gamma}(\zeta\, v_x)=-\sum_{n=1}^{\infty}\binom{\gamma}{n}D_p^{n}(\zeta)D_p^{\gamma-n}(v_x) \qquad (17)$$

$$\tau\, D_p^{\gamma}(v_p)-D_p^{\gamma}(\tau\, v_p)=-\alpha D_p(\tau)\,\partial_p^{\gamma}v-\sum_{n=1}^{\infty}\binom{\gamma}{n+1}D_p^{n+1}(\tau)\,D_p^{\gamma-n}(v) \qquad (18)$$

Generalized chain rule for composition of mappings is defined as

$$\frac{d^{\gamma}}{dp^{\gamma}}\left(f(g(p))\right)=\frac{\sum_{k=0}^{n}(-1)^{n}\binom{n}{k}g^{k}(p)\partial_t^{\gamma}\left(g^{n-k}(p)\right)}{n!}\frac{d^{n}f(z)}{dz^{n}}\, where z=g(p) \qquad (19)$$

Using generalized Leibnitz rule, we have

$$D_p^{\gamma}(\eta)=\partial_p^{\gamma}(\eta)+\eta_v\partial_p^{\gamma}(v)-v\partial_p^{\gamma}(\eta_v)+\sum_{g=0}^{\infty}\binom{\gamma}{g}\partial_p^{g}(\eta_v)\partial_p^{\gamma-g}(v)+\psi \qquad (20)$$
$$where\ \psi=\sum_{g=2}^{\infty}\sum_{m=2}^{n}\sum_{j=2}^{m}\binom{\gamma}{g}\binom{g}{m}\frac{p^{g-\gamma}V_j}{j!\Gamma(g+1-\gamma)}\frac{\partial^{g-m+j}\eta}{\partial p^{g-m}\partial v^{j}}$$

$$\begin{aligned}\eta^{\gamma,p}&=\partial_p^{\gamma}(\eta)+\eta_v\partial_p^{\gamma}(v)-v\partial_p^{\gamma}(\eta_v)+\sum_{g=0}^{\infty}\binom{\gamma}{g}\partial_p^{g}(\eta_v)\partial_p^{\gamma-g}(v)+\psi-\sum_{n=1}^{\infty}\binom{\gamma}{n}D_p^{n}(\zeta)D_p^{\gamma-n}(v_x)\\&-\gamma D_p(\tau).\partial_p^{\gamma}v-\sum_{n=1}^{\infty}\binom{\gamma}{n+1}D_p^{n+1}(\tau)D_p^{\gamma-n}(v)\\&=\partial_p^{\gamma}(\eta)+(\eta_v-\gamma D_p(\tau))\partial_p^{\gamma}(v)-v\partial_p^{\gamma}(\eta_v)+\sum_{n=1}^{\infty}\left[\binom{\gamma}{n}\partial_p^{n}(\eta_v)-\binom{\gamma}{n+1}D_p^{n+1}(\tau)\right]\partial_p^{\gamma-n}(v)\\&-\sum_{n=1}^{\infty}\binom{\gamma}{n}D_p^{n}(\zeta)\partial_p^{\gamma-n}(v_x)+\psi\end{aligned} \qquad (21)$$

After using (12) and (8), substitute (13-21) and comparing the coefficients of v_x, v_{xx} and solve the set of obtained PDEs and FDEs.

3.1 FRACTIONAL EQUAL WIDTH WAVE EQUATION

Application of equation (12) to equation (1), we get

$$\eta^{\gamma,t}+6\eta uu_x+3\eta^{x}u^{2}-\mu\eta^{xxt}=0 \qquad (22)$$

Substituting (13-21) in (22)

$$\partial_t^{\gamma}(\eta)-u\partial_t^{\gamma}(\eta_u)+3u^2\eta_x-\mu\eta_{xxt}=0 \tag{23}$$

$$\binom{\gamma}{n}\partial_t^{n}(\eta_u)-\binom{\gamma}{n+1}D_t^{n+1}(\tau)=0, n\in N \tag{24}$$

$$6\eta u+3u^2(\eta_u-\zeta_x)-\mu(2\eta_{utx}-\zeta_{xxt})=0 \tag{25}$$

$$3u^2\tau_x+\mu(\eta_{uxx}-\tau_{xxt})=0 \tag{26}$$

$$3u^2\zeta_u+\mu(\eta_{uut}-2\zeta_{utx})=0 \tag{27}$$

$$3u^2\tau_u+\mu(2\eta_{uux}-\zeta_{uxx}-2\tau_{utx})=0 \tag{28}$$

$$\zeta_t=\zeta_u=0=\tau_u=\tau_x;\ \eta_u-\gamma D_t(\tau)=0 \tag{29}$$

Solving (23-29), infinitesimals with constants p, q and r found to be

$$\zeta=\frac{q(\gamma-1)x}{2}+p;\ \tau=qt+r;\ \eta=\frac{-q(\gamma+1)u}{4} \tag{30}$$

