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EXISTENCE THEOREMS OF SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. This paper establishes a study on some important latest innovations in the existence of mild solution of semilinear for differential and fractional differential equations subject to nonlocal initial conditions. To apply this, the study uses Hausdorff measure of non-compactness and fixed point theorems. A wider applicability of these techniques are based on their reliability and reduction in the size of the mathematical work.

Keywords: Riemann-Liouville fractional derivative; differential equation; nonlocal condition; Banach space.

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1. Introduction

In recent years there has been a growing interest in the differential equation. The differential equations be an important branch of modern mathematics. It arises frequently in many applied areas which include engineering, electrostatics, mechanics, the theory of elasticity, potential, and mathematical physics [11, 12, 14, 15, 16, 33].

During the last decades, mathematical modeling has been supported by the field of fractional calculus, with several successful results and fractional operators showing to be an excellent

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tool to describe the hereditary properties of various materials and processes. Recently, this combination has gained a lot of importance, mainly because fractional differential equations have become powerful tools in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [4, 5, 6, 9, 20, 27].

The concept of nonlocal initial condition has been introduced to extend the study of classical initial value problems. The earliest works related with problems submitted to nonlocal initial conditions were made by Byszewski [1, 2, 3].

Recently, in so many published works focus on the development of techniques for discussing the solutions of nonlocal differential equations. For instance, we can remember the following works: The exsistence and uniqueness of solution to fractional ADEs were studied e.g. in [11, 12, 13, 16, 17, 18, 19, 21, 22, 23, 24]. Regularity properties of nonlocal ADEs were investigated e.g. in [3, 7, 8, 10, 30, 31, 32, 34, 35].

Motivated by above works, in this paper, we discuss new existence results for nonlocal differential equations of the form:

$$\begin{cases} y'(t) = Ay(t) + f(t, y(t)),\ t \in J := [0,1], \\ y(0) = g(y), \end{cases} \tag{1}$$

and

$$\begin{cases} D^q y(t) = Ay(t) + f(t, y(t)),\ t \in J, \\ y(0) = g(y), \end{cases} \tag{2}$$

where D^q is the standard Riemann-Liouville fractional derivative of order q, $0 < q \leq 1$ and $f : J \times X \longrightarrow X$, $g : C[0,1] \longrightarrow X$ are functions and A is a semi-group of bounded linear operators strongly continuous that generated by A in the Banach space X, with norm $\|.\|$.

The main objective of the present paper is to study the new existence results of the solution for nonlocal differential equations and nonlocal fractional differential equations.

The rest of the paper is organized as follows: In Section 2, some preliminaries, basic definitions and Lemma related to fractional calculus are recalled. In Section 3, the new existence results of the solution for nonlocal differential equations and nonlocal fractional differential equations have been proved. Finally, we will give a report on our work and a brief conclusion is given in Section 4.

2. Preliminaries

Let $(X,\|.\|)$ be a real Banach space. We denote by $C(J,X)$ the space of X-valued continuous functions on J with the norm $\|y\| = \sup\{\|y(t)\|,\ t\in J\}$, and by $L^1(J,X)$ the space of X-valued Bochner functions on J with the norm $\|y\| = \int_0^1 \|y(s)\|ds$. A C_0-semigroup $T(t)$ is said to be compact if $T(t)$ is compact for any $t>0$. If the semigroup $T(t)$ is compact then $t \longrightarrow T(t)y$ are equicontinuous at all $t>0$ with respect to y in all bounded subsets of X, i.e. the semigroup $T(t)$ is equicontinuous. In this paper, we suppose that A generates a C_0 semigroup $T(t)$ on X. Since no confusion may occur, we denote by α the Hausdorff measure of noncompactness on both X and $C(J,X)$.

