Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 5056-5062
https://doi.org/10.28919/jmcs/5975
ISSN: 1927-5307

FUZZY REGULAR OPEN SETS VIA OPERATIONS

G. DURGADEVI, P. GOMATHI SUNDARI, N. RAJESH*

Department of Mathematics, Rajah Serfoji Government College, (Affiliated to Bharathdasan University), Thanjavur-613005, Tamilnadu, India

Abstract. The aim of this paper is to introduce and study the concepts of r-fuzzy γ-regular open sets and their related notions in topological spaces.

Keywords: smooth fuzzy topological spaces; r-fuzzy γ-regular open set; r-fuzzy γ-regular closed set.

2010 AMS Subject Classification: 54D10.

1. INTRODUCTION

The fuzzy concept has invaded almost all branches of Mathematics since its introduction by Zadeh [11]. Fuzzy sets have applications in many fields such as information [7] and control [10]. The theory of fuzzy topological spaces was introduced and developed by Chang [1] and since then various notions in classical topology have been extended to fuzzy topological spaces. Sŏstak [8] and Kubiak [4] introduced the fuzzy topology as an extension of Chang's fuzzy topology. It has been developed in many directions. Sŏstak [9] also published a survey article of the developed areas of fuzzy topological spaces. In this paper, we introduce and study the notion of regular open sets by using the operation γ on operation fuzzy topological space.

*Corresponding author
E-mail address: nrajesh_topology@yahoo.co.in
Received May 03, 2021

2. Preliminaries

Definition 2.1. [4, 8] *A function $\tau : I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions:*

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$;

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ *for any* $\mu_1, \mu_2 \in I^X$.

(3) $\tau(\bigvee_{j\in\Gamma} \mu_j) \geq \bigvee_{j\in\Gamma} \tau(\mu_j)$ *for any* $\{\mu_j\}_{j\in\Gamma} \in I^X$.

The pair (X, τ, γ) is called an operation smooth fuzzy topological space.

Definition 2.2. [2] *Let (X, τ) ba a smooth fuzzy topological space. Let $\gamma : \tau \to \tau_r$ be an operation such that $\lambda^\gamma = \wedge\mu$, where $\lambda \leq \mu$ for $\tau(\mu) \geq r$, $\tau(\lambda) \geq r$ and $r \in I_0$.*

Definition 2.3. [2] *A smooth topological space (X, τ) with an operation γ is said to be an operation smooth fuzzy topological space and is denoted by (X, τ, γ).*

Definition 2.4. [2] *Let (X, τ) ba a smooth fuzzy topological space. Let $\gamma : \tau \to \tau_r$ be an operation. For $\delta \in I^X$ and $r \in I_0$, δ is called r-fuzzy γ-open if for each $\alpha \in I^X$ with $\alpha \leq \delta$ there exists an r-fuzzy open set $\lambda \in I^X$ such that $\alpha \leq \lambda$ and $\lambda^\gamma \leq \delta$. The complement of an r-fuzzy γ-open set is called an r-fuzzy γ-closed.*

Definition 2.5. [2] *Let (X, τ) ba a smooth fuzzy topological space. Let $\gamma : \tau \to \tau_r$ be an operation. For any $\lambda \in I^X$, the r-fuzzy γ-interior of λ is defined by, $\gamma\text{-}I_\tau(\lambda, r) = \vee\{\mu : \mu \leq \lambda$, μ is r-fuzzy γ-open$\}$.*

Definition 2.6. [2] *Let (X, τ) ba a smooth fuzzy topological space. Let $\gamma : \tau \to \tau_r$ be an operation. For any $\lambda \in I^X$, the r-fuzzy γ-closure of λ is defined by, $\gamma\text{-}C_\tau(\lambda, r) = \wedge\{\mu : \mu \geq \lambda$, μ is r-fuzzy γ-closed$\}$.*

Definition 2.7. [3] *A fuzzy set λ of an operation smooth fuzzy topological space (X, τ, γ) is called:*

(1) r-fuzzy semiopen if $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$;

(2) r-fuzzy γ-preopen if $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$;

(3) r-fuzzy γ-α-open if $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r), r)$;

(4) r-fuzzy semi-preopen if $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$.

(5) r-fuzzy semiopen if $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r)$;

(6) r-fuzzy γ*-preclosed if* $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r) \leq \lambda$;

(7) r-fuzzy γ*-*α*-closed if* $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) \leq \lambda$.

