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L-MOMENTS METHOD TO ESTIMATE GAMMA/GOMPERTZ DISTRIBUTION BASED ON RANKED SET SAMPLING DESIGNS

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Abstract. This paper is concerned with the estimation problem using L-moments method of estimation for the unknown parameters of Gamma/Gompertz distribution based ranked set sampling methods. Computer simulation results are given to compare the efficiencies for the Neoteric ranked set sampling (NRSS) as a dependent sampling technique, with ranked set sampling (RSS), median ranked set sampling (MRSS), percentile ranked set sampling (PRSS) as an independent sampling techniques, and simple random sampling (SRS).

Keywords:ranked set sampling; neoteric ranked set sampling; percentile ranked set sampling; median ranked set sampling; Gamma gompertz distribution; L-moments method.

2010 AMS Subject Classification: 62D05.

1. Introduction

McIntyre [13] submitted a sampling method for the estimation of pasture and forage yields in the field of agriculture it is called RSS. He aim was to maintain the unbiasedness property of the estimators like in SRS estimator with take minimum information about the estimator provided

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by visual inspection or by any inexpensive method. The McIntyre's technique was used first by Halls and Dell [7]. They found experimentally that RSS was more efficient than SRS. They also stated the name ranked set sampling that is in current use. Dell and Clutter [5] appeared that the mean estimator under imperfect RSS remains an unbiased estimator of a population mean.

RSS method as suggested by McIntyre [13] may be amended to come up with new sampling methods that can be made more efficient than the usual RSS method with small mean square errors (MSE). Muttlak [15] used MRSS to estimate the population mean and he showed that it is more efficient than the usual RSS method. Muttlak[16] introduced PRSS procedure with different value of $0 \leq p \leq 1$ for estimating the population mean. Al-Saleh and Al-Hadrami [2] used different set sizes for RSS technique for estimating the mean of the Normal and Exponential distributions, they explored that this procedure is more useful than RSS for estimating the mean of symmetric distributions. Khamnei and Mayan [12] estimated the parameters of Gumbel distribution based on SRS and RSS with compared the estimators of these two methods. A recently developed modification of RSS, Zamanzade and Al-Omari [21] introduced the NRSS technique and they showed that the efficiency of the estimators based on NRSS are greater than their estimators using RSS and SRS techniques. Esemen and Grler[8] estimated the parameters of Generalized Rayleigh distribution based on RSS and it is some modification.

The Gamma/Gompertz (GG) distribution has been used to estimate customer lifetime and a model of death rate risks. GG distribution was introduced by Bemmaor and Glady [4]. The probability density function (PDF), the cumulative distribution function (CDF), and quantile function (QF) of the GG distribution are, respectively, given by[14]

$$f(x;b,s,\beta)=\frac{bse^{bx}\beta^{s}}{(\beta-1+e^{bx})^{s+1}};\ \ 0\leq x\leq\infty, b>0, s>0, \beta>0, \tag{1}$$

$$F(x;b,s,\beta)=1-\frac{\beta^{s}}{(\beta-1+e^{bx})^{s}};\ \ 0\leq x\leq\infty, b>0, s>0, \beta>0, \tag{2}$$

and

$$x(F)=\frac{1}{b}ln[\beta(1-F)^{\frac{-1}{s}}-\beta+1] \tag{3}$$

where b is the scale parameter, s and β are the shape parameters. (see Fig. 1)