By a mild solution of the nonlocal initial value problems (1) and (2), we mean the function $y\in C(J,X)$ which satisfies

$$y(t) = T(t)g(y) + \int_0^t T(t-s)f(s,y(s))ds,\ t\in J \tag{3}$$

$$y(t) = T(t)g(y) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,y(s))ds,\ t\in J, \tag{4}$$

Lemma 1. [8] *If $W\subseteq C(J,X)$ is bounded, then $\alpha(W(t)) \le \alpha(W)$ for all $t\in J$, where $W(t) = \{y(t), y\in W\}\subseteq X$. Furthermore if W is equicontinuous on $[0,1]$, then $\alpha(W(t))$ is continuous on J, and $\alpha(W) = \sup\{\alpha(W(t)),\ t\in J\}$.*

Lemma 2. [30] *If $\{u_n\}_{n=1}^{\infty}\subset L^1(J,X)$ is uniformly integrable, then $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\alpha\Big(\Big\{\int_0^t u_n(s)ds\Big\}_{n=1}^{\infty}\Big) \le 2\int_0^t \alpha\Big(\{u_n(s)\}_{n=1}^{\infty}\Big)ds. \tag{5}$$

Lemma 3. [30] *If the semigroup $T(t)$ is equicontinuous and $\eta\in L^1(J,R^+)$, then the set $\{t\longrightarrow \int_0^t T(t-s)y(s)ds,\ y\in L^1(J,R^+),\ \|y(s)\|\le\eta(s)$, for a.e. $s\in J\}$ is equicontinuous on J.*

Lemma 4. [7] *If W is bounded, then for each $\varepsilon>0$, there is a sequence $\{u_n\}_{n=1}^{\infty}\subseteq W$, such that $\alpha(W)\le 2\alpha(\{u_n\}_{n=1}^{\infty})+\varepsilon$.*

Lemma 5. [36] *Suppose that* $0<\varepsilon<1,\ h>0$ *and let*

$$S=\varepsilon^n+C_n^1\varepsilon^{n-1}h+C_n^2\varepsilon^{n-2}\frac{(h)^2}{2!}+\cdots+\frac{(h)^n}{n!},\ n\in N^+.$$

Then $S=o(\frac{1}{n^s})(n\longrightarrow+\infty)$, *where* $s>1$ *is an arbitrary real number.*

Lemma 6. *(*[26] *Fixed Point Theorem). Let* F *be a closed and convex subset of a real Banach space* X, *let* $A:F\longrightarrow F$ *be a continuous operator and* $A(F)$ *be bounded. For each bounded subset* $B\subset F$, *set*

$$A^1(B)=A(B),\ A^n(B)=A(\bar{co}(A^{n-1}(B))),\ n=2,3,...$$

If there exist a constant $0\le k<1$ *and a positive integer* n_0 *such that for each bounded subset* $B\subset F$,

$$\alpha(A^{n_0}(B))\le k\alpha(B),$$

then A *has a fixed point in* F.

Definition 1. [25] ***(Riemann-Liouville fractional integral).*** *The Riemann-Liouville fractional integral of order* $\alpha>0$ *of a function* h *is defined as*

$$\begin{aligned} J^\alpha h(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}h(t)dt, \qquad x>0, \quad \alpha\in\mathbb{R}^+,\\ J^0h(x) &= h(x), \end{aligned}$$

where $\mathbb{R}^+$ *is the set of positive real numbers.*

Definition 2. [25, 28] ***(Caputo fractional derivative).*** *The fractional derivative of* $h(x)$ *in the Caputo sense is defined by*

$$\begin{aligned} {}^cD_x^\alpha h(x) &= J^{m-\alpha}D^mh(x)\\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}\frac{d^mh(t)}{dt^m}dt, & m-1<\alpha<m,\\ \frac{d^mh(x)}{dx^m}, & \alpha=m, \quad m\in N, \end{cases} \end{aligned} \tag{6}$$

where the parameter α *is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive* α *will be considered.*

Hence, we have the following properties:

(1) $J^{\alpha}J^{v}h = J^{\alpha+v}h, \quad \alpha, v > 0.$

(2) $J^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha},$

(3) $D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0.$

(4) $J^{\alpha}D^{\alpha}h(x) = h(x) - \sum_{k=0}^{m-1} h^{(k)}(0^{+})\frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 3. [25, 29] ***(Riemann-Liouville fractional derivative).*** *The Riemann Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^{\alpha}h(x) = D^{m}J^{m-\alpha}h(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{7}$$

Proposition 1. [10] *Let G be the Cauchy operators, $\{f_n\}_{n=1}^{\infty}$ a sequence of functions in $L^1([0,T],X)$. Assume that there is $\mu, \eta \in L^1([0,T],R^+)$ satisfying $\sup_{n\geq 1}\|f_n(t)\| \leq \mu(t)$ and $\chi(\{f_n\}_{n=1}^{\infty}) \leq \eta(t)$, a.e $t \in [0,T]$. Then for all $t \in [0,T]$, we have*

$$\chi(\{(Gf_n)(t)\}_{n=1}^{\infty}) \leq 2\mu \int_0^t \eta(s)ds,$$

where μ equals to $\sup_{0\leq t\leq T}\|T(t)\|$ and is the Hausdorff MNC.

Theorem 1. [32] *Suppose the continuous map $F : U \longrightarrow E$ verifiers the condition (if $D \subseteq \bar{U}$ is countable and $D \subseteq (\bar{co})(\{O\} \cup F(D))$ then $\bar{D}$ is compact), where E is a Banach space, U an open subset of E and $O \in U$, and assumes that $y \neq \lambda F(y)$ for all $y \in \delta U$ and $\lambda \in (0,1)$ holds. Then F has a fixed point in $\bar{U}$.*

3. Main Results

In this section, we shall give an existence results of Eq.(1), with the initial condition (2) and prove it.

Before starting and proving the main results, we introduce the following hypotheses:

- **(H1):** The C_0 semigroup $T(t)$ generated by A is equicontinuous. We denote $N = \sup\{\|T(t)\|, t \in J\}$.
- **(H2):** The function $g : C(J,X) \to X$ is continuous and compact, there exists positive constants c and d such that $\|g(y)\| \leq c\|y\| + d$, for all $y \in C(J,X)$.
- **(H3):** The function $f(.,y)$ is measurable for all $y \in X$, and $f(t,)$ is continuous for a.e. $t \in J$.

(H4): There exists a function $m \in L^1(J,R^+)$ and a nondecreasing continuous function $\Omega : R^+ \longrightarrow R^+$ such that $\|f(t,y)\| \leq m(t)\Omega(\|y\|)$, for all $y \in X$, and a.e. $t \in J$.

(H5): There exists $L \in L^1(J,R^+)$ such that for any bounded $D \subset X$, $\alpha(f(t,D)) \leq L(t)\alpha(D)$, for a.e. $t \in J$.

If $\|f(t,y) - f(t,z)\| \leq L(t)\|y - z\|$, $L(t) \in L^1(J,R^+)$, $y,z \in X$, then we can get $\alpha(f(t,D)) \leq L(t)\alpha(D)$, for any bounded $D \subset X$, and a.e. $t \in J$.

Theorem 2. *Assume that (H1)–(H5) hold. If there exist a constant R with*

$$N(cR+d)+N\Omega(R)\int_0^1 m(s)ds \leq R. \tag{8}$$

Then there is at least one mild solution of the problem (1).

Proof. Firstly, we transform (3) into fixed point problem as $y = Ty$, where the operator

$$T : C(J,X) \longrightarrow C(J,X),$$

is defined by

$$(Ty)(t) = T(t)g(y) + \int_0^t T(t-s)f(s,y(s))ds,\ t \in J, \tag{9}$$

for all $y \in C(J,X)$. We can show that F is continuous by the usual techniques (see, e.g. [6,7]). We denote $W = \{y \in C(J,X), \|y(t)\| \leq R,\ \text{for all}\ t \in J\}$, then $W \subseteq C(J,X)$ is bounded and convex. For any $y \in W$, we have

$$\begin{aligned}\|(Ty)(t)\| &\leq \|T(t)g(y)\| + \|\int_0^t T(t-s)f(s,y(s))ds\| \\ &\leq N(cR+d)+N\Omega(R)\int_0^1 m(s)ds \\ &\leq R,\end{aligned}$$

which implies $T : W \longrightarrow W$ is a bounded operator.

