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FIXED POINT RESULTS FOR GERAGHTY CONTRACTION TYPE MAPPINGS IN $b-$METRIC SPACES

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Abstract. In this paper, we introduce some new results on the fixed point and common fixed points of Geraghty contraction mappings in $b-$metric $b-$complete spaces. Moreover, we give a representative example to illustrate the compatibility of our results.

Keywords: $b-$metric space; Geraghty; fixed point.

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1. Introduction

The famous extensions of the concept of metric spaces have been done by Czerwik [1] where he introduced and studied the concepts of $b-$metric spaces. Bakhtin [2] uses b-metric spaces as a generalization of metric spaces for find fixed point. After that, several papers have been published on the theory of the fixed point in this space. For additional works and results in b-metric spaces, we encourage readers to refer to the reference ([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

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In this section, we recall some basic known definitions, notations and results in $b-$metric spaces which will be used in the sequel. Throughout this article, N, R, R^+ denote the set of natural numbers, the set of real numbers and the set of positive real numbers, respectively.

Definition 1.1. [1]. *Let Xbe a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is said to be a $b-$metric on X if the following conditions hold:*

(i) $d(x,y) = 0$ *if and only if* $x = y$;

(ii) $d(x,y) = d(y,x)$ *for all* $x,y \in X$;

(iii) $d(x,y) \leq s(d(x,z)+d(z,y))$ *for all* $x,y,z \in X$.

In this case, the pair (X,d) is called a $b-$metric space.

It is worth mentioning that the class of $b-$metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above fact.

Example 1. [16]. *Let (X,d) be a metric space and let $\beta > 1$, $\lambda \geq 0$ and $\mu > 0$. For $x,y \in X$, set $\rho(x,y) = \lambda d(x,y) + \mu d(x,y)^{\beta}$. Then $(X,\rho)$is a b-metric space with the parameter $s = 2^{\beta-1}$ and not a metric space on X.*

Definition 1.2. [17]. *Let (X,d) be a $b-$metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) $\{x_n\}$ *converges to x if and only if* $\lim_{n\to\infty} d(x_n,x) = 0$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$.

(ii) $\{x_n\}$ *is Cauchy if and only if* $\lim_{n,m\to\infty} d(x_n,x_m) = 0$.

(iii) (X,d) *is complete if and only if every Cauchy sequence in X is convergent.*

Remark 1.1. [17]. *In a $b-$metric space (X,d), the following assertions hold:*

(i) *A convergent sequence has a unique limit.*

(ii) *Each convergent sequence is Cauchy.*

(iii) *In general, a $b-$metric is not continuous.*

Theorem 1.1. [18]. *Let (X,d) be a complete metric space. Let $f : X \to X$ be given mapping satisfying:*

$$d(fx, fy) \leq \alpha(d(x,y))d(x,y), \quad \forall x,y \in X, \tag{1.1}$$

where $\alpha \in \mathscr{A}$. Then f has a unique fixed point.

At $\mathscr{A}$ be the family of all functions $\alpha : [0,\infty) \to [0,1)$ satisfying the property:

$$\lim_{n\to\infty} \alpha(t_n) = 1 \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

Theorem 1.2. [19] *Let (X,d) be a $b-$complete $b-$metric space with parameter s self-map. Suppose that there exists $\beta \in \mathscr{B}$ such that:*

$$d(fx, fy) \leq \alpha(d(x,y))d(x,y), \quad \forall x,y \in X, \tag{1.2}$$

where $\alpha \in \mathscr{B}$. Then f has a unique fixed point.

At (X,d) be a $b-$metric space with parameter $s \geq 1$ and $\mathscr{B}$ denote the set of all functions $\beta : [0,\infty) \to [0,1)$, satisfing the following condition:

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s} \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

2. Main Results

Theorem 2.1. *Let (X,d) be a $b-$complete $b-$metric space with parameter $s \geq 1$. Let $f : X \to X$ be a self-mapping satisfying:*

$$d(fx, fy) \leq \beta(\mathscr{L}(x,y))\mathscr{L}(x,y), \quad \forall x,y \in X, \tag{2.1}$$

where

$$\mathscr{L}(x,y) = \max\left\{ d(x,y), \frac{d(x,fx)[1+d(y,fy)]}{1+d(x,y)}, \frac{d(x,fx)[1+d(y,fy)]}{1+d(fx,fy)}, \frac{d(x,fy)+d(y,fx)}{2s} \right\},$$

and $\beta \in \mathscr{B}$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ such that

$$x_n = fx_{n-1} = f^n x_0, \quad \forall n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then x_n is a fixed point of f and the proof is finished. Otherwise, we have $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. Using (2.1), we obtain

