Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:95
https://doi.org/10.28919/jmcs/5994
ISSN: 1927-5307

COMMON FIXED POINT RESULTS FOR HYBRID CONTRACTION IN HAUSDORFF FUZZY METRIC SPACE

RAJESH KUMAR SAINI*, MUKESH KUSHWAHA

Deptt. of Mathematical Sciences and Computer Applications, Bundelkhand University,Jhansi, India

Abstract. Hybrid contraction of single and multi-valued fuzzy mappings in Hausdorff fuzzy metric space is discussed in the present article. Here, we introduced the concept of $\alpha^* - \eta_* - \psi-$hybrid contraction for single and multi-valued fuzzy mappings and prove the common fixed point results in Hausdorff fuzzy metric space.

Keywords: common fixed Point; $\alpha^* - \eta_* - \psi$-hybrid contraction; Hausdorff fuzzy metric space.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

The existence of fixed points [FP] of set-valued contractions in metric spaces was initiated by Nadler [28] and subsequently results in (see [27, 29, 31, 42]). Zadeh [45] was first to introduced the notion of fuzzy set [FS] in 1965. After that a number of researchers work on FS and prove fixed point theorems [FPT] with fuzzy mappings [FM]. For some results on FPT (see [3, 6, 7, 9, 16, 16, 25, 34, 44] and references therein). Kramosil and Michalek [20] first to introduced the concept of fuzzy metric space [FMS], which was modified by George and Veeramani [10]. The Banach [4] contractive FPT extended by Gregori and Sapenal [13] to fuzzy contractive mapping

*Corresponding author
E-mail address: prof.rksaini@bujhansi.ac.in
Received May 09, 2021

[FCM] of complete metric space. Recently more results in FMS (see [1, 8, 11, 15, 17, 18, 23, 24).

In nonlinear analysis the FPT play a key role. For the existing of FP in FMS, the contractive conditions and implicit function play a key role (see[5, 30, 37]). Samet et al. [39] first introduced the concept of admissible mapping [AM] for single valued mapping [SVM] and Asl et al. [2] extended the concept of admissible for SVM to multi-valued mappings [MVM]. Latter, Salimi et al. [38], defined an α-AM with respect to η on MS. Afterwards, a number of authors investigated FPT for α^* and η_* type's AM's in FMS (see [32, 43]). Recently, Hong [19] introduced the concept of $\alpha^* - \eta_* -$admissible for set valued mappings in FMS. Motivated by result of Phiangsungnoen [33] as some new fuzzy FPT's for FCM's in Hausdorff fuzzy metric spaces [HFMS], we introduce the concept of $\alpha^* - \eta_* - \psi -$hybrid contraction [HC] of SVM and MVFM in HFMS. We also give illustrative examples to support our main results.

2. Preliminaries

Recall that a continuous triangular norm (t-norm[40]) with unit 1 is an associative and commutative binary operation $\odot : [0,1] \times [0,1] \to [0,1]$, if $a \odot 1 = a$ and $a \leq c$, $b \leq d$ then $a \odot b \leq c \odot d$ for all $a,b,c,d \in [0,1]$. The typical examples of continuous t-norm are $a \odot b = ab$ or $a \odot b = min(a,b)$, and $a \odot b = \frac{ab}{max\{a,b,\lambda\}}$, for $0 < \lambda < 1$. For Lukasievicz t-norm that is, $a \odot_L b = max\{a+b-1\}$.

Definition 2.1. *[10] For a non-empty set $\mathscr{X}$ and a continuous t-norm $\odot$, an ordered triple $(\mathscr{X}, \mathscr{M}, \odot)$ is called a FMS such that, $\mathscr{M}$ is a FS on $\mathscr{X} \times \mathscr{X} \times (0,\infty)$ with conditions,*

$(\mathscr{G}\mathscr{V}_1)$ $\mathscr{M}(\Theta,\Xi,t) > 0, \forall\, \Theta,\Xi,\Sigma \in \mathscr{X}$ *and* $t > 0$,

$(\mathscr{G}\mathscr{V}_2)$ $\mathscr{M}(\Theta,\Theta,t) = 1,\ \forall\, t > 0,$ *and* $\mathscr{M}(\Theta,\Xi,t) = 1,\ for\ some\ t > 0\ \Rightarrow \Theta = \Xi,$

$(\mathscr{G}\mathscr{V}_3)$ $\mathscr{M}(\Theta,\Xi,t) = \mathscr{M}(\Xi,\Theta,t),\ \forall\, t > 0,$