The corresponding generator of lie algebra is described as

$$T=\left(\frac{q(\gamma-1)x}{2}+p\right)\frac{\partial}{\partial x}+(qt+r)\frac{\partial}{\partial t}+\left(\frac{-q(\gamma+1)u}{4}\right)\frac{\partial}{\partial u} \tag{31}$$

Taking standard generators of (31)

$$T_1=\frac{\partial}{\partial x}\ T_2=\frac{x(\gamma-1)}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{u(\gamma+1)}{4}\frac{\partial}{\partial u}\ and\ T_3=\frac{\partial}{\partial t} \tag{32}$$

Corresponding to T_1, simple characteristic equation is

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{0} \tag{33}$$

And, its invariant solution is obtained as $T=t, u=F(T)$

By putting in equation (1), it reduced to FODE.

$$\partial_t^{\gamma}F(t)=0 \tag{34}$$

Corresponding to T_2, characteristic equation found

$$\frac{2.dx}{(\gamma-1)x}=\frac{dt}{t}=\frac{4du}{-(\gamma+1)u} \tag{35}$$

On solving; explicit solution is obtained as

$$u = t^{\frac{-(\gamma+1)}{4}} F(z) \, and \, z = xt^{\frac{(1-\gamma)}{2}} \tag{36}$$

By putting in equation (1) along with the use of theorem explained below, it reduced to FODE given by (37)

Theorem1. The Similarity transformations form by variables ‘u’ and ‘z’ in (36), reduce time fractional MEWW (1) to nonlinear FODE depends on ‘z’ as

$$\left(P_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{4},\gamma} F\right)(z) + 3F^2 F_Z - \mu\left[\left(\frac{3-5\gamma}{4}\right)F_{ZZ} + \frac{1-\gamma}{2} zF_{ZZZ}\right] = 0 \tag{37}$$

Where Erdelyi-Kober fractional differ-integral operators have been explained in (7)

Proof:

Let $(g-1) \le \gamma \le g, g = 1,2,\dots\dots\dots$

The fractional derivative approach in R-L sense taken as

$$D_t^{\gamma} u = D_t^{g}\left[\frac{1}{\Gamma(g-\gamma)} \int_0^t (t-s)^{g-\gamma-1} s^{-(\gamma+1)/4} F(xs^{(1-\gamma)/2}) ds\right] \tag{38}$$

Putting $s = \frac{t}{w}$, to obtain

$$D_t^{\gamma} u = D_t^{g}\left[\frac{1}{\Gamma(g-\gamma)} \int_1^{\infty} (t-\frac{t}{w})^{g-\gamma-1} \left(\frac{t}{w}\right)^{-(\gamma+1)/4} F\{x\left(\frac{t}{w}\right)^{(1-\gamma)/2}\} \frac{t}{w^2} dw\right] \tag{39}$$

$$= D_t^{g}\left[\frac{t^{g-\frac{1}{4}-\frac{5\gamma}{4}}}{\Gamma(g-\gamma)} \int_1^{\infty} (w-1)^{g-\gamma-1} w^{-(g+\frac{3}{4}-\frac{5\gamma}{4})} F(zw^{\frac{\gamma-1}{2}}) dw\right] \tag{40}$$

$$D_t^{\gamma} u = D_t^{g}\left[t^{g-\frac{1}{4}-\frac{5\gamma}{4}}\left(K_{\frac{2}{\gamma-1}}^{\frac{3-\gamma}{4},g-\gamma} F\right)(z)\right] \tag{41}$$

If $z = xt^{\frac{1-\gamma}{2}}, F \in C'(0,\infty)$

$$tD_t F(z) = tx\frac{1-\gamma}{2} t^{\frac{-(\gamma+1)}{2}} D_Z F(z) = \frac{1-\gamma}{2} zD_Z F(z) \tag{42}$$