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq \beta(\mathscr{L}(x_{n-1}, x_n))\mathscr{L}(x_{n-1}, x_n), \tag{2.2}$$

where

$$\begin{aligned}
\mathscr{L}(x_{n-1},x_n) &= \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},fx_{n-1})[1+d(x_n,fx_n)]}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},fx_{n-1})[1+d(x_n,fx_n)]}{1+d(fx_{n-1},fx_n)},\\
&\quad \frac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{2s}\Big\}\\
&= \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,x_{n+1})},\\
&\quad \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2s}\Big\}\\
&= \max\Big\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}{1+d(x_{n-1},x_n)}, d(x_{n-1},x_n), \frac{d(x_{n-1},x_{n+1})}{2s}\Big\}\\
&\leq \max\Big\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2s}\Big\}\\
&= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}
\end{aligned}$$

If $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then from (2.2) we would have

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \beta(d(x_n,x_{n+1}))d(x_n,x_{n+1})\\
&\leq \frac{1}{s}d(x_n,x_{n+1})\\
&< d(x_n,x_{n+1}),
\end{aligned} \tag{2.3}$$

which is a contradiction. Hence, $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$,

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \beta(d(x_{n-1},x_n))d(x_{n-1},x_n)\\
&\leq \frac{1}{s}d(x_{n-1},x_n)\\
&< d(x_{n-1},x_n).
\end{aligned} \tag{2.4}$$

Since $\{d(x_{n-1},x_n)\}$ is a decreasing sequence of non-negative reals. Hence, there exists $\rho \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1},x_n) = \rho$. We will prove that $\rho = 0$. Suppose on contrary that $\rho > 0$. Then, taking $n \to \infty$ in (2.4) we have

$$\rho \leq \limsup_{n\to\infty} \beta(\mathscr{L}(x_{n-1},x_n))\rho.$$

Then,

$$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(\mathscr{L}(x_{n-1},x_n)) \leq \frac{1}{s}.$$

From $\beta \in \mathscr{B}$, then $\limsup_{n\to\infty} \beta(\mathscr{L}(x_{n-1},x_n)) = 0$. Hence, $\lim_{n\to\infty} d(x_{n-1},x_n) = 0$, which is a contradiction, that is, $\rho = 0$. Now, we prove that the sequence $\{x_n\}$ is a $b-$Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(x_{m(k)},x_{n(k)}) \geq \varepsilon. \tag{2.5}$$

This means that

$$d(x_{m(k)},x_{n(k)-1}) < \varepsilon. \tag{2.6}$$

Using (2.5) and the triangular inequality, we get

$$\varepsilon \leq d(x_{m(k)},x_{n(k)}) \leq s[d(x_{m(k)},x_{m(k)+1}) + d(x_{m(k)+1},x_{n(k)})].$$

Then, we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1},x_{n(k)}). \tag{2.7}$$

From the definition of $\mathscr{L}(x,y)$ and the above limits,

$$\begin{aligned}
&\limsup_{k\to\infty} \mathscr{L}(x_{m(k)},x_{n(k)-1}) \\
&= \limsup_{k\to\infty} \max\Bigg\{ d(x_{m(k)},x_{n(k)-1}), \frac{d(x_{m(k)},fx_{m(k)})[1+d(x_{n(k)-1},fx_{n(k)-1})]}{1+d(x_{m(k)},x_{n(k)-1})}, \\
&\quad \frac{d(x_{m(k)},fx_{m(k)})[1+d(x_{n(k)-1},fx_{n(k)-1})]}{1+d(fx_{m(k)},fx_{n(k)-1})}, \frac{d(x_{m(k)},fx_{n(k)-1})+d(x_{n(k)-1},fx_{m(k)})}{2s} \Bigg\} \\
&= \limsup_{k\to\infty} \max\Bigg\{ d(x_{m(k)},x_{n(k)-1}), \frac{d(x_{m(k)},x_{m(k)+1})[1+d(x_{n(k)-1},x_{n(k)})]}{1+d(x_{m(k)},x_{n(k)-1})}, \\
&\quad \frac{d(x_{m(k)},x_{m(k)+1})[1+d(x_{n(k)-1},x_{n(k)})]}{1+d(x_{m(k)+1},x_{n(k)})}, \frac{d(x_{m(k)},x_{n(k)})+d(x_{n(k)-1},x_{m(k)+1})}{2s} \Bigg\}
\end{aligned}$$