$(\mathscr{G}\mathscr{V}_4)$ $\mathscr{M}(\Theta,\Xi,t) \odot \mathscr{M}(\Xi,\Sigma,s) \leq \mathscr{M}(\Theta,\Sigma,t+s)\ \forall\, s,t > 0,$

$(\mathscr{G}\mathscr{V}_5)$ $\mathscr{M}(\Theta,\Xi,.) : (0,\infty) \to [0,1]$ *is continuous.*

According to Kramosil and Michalek [20], $\mathscr{M}$ is a fuzzy set on $\mathscr{X} \times \mathscr{X} \times (0,+\infty)$ which satisfies $(\mathscr{G}\mathscr{V}_3)$ and $(\mathscr{G}\mathscr{V}_4)$ while $(\mathscr{G}\mathscr{V}_1)$, $(\mathscr{G}\mathscr{V}_2)$ and $(\mathscr{G}\mathscr{V}_5)$ replaced by $(\mathscr{K}\mathscr{M}_1)$, $(\mathscr{K}\mathscr{M}_2)$ and $(\mathscr{K}\mathscr{M}_5)$, as follows:

$(\mathscr{K}\mathscr{M}_1)$ $\mathscr{M}(\Theta,\Xi,0)=0$,

$(\mathscr{K}\mathscr{M}_2)$ $\mathscr{M}(\Theta,\Xi,t)=1,\ \forall\, t>0,\ iff\ \Theta=\Xi$,

$(\mathscr{K}\mathscr{M}_5)$ $\mathscr{M}(\Theta,\Xi,.)=0,:[0,\infty)\to[0,1]\ is\ left\ continuous.$

$(\mathscr{K}\mathscr{M}_6)$ $\lim_{t\to\infty}\mathscr{M}(\Theta,\Xi,t)=1,\forall\,\Theta,\Xi\in\mathscr{X}$.

Remark 2.1. *[26] It is worth pointing out that* $0<\mathscr{M}(\Theta,\Xi,t)<1,\forall t>0$*, provided* $\Theta\neq\Xi$.

Example 2.1. *[10] Let* $(\mathscr{X},d)$ *be a metric space. Define* $a\odot b=ab$ *or* $a\odot b=min(a,b),\forall$ $a,b\in[0,1]$ *and define* $\mathscr{M}_d:\mathscr{X}\times\mathscr{X}\times[0,\infty]\to[0,1]$ *as* $\mathscr{M}_d(\Theta,\Xi,t)=t/(t+d(\Theta,\Xi))$*, for all* $\Theta,\Xi\in\mathscr{X}$ *and* $t>0$*, then fuzzy metric* $\mathscr{M}_d$ *induced by the metric d and* $(\mathscr{X},\mathscr{M}_d,\odot)$ *is called a FMS*

Example 2.2. *[12] Let* $(\mathscr{X},d)$ *be a metric space. Define* $\mathscr{G}:\mathscr{R}^+\to(k,\infty)$ *an increasing continuous with* $d(\Theta,\Xi)<k$, $\forall\Theta,\Xi\in\mathscr{X}$ *and k is a fixed constant on* $(0,\infty)$*. Define* $\mathscr{M}_d:\mathscr{X}\times\mathscr{X}\times[0,\infty]\to[0,1]$ *as* $\mathscr{M}_d(\Theta,\Xi,t)=1-\frac{d(\Theta,\Xi)}{\mathscr{G}(t)}$, $\forall\Theta,\Xi\in\mathscr{X}$ *and* $t>0$*. Then* $(\mathscr{X},\mathscr{M}_d,\odot)$ *is called a FMS on* $\mathscr{X}$ *wherein* $\odot$ *is a Lukasievicz t-norm.*

Song [41] gives two important facts that $\mathscr{M}(.,.,t)$ *is continuous function on* $\mathscr{X}\times\mathscr{X}$ *for* $t\in(0,\infty)$ *and* $\mathscr{M}(\Theta,\Xi,.)$ *is non-decreasing for all* $\forall\Theta,\Xi\in\mathscr{X}$.

Definition 2.2. *[10] Let* $(\mathscr{X},\mathscr{M},\odot)$ *be a fuzzy metric space. Then*

(a) *If* $\exists n\in\mathbb{N}$*, s.t.* $\mathscr{M}(\Theta_n,\Theta,t)>1-\varepsilon,\ \forall\ n_0\in n\ ,\varepsilon>0,\ t>0$ *, then a sequence* $\{\Theta_n\}$ *in* $\mathscr{X}$ *is said to be convergent at* Θ *in* $\mathscr{X}$.