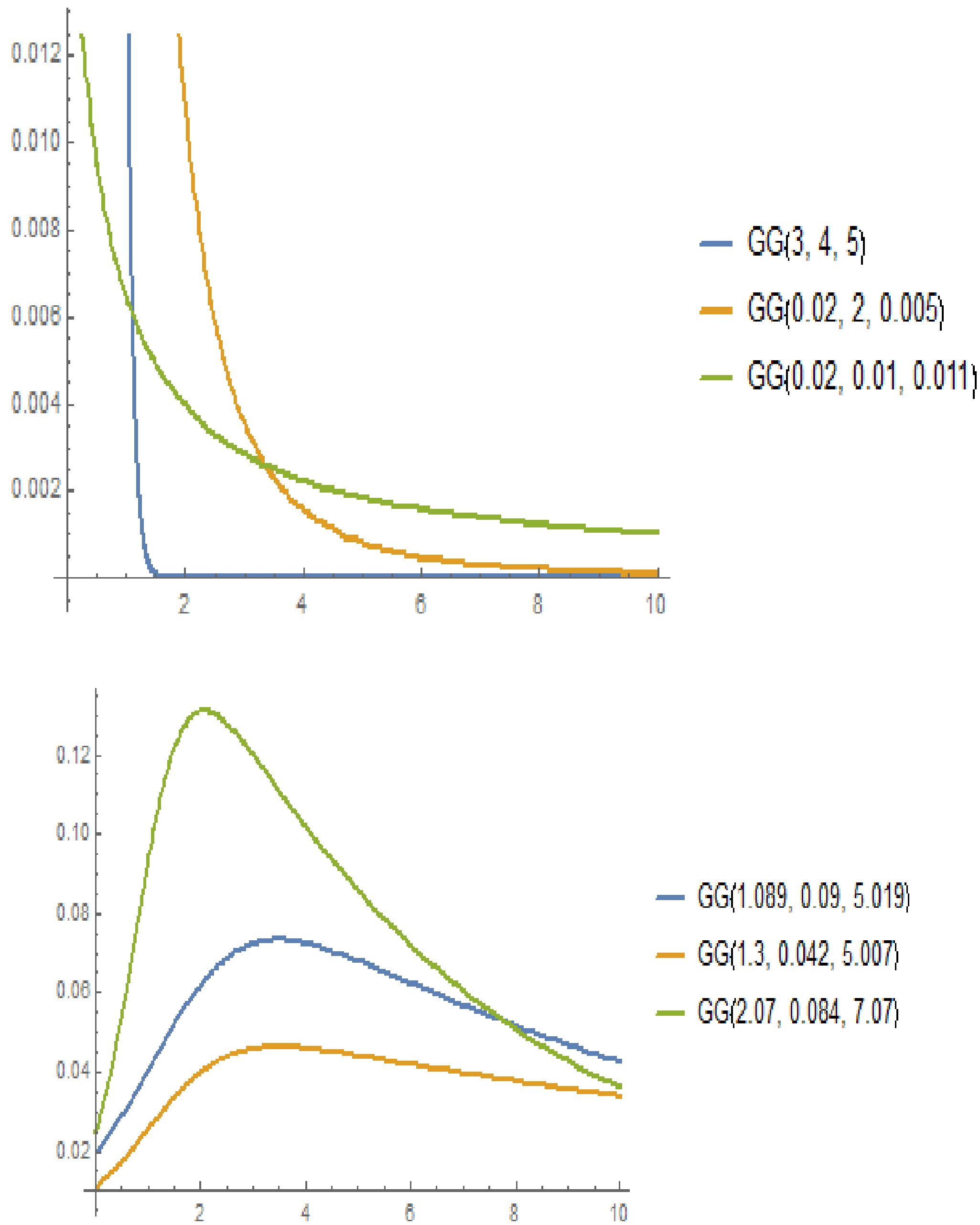


FIGURE 1. Three parameters Gamma/Gompertz distribution density function.

In this paper, the performance of NRSS method has been compared with the other ranked set sampling methods using L-moment method of estimation with the usual SRS technique for GG distribution parameters. The remaining part of this paper is organized as follows: In Section 2, introduced ranked set sampling methods. L-moment method of estimation will be discussed in

Section 3. In Section 4, computer simulation results are given to compare the efficiency for the estimators based on SRS with its counterparts RSS MRSS, PRSS, and NRSS. It is shown that for the GG distributions considered in this study, NRSS has less mean square error and more efficiency than all the other methods. Conclusions are presented in Section 5.

2. SOME RANKED SET SAMPLING TECHNIQUES

In this section, different sampling techniques for selection of units in the sample are RSS, MRSS, PRSS and NRSS will be considered with explanation figures for each case for each one of these techniques.

2.1. Ranked set sampling

The original idea of ranked set sampling (RSS) was showed by McIntyre's [13]. Ranked set sampling is a methodology that can improve the efficiency of techniques such as estimation without taking large a number of substantial observations. In addition, it is designed to decrease the number of measured observations required to achieve the desired precision in making inferences. The RSS scheme can be described as follows:[3]

(1) In order to draw a sample of size n, we identify n^2 units from the target population.

(2) Randomly allocate these units to n sets each of size n. The n units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.

(3) From the first set of n units, the smallest ranked unit is measured. From the second set of n units, the second smallest ranked unit is measured. The process is continued until from the n^{th} set of n units the largest ranked until is measured. (see Fig. 2)

(4) Repeat steps 1 through 3 for m cycles to obtain a sample of size nm.

Let $\{X_1, X_2, .., X_n\}$ be a random sample with probability density function $f(x)$with mean μ and variance σ^2. Let $\{X_{(1)1}, X_{(1)2}, .., X_{(1)n_1}; X_{(2)1}, X_{(2)2}, ..., X_{(2)n_2}$; ..., $X_{(n)1}, X_{(n)2}, ..., X_{(n)n_n}\}$ be independent random variables. Let $X_{(i;n)j}$ denotes the i^{th}order statistic from the i^{th}sample of size n where $(i = 1, 2, ..., n)$. The cycle may be repeated m times to get nm units. Let$X_{(i)j}$ denotes the j^{th} cycle of size m where $(j = 1, 2, ..., m)$. Then PDF and CDF of $X_{(i)j}$ are given by [19]

$$f_n(x_{(i)j}) = \frac{n!}{(i-1)!(n-i)!} f(x_{(i)j}; \theta)\ [F(x_{(i)j}; \theta)]^{i-1}\ [1 - F(x_{(i)j}; \theta)]^{n-i}, \tag{4}$$

and

$$F_n(x_{(i)j}) = \sum_{t=i}^{n} \frac{n!}{t!(n-t)!} \left[F(x_{(i)j};\theta)\right]^{t} \left[1-F(x_{(i)j};\theta)\right]^{n-t}. \tag{5}$$

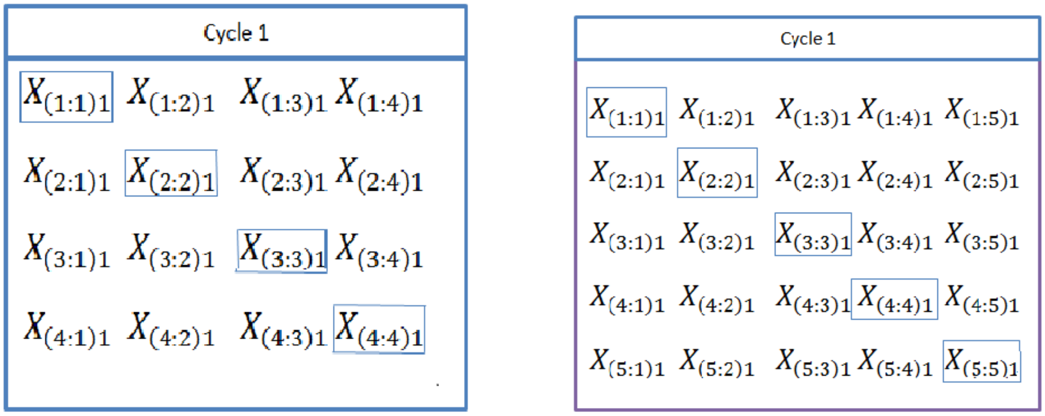


FIGURE 2. RSS design in cases of a) n is an even b) n is an odd

2.2. Median ranked set sampling

Muttlak [15] introduced Median ranked set sampling (MRSS) scheme for estimating the population mean. MRSS procedure is explained as follows:

(1) In order to draw a sample of size n, we identify n^2 units from the target population.

(2) Randomly allocate these units to n sets each of size n. The n units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.