$$= \limsup_{k\to\infty} \max\Big\{ d(x_{m(k)},x_{n(k)-1}), \frac{d(x_{m(k)},x_{m(k)+1})[1+d(x_{n(k)-1},x_{n(k)})]}{1+d(x_{m(k)},x_{n(k)-1})},$$
$$\frac{d(x_{m(k)},x_{m(k)+1})[1+d(x_{n(k)-1},x_{n(k)})]}{1+d(x_{m(k)+1},x_{n(k)})}, \frac{s[d(x_{m(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{n(k)})]}{2s},$$
$$\frac{s[d(x_{n(k)-1},x_{m(k)})+d(x_{m(k)},x_{m(k)+1})]}{2s}\Big\}$$
$$\leq \varepsilon.$$

Using (2.7) and (2.1), we get

$$\begin{aligned}\varepsilon = s(\frac{\varepsilon}{s}) &\leq s \limsup_{k\to\infty} d(x_{m(k)+1},x_{n(k)}) \\ &\leq \limsup_{k\to\infty} \beta(\mathscr{L}(x_{m(k)},x_{n(k)-1}))\mathscr{L}(x_{m(k)},x_{n(k)-1}) \\ &\leq \varepsilon \limsup_{n\to\infty} \beta(\mathscr{L}(x_{m(k)},x_{n(k)-1}))\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{k\to\infty} \beta(\mathscr{L}(x_{m(k)},x_{n(k)-1})) \leq \frac{1}{s}$. From $\beta \in \mathscr{B}$ we conclude that $\mathscr{L}(x_{m(k)},x_{n(k)-1}) \to 0$, as a result, $d(x_{m(k)},x_{n(k)-1}) \to 0$. Using (2.5) and the $b-$triangular inequality, we get

$$\varepsilon \leq d(x_{m(k)},x_{n(k)}) \leq s[d(x_{m(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{n(k)})].$$

Hence, $\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = 0$, a contradiction to (2.5). Thus, $\{x_n\}$ is a $b-$Cauchy sequence. The completeness of X implies that there exists $\theta \in X$ such that $x_n \to \theta$. Next, We will show that θ is a fixed point of f. Using $b-$triangular inequality and (2.1), we get

$$\begin{aligned}d(\theta,f\theta) &\leq s[d(\theta,fx_n)+d(fx_n,f\theta)] \\ &\leq sd(\theta,fx_n)+s\beta(\mathscr{L}(x_n,\theta))\mathscr{L}(x_n,\theta).\end{aligned}$$

Taking $n\to\infty$ in the above inequality, we obtain

$$d(\theta,f\theta) \leq s\limsup_{n\to\infty} d(\theta,x_{n+1}) + s\limsup_{n\to\infty} \beta(\mathscr{L}(x_n,\theta)) \limsup_{n\to\infty} \mathscr{L}(x_n,\theta), \tag{2.8}$$