(b) *A sequence* $\{\Theta_n\}$ *in* $\mathscr{X}$*, called Cauchy sequence if* $\exists n\in\mathbb{N},\ \&\ \forall\varepsilon>0,\ t>0$ *s.t.* $\mathscr{M}(\Theta_n,\Theta_m,t)>1-\varepsilon,\ \forall\ n,m\geq n_0\in\mathbb{N}$.

(c) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Lemma 2.1. *[36] Let* $(\mathscr{X},\mathscr{M},\odot)$ *be a fuzzy metric space and* $\{\Xi_n\}$ *be a sequence in* $\mathscr{X}$ *with* $(\mathscr{K}\mathscr{M}_6)$*. If* $\exists k\in(0,1)$ *s.t.* $\mathscr{M}(\Xi_n,\Xi_{n+1},kt)>\mathscr{M}(\Xi_{n-1},\Xi_n,t)$, $\forall\ t>0$ *and* $n\in\mathbb{N}$*, then* $\{\Xi_n\}$ *is a Cauchy sequence in* $\mathscr{X}$.

Lemma 2.2. *[35] Let* $(\mathscr{X},\mathscr{M},\odot)$ *be a fuzzy metric space and* $\{\Xi_n\}$ *be a sequence in* $\mathscr{X}$ *with* $(\mathscr{K}\mathscr{M}_6)$*. If* $\exists k\in(0,1)$ *s.t.* $\mathscr{M}(\Theta_{n+1},\Theta_{n+2},kt)\geq\mathscr{M}(\Theta_n,\Theta_{n+1},kt)$, $\forall\ t>0$ *and* $n=0,1,2,3....,$ *then* $\{\Theta_n\}$ *is a Cauchy sequence in* $\mathscr{X}$.

Lemma 2.3. *Let $(\mathscr{X},\mathscr{M},\odot)$ be a fuzzy metric space. Then $\mathscr{M}$ is said to be continuous function on $\mathscr{X}\times\mathscr{X}\times[0,\infty)$, if $\lim_{n\to\infty}\mathscr{M}(\Theta_n,\Xi_n,t_n)=\mathscr{M}(\Theta,\Xi,t)$, whenever $\{(\Theta_n,\Xi_n,t_n)\}$ is a sequence in $\mathscr{X}\times\mathscr{X}\times[0,\infty)$, which converges to a point $(\Theta,\Xi,t)\in\mathscr{X}\times\mathscr{X}\times[0,\infty)$, i.e. $\lim_{n\to\infty}\mathscr{M}(\Theta_n,\Theta,t)=\mathscr{M}(\Xi_n,\Xi,t)=1$ and $\lim_{n\to\infty}\mathscr{M}(\Theta,\Theta,t_n)=\mathscr{M}(\Xi,\Xi,t)=1$.*

Lemma 2.4. *[35] If $\forall\ \Theta,\Xi\in\mathscr{X},\ t>0$ and for a number $k\in(0,1)$ in fuzzy metric space $(\mathscr{X},\mathscr{M},\odot)$ then $\mathscr{M}(\Theta,\Xi,kt)\geq\mathscr{M}(\Theta,\Xi,t)$ implies $\Theta=\Xi$.*

Definition 2.3. *[41] Let $\mathscr{X}$ be a non-empty set and $\alpha,\eta:\mathscr{X}\times\mathscr{X}\times(0,\infty)\to[0,\infty)$ be two functions. A mapping and $\mathscr{T}:\mathscr{X}\to 2^{\mathscr{X}}$ is said to be $\alpha^*-\eta_*-$admissible mapping if for all $\Theta,\Xi\in\mathscr{X}$ we have $\alpha(\Theta,\Xi,t)\leq\eta(\Theta,\Xi,t)\Rightarrow\alpha^*(\mathscr{T}\Theta,\mathscr{T}\Xi,t)\leq\eta_*(\mathscr{T}\Theta,\mathscr{T}\Xi,t),\ \forall\ \Theta,\Xi\in\mathscr{X},\ t>0$, where*

$\alpha^*(\mathscr{T}\Theta,\mathscr{T}\Xi,t)=\sup_{\Theta\in\mathscr{T}\Theta,\Xi\in\mathscr{T}\Xi}\alpha(\Theta,\Xi,t)$ *and* $\eta_*(\mathscr{T}\Theta,\mathscr{T}\Xi,t)=\inf_{\Theta\in\mathscr{T}\Theta,\Xi\in\mathscr{T}\Xi}\eta(\Theta,\Xi,t)$