(3) for even sample size n, identify n^2units from the target population. Each sample is ranked in itself as in ranked set sampling design. Select the $\frac{n}{2}th$ smallest rank from the first $\frac{n}{2}$ sets and select the $\frac{n+2}{2}th$ smallest rank from the other $\frac{n}{2}$ sets. Similarly, for odd sample size n, Select the $\frac{n+1}{2}th$ smallest rank from all sets.

(4) Repeat steps 1 through 3 for m cycles to draw a sample of size nm. (see Fig. 3)

The PDF of $X_{(i)j}$ that ranked by MRSS when n is an even is given by [19]

$$\begin{aligned} f_n(x_{(i)j}) = {} & \Big[\frac{n!}{(\frac{n-2}{2})!(\frac{n}{2})!} f(x_{(\frac{n}{2})j};\theta) \left[F(x_{(\frac{n}{2})j};\theta)\right]^{\frac{n-2}{2}} \left[1-F(x_{(\frac{n}{2})j};\theta)\right]^{\frac{n}{2}}\Big] \\ & \Big[\frac{n!}{(\frac{n}{2})!(\frac{n-2}{2})!} f(x_{(\frac{n+2}{2})j};\theta) \left[F(x_{(\frac{n+2}{2})j};\theta)\right]^{\frac{n}{2}} \left[1-F(x_{(\frac{n+2}{2})j};\theta)\right]^{\frac{n-2}{2}}\Big]. \end{aligned} \tag{6}$$

Otherwise when n is an odd is given by

$$(7) \qquad f_n(x_{(i)j}) = \frac{n!}{(\frac{n-1}{2})!(\frac{n-1}{2})!} f(x_{(\frac{n+1}{2})j};\theta)\,[F(x_{(\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}\,[1-F(x_{(\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}.$$

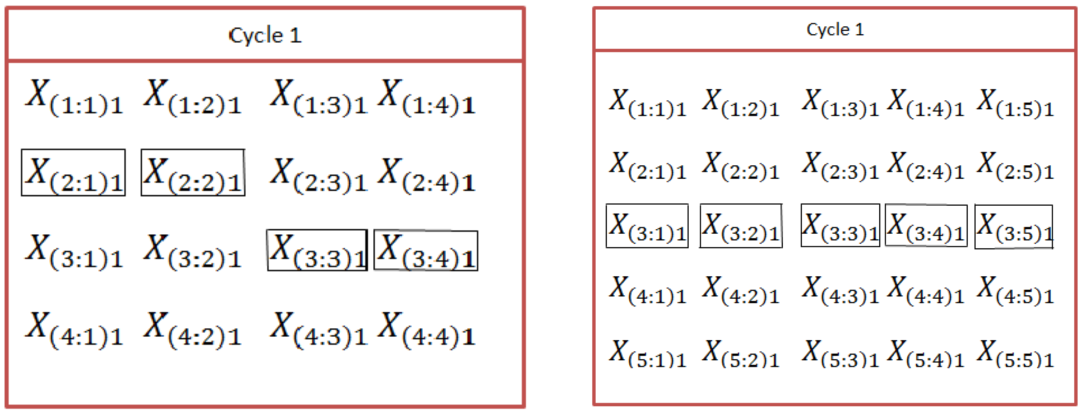


FIGURE 3. MRSS design in cases of a) n is an even b) n is an odd

2.3. Percentile ranked set sampling

Muttlak [16] used percentile ranked set sampling (PRSS) technique with a different values of $0 \leq p \leq 1$ for estimating the population mean. PRSS procedure is explained as follows:[18]

(1) In order to draw a sample of size n, we identify n^2 units from the target population.

(2) Randomly allocate these units to n sets each of size n. The n units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.

(3) For even sample size n, select the $(p(n+1))th$ smallest ranked unit from the first $\frac{n}{2}$ sets and select the $(q(n+1))th$ smallest ranked unit from the other $\frac{n}{2}$ sets. Similarly, for odd sample size n, select the $(p(n+1))th$ smallest ranked unit from first $\frac{n-1}{2}$ sets. Select the $(q(n+1))th$ smallest ranked unit from second $\frac{n-1}{2}$ sets and the median with rank $\frac{n+1}{2}$ from the remaining set.

(4) Repeat steps 1 through 3 for m cycles to draw a sample of size nm. (see Fig. 4)

The PDF of $X_{(i)j}$ that ranked by PRSS when n is an even is given by

$$f_n(x_{(i)j}) = [\frac{n!}{(p(n+1)-1)!(n-p(n+1))!}$$

$$f(x_{(p(n+1))j};\theta)\,[F(x_{(p(n+1))j};\theta)]^{p(n+1)-1}\,[1-F(x_{(p(n+1))j};\theta)]^{n-p(n+1)}]$$

$$[\frac{n!}{(q(n+1)-1)!(n-q(n+1))!} f(x_{(q(n+1))j};\theta)\,[F(x_{(q(n+1))j};\theta)]^{q(n+1)-1}$$
$$[1-F(x_{(q(n+1))j};\theta)]^{n-q(n+1)}], \tag{8}$$

Otherwise when n is an odd is given by

$$f_n(x_{(i)j}) =$$
$$[\frac{n!}{(p(n+1)-1)!(n-p(n+1))!} f(x_{(p(n+1))j};\theta)\,[F(x_{(p(n+1))j};\theta)]^{p(n+1)-1}\,[1-F(x_{(p(n+1))j};\theta)]^{n-p(n+1)}]$$
$$[\frac{n!}{(q(n+1)-1)!(n-q(n+1))!} f(x_{(q(n+1))j};\theta)\,[F(x_{(q(n+1))j};\theta)]^{q(n+1)-1}[1-F(x_{(q(n+1))j};\theta)]^{n-q(n+1)}]$$

$$[\frac{n!}{(\frac{n-1}{2})!(\frac{n-1}{2})!} f(x_{(\frac{n+1}{2})j};\theta)\,[F(x_{(\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}\,[1-F(x_{(\frac{n+1}{2})j};\theta)]^{\frac{n-1}{2}}]. \tag{9}$$