where

$$\begin{aligned}
&\limsup_{n\to\infty}\mathscr{L}(x_n,\theta)\\
&=\limsup_{n\to\infty}\max\Big\{d(x_n,\theta),\frac{d(x_n,fx_n)[1+d(\theta,f\theta)]}{1+d(x_n,\theta)},\frac{d(x_n,fx_n)[1+d(\theta,f\theta)]}{1+d(fx_n,f\theta)},\\
&\quad\frac{d(x_n,f\theta)+d(\theta,fx_n)}{2s}\Big\}\\
&\leq\limsup_{n\to\infty}\max\Big\{d(x_n,\theta),\frac{d(x_n,x_{n+1})[1+d(\theta,f\theta)]}{1+d(x_n,\theta)},\frac{d(x_n,x_{n+1})[1+d(\theta,f\theta)]}{1+d(x_{n+1},f\theta)},\\
&\quad\frac{d(x_n,f\theta)+d(\theta,x_{n+1})}{2s}\Big\}\\
&\leq\limsup_{n\to\infty}\max\Big\{d(x_n,\theta),\frac{d(x_n,x_{n+1})[1+d(\theta,f\theta)]}{1+d(x_n,\theta)},\frac{d(x_n,x_{n+1})[1+d(\theta,f\theta)]}{1+d(x_{n+1},f\theta)},\\
&\quad\frac{s[d(x_n,\theta)+d(\theta,f\theta)]+d(\theta,x_{n+1})}{2s}\Big\}\\
&\leq d(\theta,f\theta).
\end{aligned}\tag{2.9}$$

Using (2.8), we get

$$d(\theta,f\theta)\leq s\limsup_{n\to\infty}\beta(\mathscr{L}(x_n,\theta))d(\theta,f\theta).\tag{2.10}$$

which implies that $\frac{1}{s}\leq\limsup_{n\to\infty}\beta(\mathscr{L}(x_n,\theta))\leq\frac{1}{s}$. From $\beta\in\mathscr{B}$ we conclude that $\lim_{n\to\infty}\mathscr{L}(x_n,\theta)=0$. Hence, $f\theta=\theta$.

Finally, suppose that the set of fixed point of f is well ordered. Assume on contrary, that θ and Φ are two fixed points of f such that $\theta\neq\Phi$. Using (2.1), we get

$$d(\theta,\Phi)=d(f\theta,f\Phi)\leq\beta(\mathscr{L}(\theta,\Phi))\mathscr{L}(\theta,\Phi),$$

where

$$\begin{aligned}
&\mathscr{L}(\theta,\Phi)\\
&=\max\Big\{d(\theta,\Phi),\frac{d(\theta,f\theta)[1+d(\Phi,f\Phi)]}{1+d(\theta,\Phi)},\frac{d(\theta,f\theta)[1+d(\Phi,f\Phi)]}{1+d(f\theta,f\Phi)},\frac{d(\theta,f\Phi)+d(\Phi,f\theta)}{2s}\Big\}
\end{aligned}$$

$$= \max\left\{d(\theta,\Phi), \frac{d(\theta,\theta)[1+d(\Phi,\Phi)]}{1+d(\theta,\Phi)}, \frac{d(\theta,\theta)[1+d(\Phi,\Phi)]}{1+d(\theta,\Phi)}, \frac{d(\theta,\Phi)+d(\Phi,\theta)}{2s}\right\}$$

$$= \max\left\{d(\theta,\Phi), \frac{d(\Phi,\theta)}{s}\right\}$$

$$= d(\theta,\Phi).$$

Hence, we have $d(\theta,\Phi) < \frac{d(\theta,\Phi)}{s}$, a contradiction. So, $\theta = \Phi$ and the fixed point of f is unique. □

Example 2. *Let $X = \{3,4,5\}$ and $d : X \times X \to [0,\infty)$ be defined as follows:*

(i) $d(3,4) = d(4,3) = 3$
(ii) $d(3,5) = d(5,3) = \frac{3}{25}$
(iii) $d(4,5) = d(5,4) = \frac{20}{25}$
(iv) $d(3,3) = d(4,4) = d(5,5) = 0.$

It is easy to check that (X,d) is a $b-$metric space with constant $s = \frac{5}{4}$. Take $f3 = f5 = 3, f4 = 5$ and $\beta(t) = \frac{4}{5}e^{-t}, t > 0$ and $\beta(0) \in [0,5)$. Then we have

$$d(f3,f4) = d(3,5) = \frac{3}{25} \le \frac{4}{5}e^{-3} = \beta(\mathscr{L}(3,4))\mathscr{L}(3,4),$$

$$d(f3,f5) = d(3,3) = 0 \le \beta(\mathscr{L}(3,5))\mathscr{L}(3,5),$$

$$d(f4,f5) = d(5,3) = \frac{3}{25} \le \frac{4}{5}e^{\frac{20}{25}} = \beta(\mathscr{L}(4,5))\mathscr{L}(4,5).$$

Hence, the conditions of Theorem 2.1 are satisfied.