Let $(\mathscr{X},d)$ be metric space, a set $E(\Theta):\mathscr{X}\to[0,1]$ is a fuzzy set. For any fuzzy set and $\Theta\in\mathscr{X}$, $E(\Theta)$ called the membership grade of E in $\mathscr{X}$. The $\lambda-$cut of fuzzy set E represented as $E_\lambda=\{\Theta:E(\Theta)\geq\lambda\}$ if $\lambda\in(0,1]$ also $E_0=\overline{\{\Theta:E(\Theta)>0\}}$. Let $F(\mathscr{X})$ be the collection of all fuzzy set in a metric space $\mathscr{X}$. For all $E,\mathscr{F}\in F(X)$, $E\subset\mathscr{F}\Rightarrow E(\mathscr{X})\subset\mathscr{F}(\mathscr{X})$. Let us consider $\mathscr{W}(\mathscr{X})$ be a compact sub-collection of all roughly value in $\mathscr{X}$. A fuzzy set E in $\mathscr{X}$ is said to be an roughly value iff E_λ is compact and convex in $\mathscr{X}\,\forall\lambda\in(0,1]$ and $\sup_{\Theta\in\mathscr{X}}E(\Theta)=1$.

Definition 2.4. *Let $(\mathscr{X},\mathscr{M},\odot)$ be a fuzzy metric space. Then $\forall E,\mathscr{F}\in\mathscr{W}(\mathscr{X})$, $\mathscr{H}_{\mathscr{M}}(E,\mathscr{F},t):\mathscr{W}(\mathscr{X})\times\mathscr{W}(\mathscr{X})\times(0,\infty)$ is a HFM function defined as*

$$\mathscr{H}_{\mathscr{M}}(E,\mathscr{F},t)=min\left\{\inf_{\Theta\in E}\left(\sup_{\Xi\in\mathscr{F}}\mathscr{M}(\Theta,\Xi,t)\right),\inf_{\Xi\in\mathscr{F}}\left(\sup_{\Theta\in E}\mathscr{M}(\Xi,\Theta,t)\right)\right\}\tag{1}$$

Lemma 2.5. *[35] Let $(\mathscr{X},\mathscr{M},\odot)$ be a fuzzy metric space. Then the 3- tuple $(\mathscr{W}(\mathscr{X}),\mathscr{H}_{\mathscr{M}},\odot)$ is a Hausdorff fuzzy metric space.*

Throughout this paper, let $(\mathscr{W}(\mathscr{X}),\mathscr{H}_{\mathscr{M}},\odot)$ be a compact HFMS and $\mathscr{W}(\mathscr{X})$ be a compact sub-collection of all roughly values. Then for all $\forall E,\mathscr{F}\in\mathscr{W}(\mathscr{X}),t>0$ and $\lambda\in(0,1]$, we have

$$\mathscr{H}_{\mathscr{M}(p_\lambda)}(E,\mathscr{F},t)=\frac{t}{t+p_\lambda(E,\mathscr{F})}=\sup_{a\in E_\lambda,b\in\mathscr{F}_\lambda}\{\mathscr{M}_d(a,b,t)\}$$
$$\mathscr{H}_{\mathscr{M}(\delta_\lambda)}(E,\mathscr{F},t)=\frac{t}{t+\delta_\lambda(E,\mathscr{F})}=\inf_{a\in E_\lambda,b\in\mathscr{F}_\lambda}\{\mathscr{M}_d(a,b,t)\}$$

and

$$\mathscr{H}_{\mathscr{M}(D_\lambda)}(E,\mathscr{F},t)=\frac{t}{t+D_\lambda(E,\mathscr{F})}=\frac{t}{t+\mathscr{H}(E_\lambda,\mathscr{F}_\lambda)}$$
$$=\min_{a\in E_\lambda,b\in\mathscr{F}_\lambda}\left(\inf_{b\in\mathscr{F}_\lambda}\{\mathscr{M}_{d_\alpha}(a,\mathscr{F}_\lambda,t)\},\inf_{a\in E_\lambda}\{\mathscr{M}_{d_\lambda}(E_\lambda,b,t)\}\right)$$

$$\mathscr{H}_{\mathscr{M}(p)}(E,\mathscr{F},t)=\inf_\lambda\left\{\mathscr{H}_{\mathscr{M}(_{p_\lambda})}(E,\mathscr{F},t)\right\}$$
$$\mathscr{H}_{\mathscr{M}(\delta)}(E,\mathscr{F},t)=\inf_\lambda\left\{\mathscr{H}_{\mathscr{M}(_{\delta_\lambda})}(E,\mathscr{F},t)\right\}$$
$$\mathscr{H}_{\mathscr{M}(D)}(E,\mathscr{F},t)=\inf_\lambda\left\{\mathscr{H}_{\mathscr{M}(_{D_\lambda})}(E,\mathscr{F},t)\right\}$$