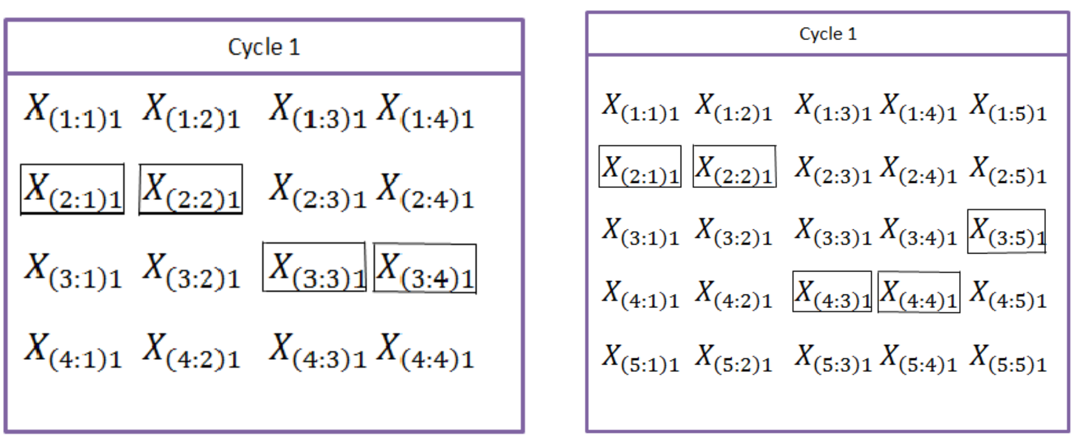


FIGURE 4. PRSS design in cases of a) n is an even b) n is an odd

2.4. Neoteric ranked set sampling

Zamanzade and Al-Omari [21] have developed modification of RSS called NRSS. NRSS differs from the original RSS scheme by the composition of a single set of n^2 units, instead of n sets of size n. This strategy has been shown to be effective, producing more efficient estimators for the population mean. The NRSS technique can be described as follows:

(1) Select a simple random sample of size n^2 units from the target finite population.

(2) Ranked the n^2selected units in an increasing magnitude based on a visual inspection or any other cost-free method with respect to a variable of interest.

(3) For odd sample size n,, select the $(\frac{n+1}{2}+(i-1)n)th$ ranked unit for $(i=1,2,...n)$. Otherwise for even sample size n, if i is an odd, then select the$(\frac{n+2}{2}+(i-1)n)th$ ranked unit. While if i is an even, then select the$(\frac{n}{2}+(i-1)n)th$ ranked unit for $(i=1,2,...n)$. (see Fig. 5)

(4) Repeat steps 1 through 3 m cycles if needed to obtain a NRSS of size nm.

The PDF of $X_{(i)j}$ that ranked by NRSS is given by [19]

$$f_n(x_{(i)j}) = \frac{n^2!}{(k(i)-k(i-1)-1)!} f(x_{k(i)j};\theta)\ [F(x_{k(i)j};\theta)-F(x_{k(i-1)j};\theta)]^{k(i)-k(i-1)-1}, \tag{10}$$

where

$$k(i) = \begin{cases} \frac{n+1}{2}+(i-1)n & n\ odd \\ \frac{n+2}{2}+(i-1)n & n\ even,\ i\,odd \\ \frac{n}{2}+(i-1)n & n\ even,\ i\,even \end{cases}$$ where $(i=1,2,...n)$, $k(0)=0$ and $x_{k(0)} \cong -\infty$.

Cycle 1															
$X_{(1)1}$	$X_{(2)1}$	$\boxed{X_{(3)1}}$	$X_{(4)1}$	$X_{(5)1}$	$\boxed{X_{(6)1}}$	$X_{(7)1}$	$X_{(8)1}$	$X_{(9)1}$	$X_{(10)1}$	$\boxed{X_{(11)1}}$	$X_{(12)1}$	$X_{(13)1}$	$\boxed{X_{(14)1}}$	$X_{(15)1}$	$X_{(16)1}$

Cycle 1								
$X_{(1)1}$	$\boxed{X_{(2)1}}$	$X_{(3)1}$	$X_{(4)1}$	$\boxed{X_{(5)1}}$	$X_{(6)1}$	$X_{(7)1}$	$\boxed{X_{(8)1}}$	$X_{(9)1}$

FIGURE 5. NRSS design in cases of a) n is an even b) n is an odd

3. ESTIMATION BASED ON L MOMENTS

Hosking [10] formalized methods for the estimation of statistics are measures of higher moments, skewness and kurtosis, are more robust with respect to sample size and the presence of outliers. These measures, called L-moments, are based on expectations of linear combinations of order statistics. Indeed, Vogel and Fennessey [20] observed that the introduction of the theory of L-moments by Hosking is probably the single most significant recent advance in relating to our understanding of extreme events.

The method of L-moments is a parameter estimation technique that is conceptually the same as the well-known "method of (product) moments". Specifically, the L-moments is a method of parameter estimation in which the parameters of a distribution are chosen so as to equate the theoretical L-moments of the distribution (λ_r) to the sample L-moments ($\hat{\lambda}_r$) for $r \leq p$ where p is the number of parameters of a probability distribution.[1]

3.1. Theoretical L-moments

The theoretical L-moments for a real-valued random variable X with a QF of x(F) are defined from the expectations of order statistics. The order statistics of X for a sample of size n are formed by the ascending order $X_{(1:n)} \leq X_{(2:n)} \leq .. \leq X_{(l:n)} \leq ... \leq X_{(n:n)}$. The theoretical L-moments (λ_r) are [11]

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} E[X_{r-k:r}], \;\; r = 1,2,... \tag{11}$$

where r is the order of the L-moment, and$E[X_{r-k:r}]$ is the expectation of the $r-k$ order statistic of a sample of size r and we obtain it as follows:

(12)

$$E[X_{r-k:r}] = \frac{6s}{b}\sum_{l=0}^{r-k-1} C^r_{r-k}C_l^{r-k-1}(-1)^l\beta^{s(r-l)}(\beta-1)^{-s(r-l)-1}\Phi_{(s(r-l)+1)}((1-\beta)^{-1},2,1)\, for\, \beta \neq 1,$$

where the Lerch transcendental function is $\Phi_{(s(r-i)+1)}((1-\beta)^{-1},2,1) = \sum_{h=0}^{\infty}\frac{(s(r-i)+1)_h}{h!}\frac{(1-\beta)^{-h}}{(h+1)^2}$ for $\beta \neq 1$. By substituting the (12) in (11), the population L-moments of order r for the GG is given by:

(13)

$$\lambda_r = \frac{6s}{b}\sum_{k=0}^{r-1}\sum_{l=0}^{r-k-1}\frac{[(r-1)!]^2}{(r-k-1-l)![k!]^2(r-k)!l!}\beta^{s(r-l)}(\beta-1)^{-s(r-l)-1}\Phi_{(s(r-l)+1)}((1-\beta)^{-1},2,1).$$

If β has a special case when equal 1. The $E[X_{r-k:r}]$ is going to be

$$E[X_{r-k:r}] = bs\sum_{l=0}^{r-k-1} C^r_{r-k}C_l^{r-k-1}(-1)^l\frac{\Gamma(2)}{(b(s(1+k)+r-k-1-i)^2}.$$

The first three theoretical L-moments can be obtained by taking $r = 1, 2\, and\, 3$ in (13) as follows:

$$\lambda_1 = E[X] = \frac{6s}{b}\beta^s(\beta-1)^{-s-1}\Phi_{(s+1)}((1-\beta)^{-1},2,1), \tag{14}$$

$$\lambda_2 = \frac{1}{2}E[X_{2:2}-X_{1:2}] = \frac{6s}{b}[(-\frac{1}{2}\beta^{2s})(\beta-1)^{-2s-1}\Phi_{(2s+1)}((1-\beta)^{-1},2,1) \\ -\frac{1}{2}\beta^s(\beta-1)^{-s-1}\Phi_{(s+1)}((1-\beta)^{-1},2,1)],$$

and

$$\lambda_3 = \frac{1}{3}E[X_{3:3}-2X_{2:3}+X_{1:3}] = \frac{6s}{b}[(\frac{-7}{3}\beta^{3s})(\beta-1)^{-3s-1}\Phi_{(3s+1)}((1-\beta)^{-1},2,1) \\ +(\frac{10}{3}\beta^{2s})(\beta-1)^{-2s-1}\Phi_{(2s+1)}((1-\beta)^{-1},2,1)+\frac{1}{3}\beta^s(\beta-1)^{-s-1}\Phi_{(s+1)}((1-\beta)^{-1},2,1)].$$

3.2. Sample L-moments

The sample L-moments are computed from the sample order statistics. Elamir and Seheult [6] proposed the following formula to calculate sample L-moments:

$$\hat{\lambda_r} = \frac{1}{r}\sum_{i=0}^{n}\sum_{k=0}^{r-1}\frac{(-1)^k C_k^{r-1} C_{r-k-1}^{i-1} C_k^{n-i}}{C_r^n}x_{i:n}, \quad r = 1,2,... \tag{15}$$

L-moment estimators for b, $s\, and\, \beta$ can be found as follows:

$$\frac{6s}{b}\beta^s(\beta-1)^{-s-1}\Phi_{(s+1)}((1-\beta)^{-1},2,1) = \bar{x}, \tag{16}$$

$$\frac{6s}{b}[(-\frac{1}{2}\beta^{2s})(\beta-1)^{-2s-1}\Phi_{(2s+1)}((1-\beta)^{-1},2,1) \\ -\frac{1}{2}\beta^s(\beta-1)^{-s-1}\Phi_{(s+1)}((1-\beta)^{-1},2,1)] = \sum_{i=1}^{n}\frac{(i-1)-(n-i)}{n(n-1)}x_{i:n}, \tag{17}$$

and

$$\frac{6s}{b}[(\frac{-7}{3}\beta^{3s})(\beta-1)^{-3s-1}\Phi_{(3s+1)}((1-\beta)^{-1},2,1)+(\frac{10}{3}\beta^{2s})(\beta-1)^{-2s-1}\Phi_{(2s+1)}((1-\beta)^{-1},2,1)$$
$$+\frac{1}{3}\beta^{s}(\beta-1)^{-s-1}\Phi_{(s+1)}((1-\beta)^{-1},2,1)]=\sum_{i=1}^{n}\frac{(i-1)(i-2)-4(i-1)(n-i)+(n-i)(n-i-1)}{n(n-1)(n-2)}x_{i:n}. \tag{18}$$

4. Simulation Study

Based on 1000 replications, a computer simulation is conduced to study the behavior of the efficiency of the sample mean using RSS, MRSS, PRSS with $p = 0.2$, and NRSS with respect to SRS. Random observations are generated from Gamma/Gompertz distribution based on different sample sizes ($n = 15, 20$ *and* 25), for cycles ($m = 1$ *and* 3), different parameters values $GG(b,s,beta) = \{GG(3,3,3), GG(2,2,4)\}$, were generated by the R package. Nadjafi [17] used 300 replications in his paper because of that the replications were extended to be more than 300. The relative efficiency (RE) of the estimator of any of the RSS methods with respect to the usual estimator SRS is defined as [9]

$$RE(\hat{\theta}_{SRS}, \hat{\theta}_{RSS's}) = \frac{MSE(\hat{\theta}_{SRS})}{MSE(\hat{\theta}_{RSS's})}, \tag{19}$$

where $MSE(\hat{\theta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}-\theta)^2$ and $bias = \hat{\theta}-\theta$.

if RE($\hat{\theta}_{SRS}$, $\hat{\theta}_{RSS's}$)>1, then $\hat{\theta}_{RSS's}$is better than $\hat{\theta}_{SRS}$. The simulation results are summarized in a list of tables and list of figures. The results of the table and these figures show the relative efficiency (the ratio of the mean square errors) of the estimators relative to SRS.

TABLE 1. MSEs of the estimators for $(b, s\, and\, beta)$ based on SRS,RSS,MRSS,PRSS, and NRSS.