(8) r-fuzzy semi-preopen if $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$.

3. r-FUZZY γ-REGULAR OPEN SETS

Definition 3.1. *A fuzzy subset* λ *of an operation fuzzy topological space* (X,τ,γ) *is said to be r-fuzzy* γ*-regular open set if* $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)) = \lambda$. *We call a subset* λ *of X is r-fuzzy* γ*-regular closed if its complement is r-fuzzy* γ*-regular open.*

Theorem 3.2. *Let* λ *be a fuzzy subset over X, then* $(1) \Rightarrow (2) \Rightarrow (3)$*, where*

(1) λ *is r-fuzzy* γ*-clopen (= r-fuzzy* γ*-open and r-fuzzy* γ*-closed).*

(2) $\lambda = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))$.

(3) $\bar{1} - \lambda$ *is r-fuzzy* γ*-regular open.*

Proof. (1)$\Rightarrow$ (2): This is obvious.
(2) $\Rightarrow$ (3): By (2), $\bar{1} - \lambda = \bar{1} - \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)) = \gamma\text{-}I_\tau(\bar{1} - \gamma\text{-}I_\tau(\lambda)) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\bar{1} - \lambda))$, and hence $\bar{1} - \lambda$ is r-fuzzy γ-regular open set. □

Lemma 3.3. *For an operation smooth fuzzy topological space* (X,τ,γ)*, we have*

(1) $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)$.

(2) $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)$.

Proof. (1). Since $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))$. Then $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))$. On the other hand, since $\gamma\text{-}I_\tau(\lambda) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))$ implies that $\gamma\text{-}I_\tau(\lambda) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))$ and hence $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))$. Therefore, $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))$.
(2). Similar to (1). □

Lemma 3.4. *Let* λ *and* μ *be fuzzy subsets of an operation smooth fuzzy topological space* (X,τ,γ). *Then the following properties hold:*

(1) γ-$I_\tau(\gamma$-$C_\tau(\lambda))$ is r-fuzzy γ-regular open.

(2) If λ and μ are r-fuzzy γ-regular open, then $\lambda \wedge \mu$ is also r-fuzzy γ-regular open.

Proof. (1). Follows from Lemma 3.3.

(2). Let λ and μ be r-fuzzy γ-regular open sets over X. Then we have $\lambda \wedge \mu = \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda)) \wedge \gamma$-$I_\tau(\gamma$-$C_\tau(\mu)) = \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda) \wedge \gamma$-$C_\tau(\mu)) \geq \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda \wedge \mu) \geq \gamma$-$I_\tau(\lambda \wedge \mu) = \lambda \wedge \mu$. Then $\lambda \wedge \mu = \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda \wedge \mu))$. Hence $\lambda \wedge \mu$ is r-fuzzy γ-regular open. □

Theorem 3.5. *The following statements are true:*

(1) A r-fuzzy γ-open set λ is r-fuzzy γ-regular open if, and only if γ-$I_\tau(\gamma$-$C_\tau(\lambda)) \leq \lambda$ holds.

(2) For every r-fuzzy γ-closed set λ, γ-$I_\tau(\lambda)$ is r-fuzzy γ-regular open.

(3) For every r-fuzzy γ-open set λ, γ-$C_\tau(\lambda)$ is r-fuzzy γ-regular closed.

Proof. (1). It suffices to prove that every r-fuzzy γ-open set λ satisfying γ-$I_\tau(\gamma$-$C_\tau(\lambda)) \leq \lambda$ is r-fuzzy γ-regular open. Since $A \leq \gamma$-$C_\tau(\lambda)$ holds, $\lambda = \gamma$-$I_\tau(\lambda) \leq \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda))$ is true, so we have $\lambda = \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda))$.

(2). If λ is r-fuzzy γ-closed, then γ-$I_\tau(\lambda) \leq \gamma$-$C_\tau(\gamma$-$I_\tau(\lambda)) \leq \gamma$-$C_\tau(\lambda) = \lambda$. Hence γ-$I_\tau(\lambda) = \gamma$-$I_\tau(\gamma$-$C_\tau(\gamma$-$I_\tau(\lambda))) \leq \gamma$-$I_\tau(\lambda)$. So, γ-$I_\tau(\lambda) = \gamma$-$I_\tau(\gamma$-$C_\tau(\gamma$-$I_\tau(\lambda))$ holds. That is, γ-$I_\tau(\lambda)$ is r-fuzzy γ-regular open.