Theorem 2.2. *Let (X,d) be a $b-$complete $b-$metric space with $s \ge 1$. Let f,g be self-mappings on X which satisfy*

$$sd(fx,gy) \le \beta(\mathscr{L}(x,y))\mathscr{L}(x,y), \quad \forall x,y \in X, \tag{2.11}$$

where

$$\mathscr{L}(x,y) = \max\left\{d(x,y), \frac{d(x,fx)[1+d(y,gy)]}{1+d(x,y)}, \frac{d(x,fx)[1+d(y,gy)]}{1+d(fx,gy)}\right\},$$

and $\beta \in \mathscr{B}$. If f or g are continuous, then f and g have a unique common fixed point.

Proof. Let x_0 be arbitrary. Define the sequence $\{x_n\}$ in X by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n = 0,1,\ldots$. Using (2.11), for all $n = 0,1,2,\ldots$, we get

$$sd(x_{2n+1},x_{2n+2}) = sd(fx_{2n},Sx_{2n+1}) \le \beta(\mathscr{L}(x_{2n},x_{2n+1}))\mathscr{L}(x_{2n},x_{2n+1}), \tag{2.12}$$

where

$$\begin{aligned}
&\mathscr{L}(x_{2n},x_{2n+1})\\
&= \max\left\{d(x_{2n},x_{2n+1}), \frac{d(x_{2n},fx_{2n})[1+d(x_{2n+1},gx_{2n+1})]}{1+d(x_{2n},x_{2n+1})}, \frac{d(x_{2n},fx_{2n})[1+d(x_{2n+1},gx_{2n+1})]}{1+d(fx_{2n},gx_{2n+1})}\right\}\\
&= \max\left\{d(x_{2n},x_{2n+1}), \frac{d(x_{2n},x_{2n+1})[1+d(x_{2n+1},x_{2n+2})]}{1+d(x_{2n},x_{2n+1})}, \frac{d(x_{2n},x_{2n+1})[1+d(x_{2n+1},x_{2n+2})]}{1+d(x_{2n+1},x_{2n+2})}\right\}\\
&\le \max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\}
\end{aligned}$$

If $\mathscr{L}(x_{2n},x_{2n+1}) = d(x_{2n+1},x_{2n+2})$, then

$$sd(x_{2n+1},x_{2n+2}) \le \beta(\mathscr{L}(x_{2n},x_{2n+1}))d(x_{2n+1},x_{2n+2}) < d(x_{2n+1},x_{2n+2}),$$

a contradiction. So, we have $\mathscr{L}(x_{2n},x_{2n+1}) = d(x_{2n},x_{2n+1})$. Using (2.12), we get

$$d(x_{2n+1},x_{2n+2}) \le \beta(\mathscr{L}(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1}) \le \frac{1}{s}d(x_{2n},x_{2n+1}). \tag{2.13}$$

Also, we get $d(x_{2n+1},x_{2n+2}) \le d(x_{2n},x_{2n+1})$. Similarly, $d(x_{2n+3},x_{2n+2}) \le d(x_{2n+2},x_{2n+1})$. Hence, we have $d(x_n,x_{n+1}) \le d(x_{n-1},x_n)$. Therefore $\{d(x_n,x_{n+1})\}$ is a nonincreasing sequence, hence there exists $\rho \ge 0$ such that $d(x_n,x_{n+1}) \to \rho$ as $n \to \infty$. We will show that $\rho = 0$. Suppose on the contrary that $\rho > 0$. Taking $n \to \infty$ in (2.13), we get

$$\rho \le \limsup_{n\to\infty}(\mathscr{L}(x_{2n},x_{2n+1})\rho \tag{2.14}$$

which implies that $\frac{1}{s} \le 1 \le \limsup(\mathscr{L}(x_{2n},x_{2n+1})) \le \frac{1}{s}$. From $\beta \in \mathscr{B}$ we conclude that $\lim_{n\to\infty} \mathscr{L}(x_{2n},x_{2n+1}) = 0$. Hence, $\rho = \lim_{n\to\infty} d(x_{2n},x_{2n+1}) = 0$, a contradiction.
This is, $\lim_{n\to\infty} d(x_{2n},x_{2n+1}) = 0$. Now, we prove that the sequence $\{x_{2n}\}$ is a $b-$Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(x_{2m(k)},x_{2n(k)}) \ge \varepsilon. \tag{2.15}$$