Distribution	n		**SRS**	**RSS**		**MRSS**		**PRSS**		**NRSS**	
				$m=1$	$m=3$	$m=1$	$m=3$	$m=1$	$m=3$	$m=1$	$m=3$
GG(3,3,3)	15		0.0199	0.0170	0.0054	0.0089	0.0049	0.0063	0.0015	0.0011	0.0010
	20	$\hat{b}$	0.0273	0.0065	0.0062	0.0019	0.0031	0.0019	0.0028	0.0017	0.0013
	25		0.0149	0.0069	0.0067	0.0055	0.0033	0.0034	0.0016	0.0005	0.0002
	15		0.0142	0.0095	0.0045	0.0059	0.0028	0.0057	0.0002	0.0003	0.0002
	20	$\hat{s}$	0.0187	0.0055	0.0050	0.0035	0.0029	0.0020	0.0039	0.0004	0.0019
	25		0.0166	0.0053	0.0053	0.0042	0.0044	0.0030	0.0023	0.0010	0.0007
	15		0.0120	0.0053	0.0020	0.0047	0.0010	0.0069	0.0004	0.0010	0.0002
	20	$b\hat{e}ta$	0.0144	0.0061	0.0053	0.0081	0.0039	0.0033	0.0072	0.0010	0.0039
	25		0.0231	0.0053	0.0055	0.0031	0.0079	0.0038	0.0048	0.0026	0.0024
GG(2,2,4)	15		0.0859	0.0685	0.0207	0.0157	0.0079	0.0028	0.0071	0.0014	0.0060
	20	$\hat{b}$	0.0031	0.0022	0.0026	0.0012	0.0029	0.0007	0.0016	0.0001	0.0018
	25		0.0300	0.0049	0.0100	0.0015	0.0061	0.0023	0.0068	0.0012	0.0019
	15		0.0558	0.0085	0.0022	0.0081	0.0034	0.0015	0.0019	0.0010	0.0011
	20	$\hat{s}$	0.0221	0.0021	0.0017	0.0051	0.0028	0.0005	0.0015	0.0004	0.0004
	25		0.0235	0.0013	0.0005	0.0009	0.0009	0.0038	0.0002	0.0006	0.0002
	15		0.0328	0.0086	0.0053	0.0059	0.0039	0.0009	0.0022	0.0057	0.0019
	20	$b\hat{e}ta$	0.0820	0.0177	0.0175	0.0102	0.0053	0.0019	0.0044	0.0017	0.0034
	25		0.0142	0.0011	0.0060	0.0011	0.0026	0.0075	0.0017	0.0009	0.0016

TABLE 2. Estimators and Biases of the estimators for $(b, s\, and\, beta)$ based on SRS, RSS, MRSS, PRSS, and NRSS.

Distribution	n			SRS	RSS $m=1$	RSS $m=3$	MRSS $m=1$	MRSS $m=3$	PRSS $m=1$	PRSS $m=3$	NRSS $m=1$	NRSS $m=3$
GG(3,3,3)	15	$\hat{b}$	*Estimator*	3.0638	3.0206	3.0047	3.0480	3.0380	3.0163	3.0902	3.0211	3.0433
			bias	−0.0638	−0.0206	−0.0191	−0.0480	−0.0380	−0.0163	−0.0902	−0.0211	−0.0433
	20		*Estimator*	3.0272	3.0090	3.0264	3.0221	3.0684	3.0105	3.0266	3.0298	3.0471
			bias	−0.0272	−0.0090	−0.0264	−0.0221	−0.0684	−0.0105	−0.0266	−0.0298	−0.0471
	25		*Estimator*	3.0631	3.0203	3.0248	3.0313	3.1006	3.0246	3.0642	3.0732	3.0481
			bias	−0.0630	−0.0203	−0.0248	−0.0313	−0.1006	−0.0246	−0.0642	−0.0732	−0.0481
	15	$\hat{s}$	*Estimator*	3.0427	3.0103	2.9929	3.0286	3.0176	3.0033	3.0570	3.0081	3.0211
			bias	−0.0427	−0.0103	−0.0043	−0.0286	−0.0176	−0.0033	−0.0570	−0.0081	−0.0211
	20		*Estimator*	3.0137	2.9986	3.0175	3.0068	3.0407	3.0015	3.0092	3.0140	3.0239
			bias	−0.0136	0.0014	−0.0091	−0.0068	−0.0407	−0.0015	−0.0092	−0.0140	−0.0239
	25		*Estimator*	3.0384	3.0056	3.0077	3.0159	3.0672	3.0097	3.0367	3.0463	3.0251
			bias	−0.0384	−0.0056	−0.0077	−0.0159	−0.0672	−0.0097	−0.0367	−0.0463	−0.0251
	15	$b\hat{e}ta$	*Estimator*	2.9780	3.0067	3.0285	2.9925	3.0100	3.0147	2.9843	3.0130	3.0104
			bias	−0.9688	−0.0067	−0.0196	0.0075	−0.0285	0.0083	0.0157	−0.0130	−0.0104
	20		*Estimator*	3.0033	3.0200	3.0175	3.0150	2.9945	3.0120	3.0167	3.0101	3.0085
			bias	−0.0033	−0.0200	−0.0175	−0.0150	0.0055	−0.0120	−0.0092	−0.0101	−0.0085
	25		*Estimator*	2.9891	3.0178	3.0188	3.0039	2.9719	3.0118	2.9992	2.9882	3.0054
			bias	0.0109	−0.0178	−0.0188	−0.0039	0.0281	−0.0118	0.0008	0.0118	−0.0054
GG(2,2,4)	15	$\hat{b}$	*Estimator*	2.0775	2.0131	2.0424	2.0848	2.0232	2.0856	2.0418	2.0307	2.0600
			bias	−0.0775	−0.0131	−0.0424	−0.0848	−0.0232	−0.0856	−0.0418	−0.0307	−0.0600
	20		*Estimator*	2.0485	2.0146	2.0756	2.0204	2.0881	2.0605	2.0474	2.0283	2.0470
			bias	−0.0485	−0.0146	−0.0756	−0.0204	−0.0881	−0.0605	−0.0432	−0.0283	−0.0470
	25		*Estimator*	2.0643	2.0366	2.0268	2.0668	2.0567	2.0561	2.0325	2.0306	2.0483
			bias	−0.0643	−0.0366	−0.0268	−0.0668	−0.0567	−0.0561	−0.0325	−0.0306	−0.0845
	15	$\hat{s}$	*Estimator*	1.9967	1.9996	1.9942	2.0083	1.9740	2.0075	1.9879	2.0098	1.9966
			bias	0.0033	0.0004	0.0058	−0.0083	0.0260	−0.0075	0.0121	−0.0097	0.0034
	20		*Estimator*	2.0104	1.9916	2.0088	1.9833	1.9987	1.9975	1.9894	2.0010	1.9899
			bias	−0.0104	0.0084	−0.0088	0.0167	0.0013	0.0025	0.0008	−0.0010	0.0101
	25		*Estimator*	2.0050	2.0088	1.9879	2.0189	2.0038	2.0001	1.9879	1.9989	1.9911
			bias	−0.0050	−0.0088	0.0121	−0.0189	−0.0038	−0.0001	0.0121	0.0011	0.0089
	15	$b\hat{e}ta$	*Estimator*	4.0992	4.0468	4.0757	4.0904	4.0905	4.0893	4.0843	4.0458	4.0851
			bias	−0.0992	−0.0468	−0.0757	−0.0904	−0.0905	−0.0893	−0.0843	−0.0458	−0.0851
	20		*Estimator*	4.0573	4.0594	4.0791	4.0763	4.1018	4.0839	4.0854	4.0565	4.0853
			bias	−0.0573	−0.0594	−0.0791	−0.0763	−0.1018	−0.0838	−0.0688	−0.0565	−0.0853
	25		*Estimator*	4.0764	4.0504	4.0735	4.0582	4.0722	4.0776	4.0771	4.0609	4.0861
			bias	−0.0764	−0.0579	−0.0735	−0.0582	−0.0722	−0.0776	−0.0771	−0.0609	−0.0861