It is noted that p_λ is non–increasing function of λ and thus

$$\mathscr{H}_{\mathscr{M}(_{p_\lambda})}(E,\mathscr{F},t)=\mathscr{H}_{\mathscr{M}(_{p_1})}(E,\mathscr{F},t).$$

In particular if $E=\{\Theta\}$ then $\mathscr{H}_{\mathscr{M}(_p)}(\{\Theta\},\mathscr{F},t)=\mathscr{H}_{\mathscr{M}(_{p_1})}(\Theta,\mathscr{F},t)=\mathscr{H}_{\mathscr{M}(_{d_1})}(\Theta,\mathscr{F},t)$. If E is a singleton i.e. $E=\{a\}$ we write

$$\mathscr{H}_{\mathscr{M}(_p)}(\{a\},\mathscr{F},t)=\mathscr{H}_{\mathscr{M}(_{p_1})}(a,\mathscr{F},t)=\mathscr{H}_{\mathscr{M}(_{d_1})}(a,\mathscr{F},t).$$

If $\mathscr{F}$ is a singleton i.e. $\mathscr{F}=\{b\}$ we write

$$\mathscr{H}_{\mathscr{M}(_p)}(E,\{b\},t)=\mathscr{H}_{\mathscr{M}(_{p_1})}(E,b,t)=\mathscr{H}_{\mathscr{M}(_{d_1})}(E,b,t).$$

It follows immediately from the definition that $\mathscr{H}_{\mathscr{M}(_{\delta_\lambda})}(E,\mathscr{F},t)=\mathscr{H}_{\mathscr{M}(_\delta)}(E,\mathscr{F},t)$.

$$\mathscr{H}_{\mathscr{M}(_\delta)}(E,\mathscr{F},t)\geq\mathscr{H}_{\mathscr{M}(_\delta)}(E,G,t)+\mathscr{H}_{\mathscr{M}(_\delta)}(G,\mathscr{F},t).$$

$$\mathscr{H}_{\mathscr{M}(_\delta)}(E,\mathscr{F},t)=1\Leftrightarrow E=\mathscr{F}=\{a\}.$$

$$\mathscr{H}_{\mathscr{M}(_\delta)}(E,\mathscr{F},t)=1\Rightarrow dim(E).$$

Note that $\mathscr{M}(p)$ is a non-increasing function for p and $\mathscr{H}_{\mathscr{M}(_{p})}$ is a Hausdorff fuzzy metric induced by fuzzy metric $\mathscr{M}$ on $\mathscr{W}(\mathscr{X})$.

Definition 2.5. *Let $E,\mathscr{F} \in \mathscr{W}(\mathscr{X})$ and $\mathscr{F}$ includes E, then $E \subseteq \mathscr{F}$ iff $E(\Theta) \leq \mathscr{F}(\Theta), \forall \Theta \in \mathscr{X}$.*

Lemma 2.6. *[35] Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a Hausdorff fuzzy metric space and $E \in \mathscr{W}(\mathscr{X})$. a set $\{\Theta\}$ denotes the membership grade of $\Theta \in \mathscr{X}$, then $\{\Theta\} \subset E$ iff*

$$\mathscr{H}_{\mathscr{M}(_{p_\lambda})}(\Theta, E, t) = 1, \forall \lambda \in (0,1].$$

Lemma 2.7. *Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a Hausdorff fuzzy metric space. Then for any $\Theta, \Xi \in \mathscr{X}$, $t > 0$, and $F \in \mathscr{W}(\mathscr{X})$.*

$$\mathscr{H}_{\mathscr{M}(_{p_\lambda})}(\Theta, \mathscr{F}, t) \geq \mathscr{H}_{\mathscr{M}(_{p_\lambda})}(\Theta, \Xi, t) + \mathscr{H}_{\mathscr{M}(_{p_\lambda})}(\Xi, \mathscr{F}, t), \forall \lambda \in [0,1]$$