This means that

$$d(x_{2m(k)}, x_{2n(k)-1}) < \varepsilon. \tag{2.16}$$

Using (2.11) and (2.15), we get

$$\begin{aligned} \varepsilon &\le d(x_{2n(k)}, x_{2m(k)}) \\ &\le sd(x_{2n(k)}, x_{2n(k)+1}) + sd(x_{2n(k)+1}, x_{2m(k)}) \\ &= sd(x_{2n(k)}, x_{2n(k)+1}) + sd(fx_{2n(k)}, gx_{2m(k)-1}) \\ &\le sd(x_{2n(k)}, x_{2n(k)+1}) + \beta(\mathscr{L}(x_{2n(k)}, x_{2m(k)-1}))\mathscr{L}(x_{2n(k)}, x_{2m(k)-1}), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \mathscr{L}(x_{2n(k)}, x_{2m(k)-1}) &= \max\Big\{d(x_{2n(k)}, x_{2m(k)-1}), \frac{d(x_{2n(k)}, fx_{2n(k)})[1 + d(x_{2m(k)-1}, gx_{2m(k)-1})]}{1 + d(x_{2n(k)}, x_{2m(k)-1})}, \\ &\qquad \frac{d(x_{2n(k)}, fx_{2n(k)})[1 + d(x_{2m(k)-1}, gx_{2m(k)-1})]}{1 + d(fx_{2n(k)}, gx_{2m(k)-1})}\Big\} \\ &= \max\Big\{d(x_{2n(k)}, x_{2m(k)-1}), \frac{d(x_{2n(k)}, x_{2n(k)+1})[1 + d(x_{2m(k)-1}, x_{2m(k)})]}{1 + d(x_{2n(k)}, x_{2m(k)-1})}, \\ &\qquad \frac{d(x_{2n(k)}, x_{2n(k)+1})[1 + d(x_{2m(k)-1}, x_{2m(k)})]}{1 + d(x_{2n(k)+1}, x_{2m(k)})}\Big\}. \end{aligned}$$

Taking $k \to \infty$, we get

$$\limsup_{k\to\infty} \mathscr{L}(x_{2n(k)}, x_{2m(k)-1}) = \limsup_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}).$$

Using the $b-$triangular inequality, we get

$$d(x_{2n(k)}, x_{2m(k)-1}) \le s(d(x_{2n(k)}, x_{2m(k)-2}) + d(x_{2m(k)-2}, x_{2m(k)-1})). \tag{2.18}$$

Taking $k \to \infty$ in (2.18), we get

$$\limsup_{k\to\infty} d(x_{2n(k)}, x_{2m(k)-1}) \le s\varepsilon. \tag{2.19}$$

Using (2.17) and (2.19), we obtain

$$
\begin{aligned}
\varepsilon &\leq \limsup_{k\to\infty}(\beta(\mathscr{L}(x_{2n(k)},x_{2m(k)-1}))\mathscr{L}(x_{2n(k)},x_{2m(k)-1})) \\
&= \limsup_{k\to\infty}\beta(\mathscr{L}(x_{2n(k)},x_{2m(k)-1}))\limsup_{k\to\infty}d(x_{2n(k)},x_{2m(k)-1}) \qquad (2.20)\\
&\leq s\varepsilon\limsup_{k\to\infty}\beta(\mathscr{L}(x_{2n(k)},x_{2m(k)-1}))
\end{aligned}
$$

which implies that $\frac{1}{s} \leq \limsup_{k\to\infty}\beta(\mathscr{L}(x_{2n(k)},x_{2m(k)-1})) \leq \frac{1}{s}$. From $\beta \in \mathscr{B}$ we conclude that $\lim_{n\to\infty}\mathscr{L}(x_{2n(k)},x_{2m(k)-1}) = 0$. Hence,

$$\lim_{n\to\infty}d(x_{2n(k)},x_{2m(k)-1}) = 0. \qquad (2.21)$$

Using (2.15) and the $b-$triangular inequality, we get

$$\varepsilon \leq d(x_{2n(k)},x_{2m(k)}) \leq s(d(x_{2n(k)},x_{2m(k)-1}) + d(x_{2m(k)-1},x_{2m(k)})). \qquad (2.22)$$