Then $D_t^{g}\left[t^{g-\frac{1}{4}-\frac{5\gamma}{4}}\left(K_{\frac{2}{\gamma-1}}^{\frac{3-\gamma}{4},g-\gamma} F\right)(z)\right] = D_t^{g-1}\left[D_t\left\{t^{g-\frac{1}{4}-\frac{5\gamma}{4}}\left(K_{\frac{2}{\gamma-1}}^{\frac{3-\gamma}{4},g-\gamma} F\right)(z)\right\}\right]$

$$= D_t^{g-1}\left[t^{g-1-\frac{1}{4}-\frac{5\gamma}{4}}\left(\left(g-\frac{1}{4}-\frac{5\gamma}{4}+\frac{(1-\gamma)}{2}zD_Z\right)K_{\frac{2}{\gamma-1}}^{\frac{3-\gamma}{4},g-\gamma}F\right)(z)\right] \tag{43}$$

Repeating process (g-1) times, we find

$$\begin{aligned} D_t^{g}\left[t^{g-\frac{1}{4}-\frac{5\gamma}{4}}\left(K_{\frac{2}{\gamma-1}}^{\frac{3-\gamma}{4},g-\gamma}F\right)(z)\right] &= t^{-\frac{1}{4}-\frac{5\gamma}{4}}\prod_{j=0}^{g-1}\left(1+j-\frac{1}{4}-\frac{5\gamma}{4}+\frac{(1-\gamma)}{2}zD_Z\right)\left(K_{\frac{2}{\gamma-1}}^{\frac{3-\gamma}{4},g-\gamma}F\right)(z) \\ &= t^{-\frac{1}{4}-\frac{5\gamma}{4}}\left(P_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{4},\gamma}F\right)(z) \end{aligned} \tag{44}$$

$$\Rightarrow D_t^{\gamma}u = t^{-\frac{1}{4}-\frac{5\gamma}{4}}\left(P_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{4},\gamma}F\right)(z) \tag{45}$$

At last, we obtain

$$\left(P_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{4},\gamma}F\right)(z)+3F^2F_Z-\mu\left[\left(\frac{3-5\gamma}{4}\right)F_{ZZ}+\frac{1-\gamma}{2}zF_{ZZZ}\right]=0 \tag{46}$$

4. EXPLICIT POWER SERIES SOLUTION

To execute the solution of obtained FODE (46), we would like to pursue power series solution to obtain explicit solution of generalized MEWW equation

Let $F(z)=\sum_{g=0}^{\infty}a_g z^g$ be the solution of (46), where coefficients 'a_g' are to be determined. Putting the value of $F(z)$ in (46), we get

$$\begin{aligned} &\sum_{g=0}^{\infty}\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)g}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)}{2}g\right)}a_g z^g+3\left(\sum_{g=0}^{\infty}a_g z^g\right)^2\left(\sum_{g=0}^{\infty}(g+1)a_{g+1}z^g\right)- \\ &\mu\left[\left(\frac{3-5\gamma}{4}\right)\sum_{g=0}^{\infty}(2+g)(1+g)a_{g+2}z^g+\left(\frac{1-\gamma}{2}\right)z\sum_{g=0}^{\infty}(1+g)(2+g)(3+g)a_{g+3}z^g\right]=0 \end{aligned} \tag{47}$$

$$\begin{aligned} &\sum_{g=0}^{\infty}\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)g}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)g}{2}\right)}a_g z^g+3\left(\sum_{g=0}^{\infty}a_g z^g\right)^2\left(\sum_{g=0}^{\infty}(g+1)a_{g+1}z^g\right)- \\ &\mu\left[\left(\frac{3-5\gamma}{4}\right)\sum_{g=0}^{\infty}(2+g)(1+g)a_{g+2}z^g+\left(\frac{1-\gamma}{2}\right)\sum_{g=1}^{\infty}g(2+g)(1+g)a_{g+2}z^g\right]=0 \end{aligned} \tag{48}$$