Lemma 2.8. *[33] Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a HFMS and $E \in \mathscr{W}(\mathscr{X})$. If $\Theta \in \mathscr{X}$ $\{\Theta_0\} \subset E$ then $\mathscr{H}_{\mathscr{M}(_{p_\lambda})}(\Theta_0, \mathscr{F}, t) \geq \mathscr{H}_{\mathscr{M}(_{D_\lambda})}(E, \mathscr{F}, t)$ $\forall \mathscr{F} \in \mathscr{W}(\mathscr{X})$,$t > 0$ and $\lambda \in [0,1]$.*

Proposition 2.1. *Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a Hausdorff fuzzy metric space and $\mathscr{F} : \mathscr{X} \to \mathscr{W}(\mathscr{X})$ be a fuzzy mapping and $\Theta_0 \in \mathscr{X}$. Then $\exists$ $\Theta_1 \in \mathscr{X}$ s.t. $\{\Theta_1\} \subset \mathscr{F}(\Theta_0)$.*

Remark 2.2. *Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a HFMS and let $\mathscr{J} : \mathscr{X} \to \mathscr{X}$ be a single valued mapping and $\mathscr{F} : \mathscr{X} \to \mathscr{W}(\mathscr{X})$ a multi-valued mapping s.t. $\cup\{\mathscr{F}\mathscr{X}\}_\lambda \subseteq \mathscr{J}(\mathscr{X})$, $\forall \lambda \in [0,1]$. Suppose $\mathscr{J}(\mathscr{X})$ is complete. By 2.1, $\exists \Theta_1 \in \mathscr{X}$ for $\Theta_0 \in \mathscr{X}$ s.t. $\{\Theta_1\} \subseteq \mathscr{F}(\Theta_0)$.*

Proposition 2.2. *Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a HFMS. If $E,\mathscr{F} \in \mathscr{W}(\mathscr{X})$ and $e \in E$ then $\exists f \in \mathscr{F}$, s.t.*

$$\mathscr{M}_d(e, f, t) \geq \mathscr{H}_{\mathscr{M}(_D)}(E, \mathscr{F}, t)$$

Lemma 2.9. *[33] Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a Hausdorff fuzzy metric space. If $E \in \mathscr{W}(\mathscr{X})$ then $\Theta \in E$ if and only if*

$$\mathscr{H}_{\mathscr{M}(\delta)}(\Theta, E, t) = 1, \ for\ t > 0.$$

3. Implicit Relations

Let $\psi \in \Psi(set of all continuous functions)$ and $\psi : [0,\infty)^5 \to [0,\infty)$ s.t.

(i) ψ is non-increasing in 2nd, 3rd, 4th and 5th variables.

(ii) If $\mu, \nu \in [0,\infty)$ are s.t. $\mu \geq \psi(\nu,\nu,\mu,\mu+\nu,1)$ or $\mu \geq \psi(\nu,\mu,\nu,1,\mu+\nu)$, then $\mu \geq h\nu$, where $0 < h < 1$, is a given constant.

(iii) If $\mu \in [0,\infty)$ is such that $\mu \geq \psi(\mu,1,1,\mu,\mu)$ or $\mu \geq \psi(1,\mu,\mu,1,1)$, then $\mu = 1$.

4. Main Results

Let $\mathscr{T} : \mathscr{X} \to \mathscr{W}(\mathscr{X})$ be a multi-valued mapping of Hausdorff fuzzy metric space $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ If $\Theta \in \mathscr{T}\Theta$ then an element $\Theta \in \mathscr{X}$ is called a fixed point of $\mathscr{T}$.

In this section, we introduced an $\alpha^* - \eta_* - \psi-$hybrid contraction for single valued mapping's and multi-valued mapping's and proved the two results of fixed point theorem in HFMS $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$

Lemma 4.1. *[32] Let $(\mathscr{X}, \mathscr{M}, \odot)$ be a Hausdorff fuzzy metric space such that $\Theta, \Xi \in \mathscr{X}, t > 0$ and $s > 1$*

(2)
$$\lim_{n\to\infty} \bigodot_{i=n}^{\infty} \mathscr{M}(\Xi_1, \Xi_2, ts^i) = 1.$$

Suppose $\{\Xi_n\}$ is a sequence in $\mathscr{X}$ s.t. $\forall n \in \mathbb{N}, \sigma \in (0,1)$, $\mathscr{M}(\Xi_n, \Xi_{n+1}, \sigma t) \geq \mathscr{M}(\Xi_{n-1}, \Xi_n, t)$, then $\{\Xi_n\}$ is CS.