Let $B_0 = \bar{co}(TW)$. For any $B \subset B_0$, we know from Lemma 4, for any $\varepsilon > 0$, there is a sequence $\{y_n\}_{n=1}^{\infty} \subset B$, such that

$$\begin{aligned}
\alpha(T^1B(t)) &= \alpha(TB(t)) \\
&\leq 2\alpha\Big(\int_0^t T(t-s)f(s,\{y_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\
&\leq 4\int_0^t \alpha\Big(T(t-s)f(s,\{y_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\
&\leq 4N\int_0^t L(s)\alpha(\{y_n(s)\}_{n=1}^{\infty})ds + \varepsilon \\
&\leq 4N\alpha(\{y_n\}_{n=1}^{\infty})\int_0^t L(s)ds + \varepsilon \\
&\leq 4N\alpha(B)\int_0^t L(s)ds + \varepsilon.
\end{aligned}$$

We know there is a continuous function $\Phi : J \longrightarrow R^+ (M = \max|\Phi(t)| : t \in J)$ such that for any $\gamma > 0,\ (\gamma < \frac{1}{N})$,

$$\int_0^t |L(s) - \Phi(s)|ds < \gamma.$$

So,

$$\begin{aligned}
\alpha(T^1B(t)) &\leq 4N\alpha(B)[\int_0^t |L(s)-\Phi(s)|ds + \int_0^t |\Phi(s)|ds] + \varepsilon \\
&\leq 4N[\gamma + Mt]\alpha(B) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from the above inequality that

$$\alpha(T^1B(t)) \leq (a+bt)\alpha(B),\ a = 4N,\ b = 4NM.$$

From Lemma 4, for any $\varepsilon > 0$, there is a sequence $\{z_n\}_{n=1}^{\infty} \subset \bar{co}(T^1B)$, such that

$$\begin{aligned}
\alpha(T^2B(t)) &= \alpha(T((\bar{co}(T^1B(t))))) \\
&\leq 2\alpha\Big(\int_0^t T(t-s)f(s,\{y_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\
&\leq 4\int_0^t \alpha\Big(T(t-s)f(s,\{y_n(s)\}_{n=1}^{\infty})\Big)ds + \varepsilon \\
&\leq 4N\int_0^t L(s)\alpha(\{y_n(s)\}_{n=1}^{\infty})ds + \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\leq 4N\int_0^t L(s)\alpha(T^1B(s))ds+\varepsilon\\
&\leq 4N\int_0^t \{|L(s)-\Phi(s)|+|\Phi(s)|\}(a+bs)\alpha(B)ds+\varepsilon\\
&\leq 4N\int_0^t |L(s)-\Phi(s)|ds(a+bt)\alpha(B)+4N\int_0^t M(a+bs)ds\alpha(B)+\varepsilon\\
&\leq \left(a^2+2abt+\frac{(bt)^2}{2!}\right)\alpha(B)+\varepsilon.
\end{aligned}$$

Hence, by the method of mathematical induction, for any positive integer n and $t \in J$, we obtain

$$\alpha(T^nB(t)) \leq \left(a^n+C_n^1a^{n-1}bt+C_n^2a^{n-2}\frac{(bt)^2}{2!}+\cdots+\frac{(bt)^n}{n!}\right)\alpha(B).$$

Therefore, by Lemma 3, we have

$$\alpha(T^nB) \leq \left(a^n+C_n^1a^{n-1}b+C_n^2a^{n-2}\frac{(b)^2}{2!}+\cdots+\frac{(b)^n}{n!}\right)\alpha(B).$$

From Lemma 5, there exists a positive integer n_0 such that

$$a^{n_0}+C_{n_0}^1a^{n_0-1}b+C_{n_0}^2a^{n_0-2}\frac{(b)^2}{2!}+\cdots+\frac{(b)^{n_0}}{n_0!}=r<1.$$

Then

$$\alpha(T^{n_0}B) \leq r\alpha(B).$$

It follows from Lemma 6, that F has at least one fixed point in B_0, i.e. the nonlocal initial value problem (1) has at least one mild solution in B_0. Thus, the proof is completed.