After rearranging the terms, we obtain:

$$\sum_{g=0}^{\infty}\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)g}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)g}{2}\right)}a_g z^g+3\sum_{g=0}^{\infty}\left[\sum_{m=0}^{g}\left\{\sum_{i=0}^{m}a_i a_{m-i}\right\}(g+1-m)a_{g+1-m}\right]z^g- \\ \mu\left[\left(\frac{3-5\gamma}{4}\right)\sum_{g=0}^{\infty}(g+2)(g+1)a_{g+2}z^g+\left(\frac{1-\gamma}{2}\right)\sum_{g=1}^{\infty}g(2+g)(1+g)a_{g+2}z^g\right]=0 \tag{49}$$

$$\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)}{2}\right)}a_0+3a_0a_1-2\mu\left(\frac{3-5\gamma}{4}\right)a_2+\sum_{g=1}^{\infty}\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}a_g z^g+ \\ 3\sum_{g=1}^{\infty}\left[\sum_{m=0}^{g}\left\{\sum_{i=0}^{m}a_i a_{m-i}\right\}(g+1-m)a_{g+1-m}\right]z^g- \\ \mu\left[\left(\frac{3-5\gamma}{4}\right)\sum_{g=1}^{\infty}(g+2)(g+1)a_{g+2}z^g+\left(\frac{1-\gamma}{2}\right)\sum_{g=1}^{\infty}g(2+g)(1+g)a_{g+2}z^g\right]=0 \tag{50}$$

$$\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)}{2}\right)}a_0+3a_0a_1-2\mu\left(\frac{3-5\gamma}{4}\right)a_2+\sum_{g=1}^{\infty}\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}a_g z^g+ \\ 3\sum_{g=1}^{\infty}\left[\sum_{m=0}^{g}\left\{\sum_{i=0}^{m}a_i a_{m-i}\right\}(g+1-m)a_{g+1-m}\right]z^g-\mu\left[\sum_{g=1}^{\infty}\left(\frac{3-5\gamma}{4}+g\frac{1-\gamma}{2}\right)(2+g)(1+g)a_{g+2}z^g\right]=0 \tag{51}$$

Comparing the powers of 'z' on both sides,

$$\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)}{2}\right)}a_0+3a_0{}^2a_1-2\mu\left(\frac{3-5\gamma}{4}\right)a_2=0$$

$$\Rightarrow a_2=\frac{2}{\mu(3-5\gamma)}\left[\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)}{2}\right)}+3a_1a_0\right]a_0 \tag{52}$$

$$a_{g+2}=\frac{4}{\mu\{3+2g-(2g+5)\gamma\}(g+1)(g+2)}\left[\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}a_g+3\sum_{m=0}^{g}\left\{\sum_{i=0}^{m}a_i a_{m-i}\right\}(g+1-m)a_{g+1-m}\right] \tag{53}$$

Hence, power series solution of the equation (46) is given below:

$$F(z)=a_0+a_1\left(xt^{\frac{1-\gamma}{2}}\right)+a_2\left(xt^{\frac{1-\gamma}{2}}\right)^2+\sum_{g=1}^{\infty}a_{g+2}\left(xt^{\frac{1-\gamma}{2}}\right)^{g+2} \tag{54}$$

Where, the explicit power series solution for time fractional modified equal width wave equation is provided below:

$$u(x,t)=\frac{1}{t^{\frac{(\gamma+1)}{4}}.}\left[a_0+a_1\left(xt^{\frac{1-\gamma}{2}}\right)+a_2\left(xt^{\frac{1-\gamma}{2}}\right)^2+\sum_{g=1}^{\infty}a_{g+2}\left(xt^{\frac{1-\gamma}{2}}\right)^{g+2}\right] \tag{55}$$

4.1 CONVERGENCE OF THE SERIES SOLUTION OF TFMEWW EQUATION

We can take a_0 and a_1 as two arbitrary constants from equation (52). So let us assume $a_0=\rho\neq 0$ and $a_1=\sigma\neq 0$ therefore, by equation (52) $a_2=\frac{2}{\mu(3-5\gamma)}\left[\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{(\gamma-1)}{2}\right)}+3\sigma\rho\right]\rho$.

Similarly, we can find other coefficient from (53). From (53), we observe that

$$|a_{g+2}|\leq N\left[|a_g|+\sum_{m=0}^{g}\sum_{i=0}^{m}|a_i||a_{m-i}||a_{g+1-m}|\right] \tag{56}$$

Where $N=\max\left[\frac{4}{\mu\{3+2g-(2g+5)\gamma\}},\frac{12}{\mu\{3+2g-(2g+5)\gamma\}}\right]$ and $\frac{\Gamma\left(\frac{3-5\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4}-\frac{g(\gamma-1)}{2}\right)}<1$ for all

g.