Proof. : For all $\sigma \in (0,1)$, $k \in \mathbb{N} \cup \{0\}$, and $t > 0$, we have

$$\mathscr{M}(\Xi_k, \Xi_{k+1}, t) \geq \mathscr{M}(\Xi_{k-1}, \Xi_k, \frac{t}{\sigma}) \geq \mathscr{M}(\Xi_{k-2}, \Xi_{k-1}, \frac{t}{\sigma^2}) \geq ..... \geq \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{\sigma^{n-1}})$$

For each $n \in \mathbb{N}$, we get, $\mathscr{M}(\Xi_k, \Xi_{k+1}, t) \geq \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{\sigma^{n-1}})$, $\forall k \in \mathbb{N}$, *and* $t > 0$. Letting $s > 1$ and $j = 1, 2, ...$ s.t.

$$s\sigma < 1\ , \sum_{i=j}^{\infty} \frac{1}{s^i} = \frac{\frac{1}{s^i}}{1 - \frac{1}{s}} < 1.$$

Now $\forall n > m, n, m \in \mathbb{N}$ and $t > 0$, we get

$$\begin{aligned}\mathscr{M}(\Xi_n, \Xi_m, t) \geq & \mathscr{M}\left(\Xi_n, \Xi_m, \left(\frac{t}{\sigma^j} + \frac{t}{\sigma^{j+1}} + ... + \frac{t}{\sigma^{j+m}}\right)\right) \\ \geq & \mathscr{M}(y_n, y_{n+1}, \frac{t}{s^j}) \odot \mathscr{M}(\Xi_{n+1}, \Xi_{n+2}, \frac{t}{s^{j+1}}) \odot ..... \odot \mathscr{M}(\Xi_{m-1}, \Xi_m, \frac{t}{s^{j+m}}) \\ \geq & \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{(s\sigma)^{n-1}}) \odot \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{(s\sigma)^{n}}) \odot ..... \odot \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{(s\sigma)^{m-2}}) \\ \geq & \bigodot_{i=n}^{\infty} \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{(s\sigma)^{i-1}}).\end{aligned}$$

Letting $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} \mathscr{M}(\Xi_n, \Xi_m, t) \geq \lim_{m,n\to\infty} \bigodot_{i=n}^{\infty} \mathscr{M}(\Xi_0, \Xi_1, \frac{t}{(s\sigma)^{i-1}}) = 1.$$

Hence $\{\Xi_n\} \in \mathscr{X}$ is a Cauchy sequence. □

Definition 4.1. *Let X be a non empty set in a Hausdorf fuzzy metric space* $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ *and* $\mathscr{F}_1, \mathscr{F}_2 : \mathscr{X} \to \mathscr{W}(\mathscr{X})$ *be two MVFM's. Let* $\alpha, \eta : \mathscr{X} \times \mathscr{X} \times (0, \infty) \to [0, \infty)$ *be two functions. We say that* $\mathscr{F}_1 and \mathscr{F}_2$ *are said to be* $\alpha^* - \eta_* -$*admissible mapping's if* $\forall \Theta, \Xi \in \mathscr{X}, t > 0$, *we have*

$$\alpha(\Theta, \Xi, t) \leq \eta(\Theta, \Xi, t) \Rightarrow \alpha^*(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, t) \leq \eta_*(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, t),$$

where

$\alpha^*(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, t) = \sup_{\Theta \in \mathscr{F}_1\Theta, \Xi \in \mathscr{F}_2\Xi} \alpha(\Theta, \Xi, t)$ *and* $\eta_*(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, t) = \sup_{\Theta \in \mathscr{F}_1\Theta, \Xi \in \mathscr{F}_2\Xi} \eta(\Theta, \Xi, t)$.

Definition 4.2. *Let* $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ *be a Hausdorff fuzzy metric space,* $\psi \in \Psi$ *and* $\mathscr{I}, \mathscr{J} : \mathscr{X} \to \mathscr{X}$ *be self mappings. The multi-valued mapping's* $\mathscr{F}_1, \mathscr{F}_2 : \mathscr{X} \to \mathscr{W}(\mathscr{X})$ *are called* $\alpha^* - \eta_* - \psi-$*hybrid contraction with single valued mapping's* $\mathscr{I}, \mathscr{J}$ *if the following implication takes place:*

(3) $\alpha(\Theta, \Xi, t) \leq \eta(\Theta, \Xi, t) \Rightarrow \mathscr{H}_{\mathscr{M}}(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, qt) \geq \mathscr{H}_{\mathscr{M}_{(D)}}(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, t) \geq \psi(m(\Theta, \Xi, t))$