Taking $k \to \infty$ in (2.21) and using (2.22), we obtain

$$\limsup_{k\to\infty}d(x_{2n(k)},x_{2m(k)}) = 0.$$

This contradicts (2.15). This implies that $\{x_{2n}\}$ is a $b-$Cauchy sequence and hence there exists $\theta \in X$ such that $\lim_{n\to\infty}x_n = \theta$. If f is continuous, we get

$$f\theta = \lim_{n\to\infty}fx_{2n} = \lim_{n\to\infty}x_{2n+1} = \theta. \qquad (2.23)$$

Using (2.11), we obtain

$$sd(\theta,g\theta) = sd(f\theta,g\theta) \leq \beta(\mathscr{L}(\theta,\theta))\mathscr{L}(\theta,\theta), \qquad (2.24)$$

where

$$
\begin{aligned}
\mathscr{L}(\theta,\theta) &= \max\left\{d(\theta,\theta), \frac{d(\theta,f\theta)[1+d(\theta,g\theta)]}{1+d(\theta,\theta)}, \frac{d(\theta,f\theta)[1+d(\theta,g\theta)]}{1+d(f\theta,g\theta)}\right\} \\
&\leq d(\theta,g\theta).
\end{aligned}
$$

From $\beta \in \mathscr{B}$ we conclude that

$$sd(\theta,g\theta) \leq \beta((\theta,\theta))d(\theta,g\theta) \leq d(\theta,g\theta).$$

Hence, $g\theta = \theta$. If g is continuous, then, by a similar argument to that of above, one can show that f, g have a common fixed point. Now, we prove the uniqueness of the common fixed point. Let $y = fy = gy$, is another common fixed point for f and g. Using (2.11), we obtain

$$sd(\theta, y) = sd(f\theta, gy) \leq \beta(\mathscr{L}(\theta, y))\mathscr{L}(\theta, y), \tag{2.25}$$

where

$$\begin{aligned}\mathscr{L}(\theta, y) &= \max\left\{d(\theta, y), \frac{d(\theta, f\theta)[1 + d(y, fy)]}{1 + d(\theta, y)}, \frac{d(\theta, f\theta)[1 + d(y, fy)]}{1 + d(f\theta, fy)}, \frac{d(\theta, fy) + d(y, f\theta)}{2s}\right\} \\ &= d(\theta, y).\end{aligned}$$

Hence, $\theta = y$ and the common fixed point f and g is unique. □

Corollary 2.1. *Let (X, d) be a b−complete b−metric space with $s \geq 1$. Let f be self-mapping on X which satisfy*

$$sd(fx, fy) \leq \beta(\mathscr{L}(x, y))\mathscr{L}(x, y), \quad \forall x, y \in X, \tag{2.26}$$

where

$$\mathscr{L}(x, y) = \max\left\{d(x, y), \frac{d(x, fx)[1 + d(y, fy)]}{1 + d(x, y)}, \frac{d(x, fx)[1 + d(y, fy)]}{1 + d(fx, fy)}\right\},$$

and $\beta \in \mathscr{B}$. If f is continuous, then f has a unique fixed point.

Proof. Taking $f = g$ in Theorem 2.2, we get the following result. □

Example 3. *Let $X = [0, 1]$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$, for all $x, y \in [0, 1]$. Apparently, (X, d) is a b−metric space with parameter $s = 2$. Take $fx = \frac{x}{6}$ for all $x \in X$ and $\beta(t) = \frac{1}{6}$ for all $t > 0$. Then,*

$$\begin{aligned}2d(fx, fy) &= 2\left|\frac{x}{6} - \frac{y}{6}\right|^2 \\ &= \frac{1}{18}|x - y|^2 \\ &\leq \frac{1}{6}|x - y|^2 \\ &\leq \beta(\mathscr{L}(x, y))\mathscr{L}(x, y).\end{aligned}$$

Then, the conditions of Corollary 2.1 are satisfied.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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