From Table 1 *and Table* 2, **it can be observed that:**

• In almost all cases, the biases are very small.

• The values of the all estimators for $(b, s\ and\ beta)$ is very close to the real values of $b, s\ and\ beta$.

• In almost all cases, MSEs of all estimators for $(b, s\ and\ beta)$ based on RSS, MRSS, PRSS and NRSS are smaller than MSEs of the estimators based on SRS.

• In almost all cases, MSEs of all estimators based on RSS, MRSS, PRSS and NRSS decrease as the cycle sizes increase.

• When the values of b and s increase, the MSEs of all estimators based on RSS, MRSS, PRSS and NRSS decrease.

• MSEs of the estimators for $(b, s\ and\ beta)$ based on NRSS have the smallest MSEs in all cases comparing with other estimators based on RSS, MRSS and PRSS.

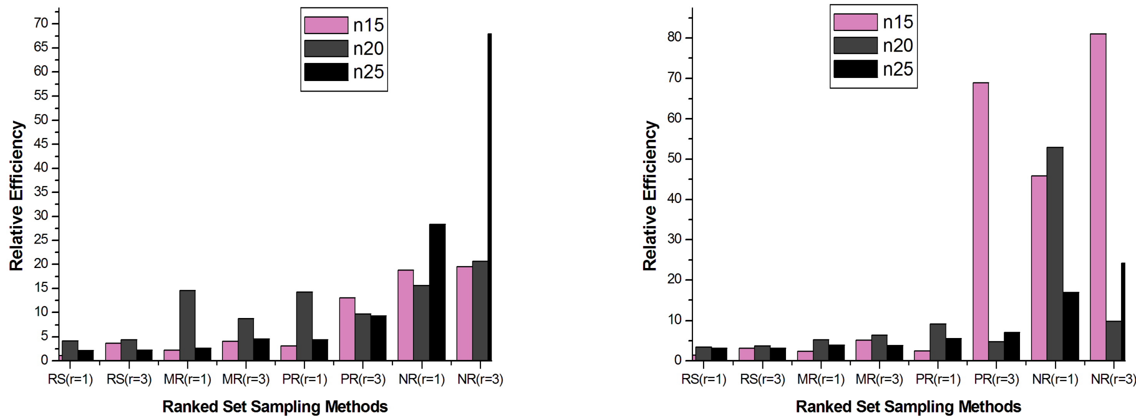
TABLE 3. Exact REs relative to SRS based on RSS, MRSS, PRSS, and NRSS.

Distribution	n		RSS		MRSS		PRSS		NRSS	
			$m=1$	$r=3$	$m=1$	$m=3$	$m=1$	$m=3$	$m=1$	$m=3$
GG(3,3,3)	15	$\hat{b}$	1.1718	3.6780	2.2389	4.0991	3.1427	13.0538	18.8661	19.5411
	20		4.1932	4.3836	14.6188	8.7689	14.2698	9.7118	15.6442	20.7295
	25		2.1648	2.2211	2.6959	4.5559	4.4036	9.3273	28.3575	67.9058
	15	$\hat{s}$	1.4955	3.1867	2.4200	5.1781	2.5107	68.9798	45.9191	81.0011
	20		3.4212	3.7668	5.3138	6.4090	9.1615	4.8103	52.9446	9.8355
	25		3.1413	3.1379	3.9230	3.7967	5.5859	7.0556	16.9299	24.1395
	15	$\hat{beta}$	2.2735	6.0463	2.5305	12.2524	1.7284	26.8113	12.5380	54.9708
	20		2.3414	2.7351	1.7776	3.6464	4.3516	2.0000	14.1599	3.6610
	25		4.3420	4.1787	7.3919	2.9307	6.0814	4.7697	8.8934	9.6184
GG(2,2,4)	15	$\hat{b}$	1.2548	4.431	5.4581	10.8605	30.4087	12.1046	59.3503	14.4328
	20		1.3873	1.1908	2.6676	1.0484	4.3238	1.9480	27.4535	1.7365
	25		6.1538	3.0074	20.3993	4.9197	12.9460	4.4248	24.0068	15.9413
	15	$\hat{s}$	6.5498	25.0192	6.8915	16.4541	38.0996	28.8354	58.6385	49.6355
	20		10.6288	12.9179	4.3012	7.9081	46.1929	14.3126	50.5101	54.7224
	25		17.7256	43.6959	26.4348	26.3662	6.1915	134.3402	40.4657	139.1280
	15	$\hat{beta}$	3.8078	6.1727	5.5194	8.3734	35.3623	14.9441	5.7389	16.8289
	20		4.6395	4.6868	8.0200	15.4738	43.6575	18.7304	48.0546	23.8312
	25		12.5568	2.3786	12.8752	5.4628	1.8842	8.1582	16.5923	9.0914