(3). Let λ be an r-fuzzy γ-open set over X. Then clearly γ-$I_\tau(\gamma$-$C_\tau(\lambda,r),r) \leq \gamma$-$C_\tau(\lambda,r)$ implies that γ-$C_\tau(\gamma$-$I_\tau(\gamma$-$C_\tau(\lambda,r),r),r) \leq \gamma$-$C_\tau(\gamma$-$C_\tau(\lambda,r),r) = \gamma$-$C_\tau(\lambda,r)$. Since λ is r-fuzzy γ-open, $\lambda = \gamma$-$I_\tau(\lambda,r)$. Also since $\lambda \leq \gamma$-$C_\tau(\lambda,r)$, $\lambda = \gamma$-$I_\tau(\lambda,r) \leq \gamma$-$I_\tau(\gamma$-$C_\tau(\lambda,r),r)$. Thus γ-$C_\tau(\lambda,r) \leq \gamma$-$C_\tau(\gamma$-$I_\tau(\gamma$-$C_\tau(\lambda,r),r),r)$. Hence γ-$C_\tau(\lambda,r)$ is an r-fuzzy γ-regular closed set. □

Theorem 3.6. *Let λ be any fuzzy subset of an operation smooth fuzzy topological space (X,τ,γ). Then*

(1) λ is γ-clopen if, and only if it is r-fuzzy γ-regular open and r-fuzzy γ-regular closed.

(2) λ is r-fuzzy γ-regular open if, and only if it is r-fuzzy γ-preopen and r-fuzzy γ-semiclosed.

(3) λ is r-fuzzy γ-regular closed if, and only if it is r-fuzzy γ-semiopen and r-fuzzy γ-preclosed.

Proof. Follows from their definitions. □

Theorem 3.7. *Let λ be any fuzzy subset of an operation smooth fuzzy topological space (X,τ,γ). Then the following statements are equivalent:*

(1) λ is r-fuzzy γ-regular open.

(2) λ is r-fuzzy γ-open and r-fuzzy γ-semiclosed.

(3) λ is r-fuzzy γ-α-open and r-fuzzy γ-semiclosed.

(4) λ is r-fuzzy γ-preopen and r-fuzzy γ-semiclosed.

(5) λ is r-fuzzy γ-open and r-fuzzy γ-semi-preclosed.

(6) λ is r-fuzzy γ-α-open and r-fuzzy γ-semi-preclosed.

Proof. (1)$\Rightarrow$ (2): Let λ be r-fuzzy γ-regular open set. Since every r-fuzzy γ-regular open set is r-fuzzy γ-open and every r-fuzzy γ-regular open set is r-fuzzy γ-semiclosed. Then λ is r-fuzzy γ-open and r-fuzzy γ-semiclosed.

(2)$\Rightarrow$ (3): Let λ be r-fuzzy γ-open and r-fuzzy γ-semiclosed set. Since every r-fuzzy γ-open set is r-fuzzy γ-α-open. Then λ is r-fuzzy γ-α-open and r-fuzzy γ-semiclosed.

(3)$\Rightarrow$ (4): Let λ be r-fuzzy γ-α-open and r-fuzzy γ-semiclosed set. Since every r-fuzzy γ-α-open set is r-fuzzy γ-preopen. Then λ is r-fuzzy γ-preopen and r-fuzzy γ-semiclosed.

(4)$\Rightarrow$ (5): Let λ be r-fuzzy γ-preopen and r-fuzzy γ-semiclosed set. Then $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda))$ and $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)) \leq \lambda$. Then $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda))$. Hence λ is r-fuzzy γ-regular open set and hence it is r-fuzzy γ-open. Since every r-fuzzy γ-semiclosed set is r-fuzzy γ-semi-preclosed. Then λ is r-fuzzy γ-open and r-fuzzy γ-semi-preclosed.

(5)$\Rightarrow$ (6): It is obvious since every r-fuzzy γ-open set is r-fuzzy γ-α-open.

(6)$\Rightarrow$ (1): Let λ be r-fuzzy γ-α-open and r-fuzzy γ-semi-preclosed set, $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))$ and $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))) \leq \lambda$. Then $\gamma\text{-}I_\tau(\lambda) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))) = \lambda$ and hence $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))) = \lambda$. Hence λ is r-fuzzy γ-regular open set. □

Corollary 3.8. *Let λ be any soft subset of an operation fuzzy topological space (X,τ,A,γ). Then the following statements are equivalent:*

(1) λ is r-fuzzy γ-regular closed.