□

We shall next discuss the existence result for the nonlocal initial value problem (2). Here we list the following hypotheses.

(A1): $f(t,.) : X \longrightarrow X$ is continuous for a.e $t \in I := [0,T]$, and $fR(.,y) : I \longrightarrow X$ is measurable for $y \in X$, where $f : I \times X \longrightarrow X$.

(A2): $\|f(t,y)\| \leq Z(t)\theta(\|y\|)$ for all $y \in X$ and $t \in I$, where $Z \in L^1(I,R^+)$ and $\theta : R^+ \longrightarrow R^+$ is nondecreasing function.

(A3): $q(R(t,D)) \leq L(t)q(D)$ for all $t \in I$, where q is the Hausdorff $MNC, D \subset X$ every bounded set and the function $L \in L^1(I,R^+)$.

(A4): $T(t)$ is equicontinuous semigroup of bounded linear operators strongly continuous generated by (T) for all $t \in I$.

(A5): $\|g(y)\| \leq a\|y\| + b, \ \forall y \in C(I,X)$ and for some positive constants a, b where $g : C(I,X) \longrightarrow X$ is a continuous compact map.

(A6): Let $\iota > 0$ is a constant, $M = \sup_{y \leq t \leq 1} \|T(t)\|$, such that

$$\frac{(1-Ma)\ell}{M(b+h\theta(\iota)\|m\|_{L^1}} > 1, \ 2M\|m\|_{L^1} < 1.$$

where $L^*_{z_1}, L^*_{z_2}$ and L^*_g are positive constants.

Theorem 3. *Assume that the hypotheses (A1)–(A6) are satisfied. Then the problem* (2) *has at least one mild solution on I.*

Proof. Let the operator $\Upsilon : C(I,X) \to C(I,X)$ be defined by

$$(\Upsilon y)(t) = T(t)g(y) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) f(s, y(s)) ds, \ t \in I, \tag{10}$$

To prove the operator T is continuous on $C(I,X)$, we suppose $y_n \longrightarrow y$ in $C(I,X)$ then by (A2), we have that.

$$f(s, y_n(s)) \longrightarrow f(s, y(s)) \text{ as } n \longrightarrow \infty, \ \forall s \in I.$$

Since

$$\|f(s, y_n(s)) \longrightarrow f(s, y(s))\| \leq 2\theta(\iota)m(s)$$

by (A2), (A3) and the dominated convergence theorem we have ,

$$\begin{aligned}
\|\Upsilon y_n - \Upsilon y\| &\leq \|T(t)g(y_n) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) f(s, y_n(s)) ds \\
&\quad -T(t)g(y) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} T(t-s) f(s, y(s)) ds\| \\
&\leq \|T(t)\| \|g(y_n) - g(y)\| + \frac{T^q \|T(t-s)\|}{\Gamma(q+1)} \|f(s, y_n(s)) - f(s, y(s))\| \\
&\leq M\|g(y_n) - g(y)\| + \frac{T^q M}{\Gamma(q+1)} \|f(s, y_n(s)) - f(s, y(s))\| \\
&\quad \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Then Υ is continuous.

Firstly, we prove that Υ is satisfied Monch's conditions , let $B_r = \{y \in C(J,X) : \|y\|_\infty \leq r\}$, and suppose $D \subseteq B_r$ is a countable such that $D \subseteq (\bar{CO})(\{0\} \cup \Upsilon(D))$. Let Q is the Hausdorff MNC.