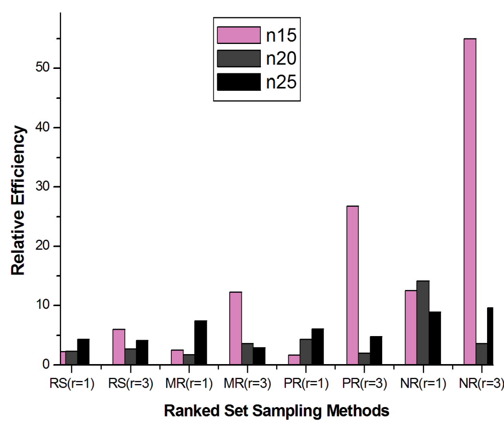
From Table 3, **it can be observed that:**

• Efficiencies of the estimators based on RSS, MRSS, PRSS and NRSS decrease as the values of (b, s and $beta$) increase.

• In almost all cases, efficiencies of the estimators for (b, s *and beta*) based on NRSS are greater than the efficiencies based on RSS, MRSS and PRSS, except in some cases.

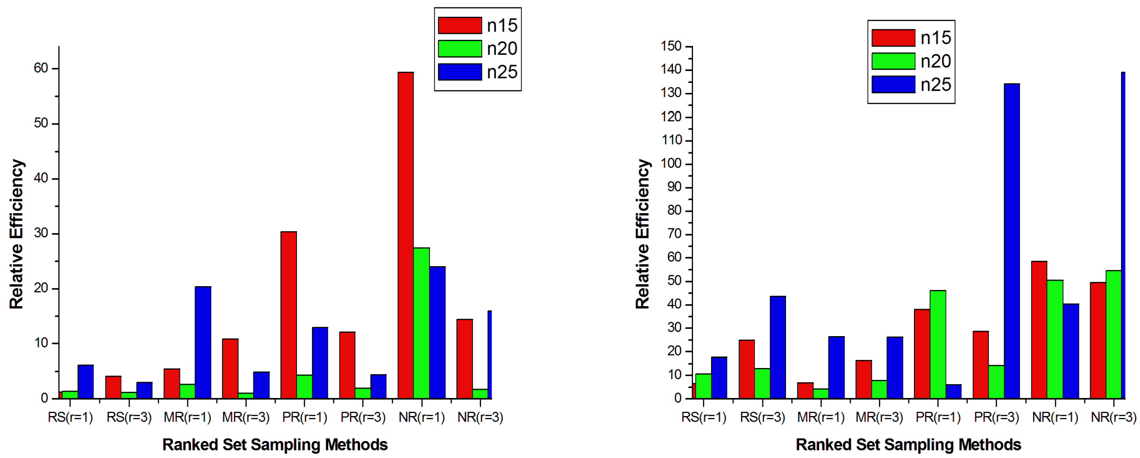


a) For $\hat{b}$ estimator b) For $\hat{s}$ estimator

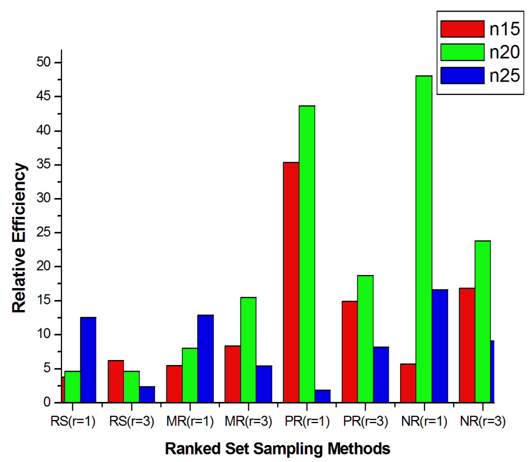


c) For $\hat{beta}$ estimator

FIGURE 6. Efficiencies of the estimators for $GG(3,3,3)$.



a) For $\hat{b}$ estimator b) For $\hat{s}$ estimator



c) For $\hat{beta}$ estimator

FIGURE 7. Efficiencies of the estimators for $GG(2,2,4)$.

$\underline{From\ Figure\, 6\ and\ Figure\, 7,}$ **it can be observed that:**

• Efficiencies of the estimators for $(b, s\ and\ beta)$ based on NRSS have the largest efficiencies in all cases, except in some cases.

• Efficiencies of the estimators for $(b, s\ and\ beta)$ based on PRSS are greater than the efficiencies based on RSS and MRSS, except in some cases.

• Efficiencies of the estimators for $(b, s\ and\ beta)$ based on RSS have the smallest efficiencies in all cases, except in some cases.

• When the cycle sizes increase, the efficiencies of all estimators based on RSS, MRSS, PRSS and NRSS increase.

5. Summary and Conclusions

Based on the numerical results from the comparative study, the findings may be summarized as follows:

(1) The MSEs based on SRS data has the largest MSEs comparing to RSS and its modifications schemes.
(2) It can be noted that, NRSS technique has superior to the rest of other sampling schemes. It has the smallest MSEs and largest efficiencies.
(3) Also it can be noted that RSS technique has inferior to the rest of other sampling schemes. It has the largest MSEs and smallest efficiencies.
(4) Generally the estimators based on NRSS, PRSS, MRSS and RSS techniques are more efficient than the estimators based on SRS techniques.
(5) L- moments method for estimation is given results very close to the real values of the parameters with very small biases.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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