We now consider the new power series

$$M=R(Y)=\sum_{g=0}^{\infty}r_gY^g \text{ as } r_0=|a_0|=|\rho|,\ \ r_1=|a_1|=|\sigma|,$$

$$r_2=|a_2|,\text{ And } r_{g+2}=N\left[r_g+\sum_{m=0}^{g}\sum_{i=0}^{m}r_i.r_{m-i}r_{g+1-m}\right],\ \ g=1,2,3,\ldots\ldots$$

We can easily show that $|a_g|\leq r_g,\ g=1,2,3,\ldots\ldots.$

Now, we claim for the positive radius of convergence of the series $M=R(Y)=\sum_{g=0}^{\infty}r_gY^g$

$$R(Y) = r_0 + r_1Y + r_2Y^2 + \sum_{g=1}^{\infty} r_{g+2}Y^{g+2}$$
$$= r_0 + r_1Y + r_2Y^2 + N[\sum_{g=1}^{\infty} r_gY^{g+2} + \sum_{g=1}^{\infty}\sum_{m=0}^{g}\sum_{i=0}^{m} r_i r_{m-i} r_{g+1-m} Y^{g+2}] \tag{57}$$
$$= r_0 + r_1Y + r_2Y^2 + N[\left(M - r_0 + 2Mr_0r_1 - M^2r_1 - r_1r_0^{\,2}\right)Y^2 + \left(M^3 + 3Mr_0(r_0 - M)\right) - r_0^{\,3})Y]$$

Now, consider the implicit functional equation,

$$H(Y,M) = M - R(Y)$$
$$= M - r_0 - r_1Y - r_2Y^2 - N[\left(M - r_0 + 2Mr_0r_1 - M^2r_1 - r_1r_0^{\,2}\right)Y^2 + \left(M^3 + 3Mr_0(r_0 - M)\right) - r_0^{\,3})Y] \tag{58}$$

Here, H is analytic function in (Y,M) plane and $H(0,r_0) = 0, H'_M(0,r_0) = 1 \neq 0$

Therefore, $M = R(Y)$ has positive radius of convergence and is analytic in the vicinity of the $(0,r_0)$.

Hence, series solution (54) of equation (1) is convergent in the vicinity of the point $(0,r_0)$ and can be verified by implicit function theorem [24].

Thus, solution of equation (46) in power series is as:

$$F(z) = \rho + \sigma Z + \frac{2}{\mu(3-5\gamma)}\left[\frac{\Gamma\left(\frac{3-5\gamma}{4} - \frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4} - \frac{(\gamma-1)}{2}\right)} + 3\sigma\rho\right]\rho(Z)^2 + \ldots \tag{59}$$

Therefore, power series solution of equation (1) is

$$u(x,t) = \frac{1}{t^{\frac{(\gamma+1)}{4}}.}\left[\rho + \sigma\left(xt^{\frac{1-\gamma}{2}}\right) + \frac{2}{\mu(3-5\gamma)}\left[\frac{\Gamma\left(\frac{3-5\gamma}{4} - \frac{(\gamma-1)}{2}\right)}{\Gamma\left(\frac{3-\gamma}{4} - \frac{(\gamma-1)}{2}\right)} + 3\sigma\rho\right]\rho\left(xt^{\frac{1-\gamma}{2}}\right)^2 + \ldots\ldots\right] \tag{60}$$

$\rho\, and\, \sigma$ are the arbitrary constants and μ is a positive parameter.

CONCLUSIONS

In present work, the time fractional MEWW equation with one parameter analyzed by means of Lie symmetry analysis in the sense of RL derivative. We have obtained the infinitesimal generators along with explicit similarity transformations by invariance analysis of the problem.

In addition, with the help of the similarity variables, FPDE are converted into nonlinear FODE by making use of EK differ-integral operators. Power series method is implemented to the FODE. In last, convergence of the power series solution is proved. Overall, we conclude that the mathematical solution with use of dual techniques Lie symmetry analysis along with power series solution is reliable and efficient tool in obtaining the solution of such kind of linear and nonlinear time fractional wave models.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, San Diego, 1999.

[2] K.B. Oldham, J. Spanier, The fractional calculus, Academic Press, New York, 1974.

[3] L. Debnath, Recent applications of fractional calculus to science and engineering, Int. J. Math. Math. Sci. 54 (2003), 3413-3442.

[4] P.J. Olver, Applications of Lie groups to differential equations, Springer, New York, 2000.