Secondly, we will show that $Q(D)=0$, let us suppose $D=\{y_n\}_{n=1}^{\infty}$ from conditions (A4) and (A5), we have $T(t)$ is equicontinuous $g(y)$ is compact and $f(s,y(s))$ is measurable function, and since

$$(\Upsilon y_n) \leq T(t)g(y_n)+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,y_n(s))ds$$

$D\subseteq (\bar{CO})(\{0\}\cup\Upsilon(D))$ is equicontinuous if we can verify that $\{\Upsilon y_n\}_{n=1}^{\infty}$ is equicontinuous .

We get from (A3), (A6), proposition 1 and properties of MNC Q, the following:

$$\begin{aligned}
Q(\{\Upsilon y_n\}_{n=1}^{\infty}) &\leq \sup_{y\leq t\leq T} Q(T(t)g(y_n)_{n=1}^{\infty})+\sup_{y\leq t\leq T} Q\Big(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,y_n(s))ds\Big)\\
&\leq M\sup_{y\leq t\leq T} Q\{y_n\}_{n=1}^{\infty}+\frac{T^qM}{\Gamma(q+1)}\sup_{y\leq t\leq T} Q\{y_n\}_{n=1}^{\infty}\\
&\leq \frac{2T^qM}{\Gamma(q+1)}\int_0^T L(s)ds\sup_{y\leq t\leq T} Q\{y_n(t)\}_{n=1}^{\infty}Q(Ty_n)_{n=1}^{\infty})\\
&= 2Mh\|L\|_{L^1}\{y_n(t)\}_{n=1}^{\infty}
\end{aligned}$$

where $h=\frac{T^q}{\Gamma(q+1)}$, Thus we get

$$\begin{aligned}
Q(D) &\leq Q(\{0\}\cup\Upsilon(D))=Q(\Upsilon(D))\\
&\leq 2M\|m\|_{L^1}Q(D),
\end{aligned}$$

and since $2M\|m\|_{L^1}<1$, we get $Q(D)=0$.

Let $\delta\in(0,1)$ and $y=\delta\Upsilon(y)$, then

$$y(t)=\delta T(t)g(y)+\frac{\delta}{\Gamma(q)}\int_0^t (t-s)^{q-1}T(t-s)f(s,y(s))ds,\ t\in J, \tag{11}$$

$$\begin{aligned}
\|y(t)\| &\leq \|\delta T(t)g(y)\|+\frac{\delta}{\Gamma(q)}\int_0^t \|(t-s)^{q-1}T(t-s)f(s,y(s))ds\|\\
&\leq \|\delta T(t)\|\|g(y)\|+\|\frac{T^q\delta M}{\Gamma(q+1)}\int_0^t \|f(s,y(s))ds\|\\
&\leq M(a\|y\|+b)+M\int_0^T m(s)\theta\|y(s)\|ds\\
&\leq Ma\|y\|+Mb+M\theta(\|y\|)\int_0^T m(s)ds\\
&\leq Ma\|y\|+Mb+M\theta(\|y\|)\|m\|_{L^1}-Ma\|y\|\\
&\leq Mb+M\theta(\|y\|)\|m\|_{L^1}.
\end{aligned}$$

Thus

$$\frac{(1-Ma)\|y\|}{Mb+M\theta\|y\|\|m\|_{L^1}} \leq 1.$$

Then by (A6) there exist $\iota > 0$ such that $\iota \neq \|y\|$, let the set $\Lambda = \{y \in C(I,X) : |y\| < \iota\}$. So for all $y \in \delta\Lambda$, we get $y \neq \delta\Upsilon(y)$ to some $\beta \in (0,1)$. Thus by theorem 1, we get a fixed point of Υ in $\bar{\Lambda}$ and this fixed point is a mild solution to the problem (2), and the proof is completed. □

4. Conclusions

The main purpose of this work was to present new existence of mild solution of semilinear for differential and fractional differential equations subject to nonlocal initial conditions. To apply this, the study uses Hausdorff measure of non-compactness and fixed point theorems. Moreover, the results of references [34, 10] appear as a special case of our results.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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