(2) λ is r-fuzzy γ-closed and r-fuzzy γ-semiopen.

(3) λ is r-fuzzy γ-α-closed and r-fuzzy γ-semiopen.

(4) λ is r-fuzzy γ-preclosed and r-fuzzy γ-semiopen.

(5) λ is r-fuzzy γ-closed and r-fuzzy γ-semi-preopen.

(6) λ is r-fuzzy γ-α-closed and r-fuzzy γ-semi-preopen.

Proof. Similar to Theorem 3.7. □

Theorem 3.9. *Let λ be any fuzzy subset of an operation fuzzy topological space (X,τ,γ). Then the following statements are equivalent:*

(1) λ is r-fuzzy γ-clopen.

(2) λ is r-fuzzy γ-regular open and r-fuzzy γ-regular closed.

(3) λ is r-fuzzy γ-open and r-fuzzy γ-α-closed.

(4) λ is r-fuzzy γ-open and r-fuzzy γ-preclosed.

(5) λ is r-fuzzy γ-α-open and r-fuzzy γ-preclosed.

(6) λ is r-fuzzy γ-α-open and r-fuzzy γ-closed.

(7) λ is r-fuzzy γ-preopen and r-fuzzy γ-closed.

(8) λ is r-fuzzy γ-preopen and r-fuzzy γ-α-closed.

Proof. (1) $\Rightarrow$ (2) See Remark 3.4 (1).

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7) $\Rightarrow$ (8) are obvious.

(5) $\Rightarrow$ (6): Let λ be r-fuzzy γ-α-open and r-fuzzy γ-preclosed set. Then $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))$ and $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))) \leq \lambda$. This implies that $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda)))$ and hence $\gamma\text{-}C_\tau(\lambda) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))))$. Then $\gamma\text{-}C_\tau(\lambda) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))$. Since $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda))) \leq \lambda$, then $\gamma\text{-}C_\tau(\lambda) \leq \lambda$. But in general $\lambda \leq \gamma\text{-}C_\tau(\lambda)$. Then $\gamma\text{-}C_\tau(\lambda) = \lambda$. It is obvious that λ is r-fuzzy γ-closed.

(8) $\Rightarrow$ (1): Let λ be r-fuzzy γ-preopen and r-fuzzy γ-α-closed. Then $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda))$ and $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda))) \leq \lambda$. Hence $\gamma\text{-}C_\tau(\lambda) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda))) \leq \lambda$ and hence $\gamma\text{-}C_\tau(\lambda) \leq \lambda$. Hence $\gamma\text{-}C_\tau(\lambda) = \lambda$. It is obvious that λ is r-fuzzy γ-closed. Since $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda))) \leq \lambda$, $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)))) \leq \gamma\text{-}I_\tau(\lambda)$. Then $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda)) \leq \gamma\text{-}I_\tau(\lambda)$ and hence $\lambda \leq \gamma\text{-}I_\tau(\lambda)$. But in general $\gamma\text{-}I_\tau(\lambda) \leq \lambda$. Then $\gamma\text{-}I_\tau(\lambda) = \lambda$. It is obvious that λ is r-fuzzy γ-open. Therefore, λ is r-fuzzy γ-clopen. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.

[2] G. Durgadevi, P. Gomathi Sundari, Fuzzy γ-open sets, Communicated.

[3] G. Durgadevi, P. Gomathi Sundari, Some subsets in operation smooth fuzzy topological spaces, Communicated.

[4] T. Kubiak, On fuzzy topologies, Ph.D. Thesis, Adam Mickiewicz University, Poznan, Poland, 1985.

[5] P.P. Ming, L.Y. Ming, Fuzzy topology. I. neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980), 571-599.

[6] F.G. Shi, J. Zhang, C.Y. Zheng, On L-fuzzy topological spaces, Fuzzy Sets Syst. 149 (2005), 473-484.

[7] P. Smets, The degree of belief in a fuzzy event, Inform. Sci. 25 (1981), 1-19.

[8] A. P. Sŏstak, On a fuzzy topological structure, Suppl. Rend. Circ. Matem. Palermo2 Ser II 11 (1985), 89–103.

[9] A. P. Sŏstak, Basic structures of fuzzy topology, J. Math. Sci. 78(6) (1996) 662-701.

[10] M. Sugeno, An introductory survey of fuzzy control, Inform. Sci. 36 (1985), 59-83.

[11] L.A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338-353.