[5] T. Bakkyaraj, R. Sahadevan, Group formalism of Lie transformations to time-fractional partial differential equations, Pramana-J. Phys. 85 (2015), 849–860.

[6] A. Biswas, M. Song, H. Triki, et al. Solitons, shock waves, conservation laws and bifurcation analysis of Boussinesq equation with power law nonlinearity and dual dispersion. Appl. Math. Inform. Sci. 8(3) (2014), 949-957.

[7] A. Biswas, C.M. Khalique, Optical quasi-solitons by Lie symmetry analysis, J. King Saud Univ.-Sci. 24(3) (2012), 271-276.

[8] K. Singla, R.K. Gupta, On invariant analysis of some-time fractional nonlinear systems of partial differential equations, I, J. Math. Phys. 57(10) (2016), 101504.

[9] I.N. Sneddon, The use in mathematical physics of Erdelyi-Kober operators and of some of their generalizations in Fractional Calculus and its applications, Springer, Berlin, Heidelberg, (1975), 37-79.

[10] Y. Zhang, Lie Symmetry analysis to general time-fractional Korteweg-De Vries equations. Fract. Differ. Calc. 5(2) (2015), 125-133.

[11] L. Kaur, A.M. Wazwaz, Painlevé analysis and invariant solutions of generalized fifth-order nonlinear

integrable equation, Nonlinear Dyn. 94(4) (2018), 2469-2477.

[12] L. Kaur, A.-M. Wazwaz, Einstein's vacuum field equation: Painlevé analysis and Lie symmetries, Waves Random Complex Media. 31 (2021), 199–206.

[13] A.M. Wazwaz, L. Kaur, Complex simplified Hirota's forms and Lie symmetry analysis for multiple real and complex soliton solutions of the modified KdV–Sine-Gordon equation. Nonlinear Dyn. 95(3) (2019), 2209-2215.

[14] L. Kaur, R.K. Gupta, Some invariant solutions of field equations with axial symmetry for empty space containing an electrostatic field, Appl. Math. Comput. 231 (2014), 560-565.

[15] V. Kumar, L. Kaur, A. Kumar, M.E. Koksal, Lie symmetry based-analytical and numerical approach for modified Burgers-KdV equation, Results Phys. 8 (2018), 1136-1142.

[16] Q. Huang, R. Zhdanov, Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann–Liouville derivative, Physica A: Stat. Mech. Appl. 409 (2014), 110-118.

[17] G. Wang, T. Xu, Symmetry properties and explicit solutions of the nonlinear time fractional KdV equation, Bound. Value Probl. 2013 (2013), 232.

[18] G.W. Wang, M.S. Hashemi, Lie symmetry analysis and soliton solutions of time-fractional K (m, n) equation, Pramana – J. Phys. 88 (2017), 7.

[19] T.M. Garrido, M.S. Bruzón, Lie point symmetries and travelling wave solutions for the generalized Drinfeld–Sokolov system, J. Comput. Theor. Transport, 45(4) (2016), 290-298.

[20] R. Arora, A. Chauhan, Lie symmetry reductions and solitary wave solutions of modified equal width wave equation, Int. J. Appl. Comput. Math. 4(5) (2018), 122.

[21] L.R.T. Gardner, G.A. Gardner, Solitary waves of the regularised long-wave equation, J. Comput. Phys. 91(2) (1990), 441-459.

[22] L.R.T. Gardner, G.A. Gardner, Solitary waves of the equal width wave equation, J. Comput. Phys. 101(1) (1992), 218-223.

[23] P.J. Morrison, J.D. Meiss, J.R. Cary, Scattering of regularized-long-wave solitary waves, Physica D: Nonlinear Phenom. 11(3) (1984), 324-336.

[24] W. Rudin, Principles of Mathematical Analysis, 3rd edn. China Machine Press, Beijing, (2004).

[25] H. Gandhi, D. Singh, A. Tomar, Explicit solution of general fourth order time fractional KdV equation by lie symmetry analysis, in: Noida, India, 2020: p. 020012.

[26] H. Gandhi, A. Tomar, D. Singh, Lie symmetry analysis to general fifth-order time-fractional Korteweg-de-Vries equation and its explicit solution, in: P. Singh, R.K. Gupta, K. Ray, A. Bandyopadhyay (Eds.), Proceedings of International Conference on Trends in Computational and Cognitive Engineering, Springer Singapore, Singapore, 2021: pp. 189–201.