$\forall \Theta, \Xi \in \mathscr{X}, t > 0$, *where*

$$\begin{aligned}m(\Theta, \Xi, t) \geq \Big(& \mathscr{H}_{\mathscr{M}d)}(\mathscr{I}\Theta, \mathscr{J}\Xi, t), \mathscr{H}_{\mathscr{M}_{(p)}}(\mathscr{I}\Theta, \mathscr{F}_1\Theta, t), \mathscr{H}_{\mathscr{M}_{(p)}}(\mathscr{J}\Xi, \mathscr{F}_2\Xi, t), \mathscr{H}_{\mathscr{M}_{(p)}} \\ & (\mathscr{I}\Theta, \mathscr{F}_2\Xi, t), \mathscr{H}_{\mathscr{M}_{(p)}}(\mathscr{J}\Xi, \mathscr{F}_1\Theta, t)\Big).\end{aligned}$$

We prove the following theorem:

Theorem 4.2. *Let $(\mathscr{W}(\mathscr{X}),\mathscr{H}_{\mathscr{M}},\odot)$ be a Hausdorff fuzzy metric space with $a\odot b=ab$ and $\psi\in\Psi$. Let $\mathscr{I},\mathscr{J}:\mathscr{X}\to\mathscr{X}$ be self mappings and $\mathscr{F}_1,\mathscr{F}_2:\mathscr{X}\to\mathscr{W}(\mathscr{X})$ are multi-valued mapping's. Suppose $\mathscr{F}_1$and$\mathscr{F}_2$ are $\alpha^*-\eta_*-\psi-$hybrid contraction with single-valued mapping's $\mathscr{I},\mathscr{J}$ satisfying conditions (i)-(iii) of implicit relations and the following assertions:*
(i) $\mathscr{F}_1,\mathscr{F}_2$ are $\alpha^*-\eta_*-$admissible mapping's,
(ii) (a) $\cup\{\mathscr{F}_1(\mathscr{X})\}_\lambda\subset\mathscr{J}(\mathscr{X})$ and (b) $\cup\{\mathscr{F}_2(\mathscr{X})\}_\lambda\subset\mathscr{I}(\mathscr{X})$ for each $\lambda\in[0,1]$
(iii) there is a $\psi\in\Psi$ s.t $\forall\Theta,\Xi\in\mathscr{X},t>0$, then from (3)

$$\mathscr{H}_{\mathscr{M}(D)}(\mathscr{F}_1\Theta,\mathscr{F}_2\Xi,t)\geq\Big(\mathscr{H}_{\mathscr{M}d)}(\mathscr{I}\Theta,\mathscr{J}\Xi,t),\mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta,\mathscr{F}_1\Theta,t),$$
$$\mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}\Xi,\mathscr{F}_2\Xi,t),\mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta,\mathscr{F}_2\Xi,t),\mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}\Xi,\mathscr{F}_1\Theta,t)\Big)$$

(iv) for any sequence $\{\Xi_n\}$in$\mathscr{X}$, converging to $\Sigma\in\mathscr{X}$ and $\alpha(\Xi_n,\Xi_{n+1},t)\leq\eta(\Xi_n,\Xi_{n+1},t)$, we have

$$\alpha(\Xi_n,\Sigma,t)\leq\eta(\Xi_n,\Sigma,t)$$

$\forall n\in\mathbb{N}\cup\{0\},\Sigma\in\mathscr{X}$ andt >0, s.t. $\mathscr{I}\Sigma\subseteq\mathscr{F}_1\Sigma$ and $\mathscr{J}\Sigma\subseteq\mathscr{F}_2\Sigma$ satisfy the condition (ii),

$$i.e.\lim_{n\to\infty}\bigodot_{i=n}^{\infty}\mathscr{M}(\Xi_1,\Xi_2,ts^i)=1.$$

If $\mathscr{I}(\mathscr{X})$or$\mathscr{J}(\mathscr{X})$ is complete, then Σ is a common fixed point of $\mathscr{I},\mathscr{J},\mathscr{F}_1and\mathscr{F}_2$.

Proof. : Let $\Theta_0\in\mathscr{X}$ and suppose that $\mathscr{J}(\mathscr{X})$ is complete. Taking $\Xi_0=\mathscr{I}\Theta_0$. Then by remark 2.1 and (ii)(a) $\exists\Theta_1,\Xi_1\in\mathscr{X}$ s.t. $\{\Xi_1\}=\{\mathscr{J}\Theta_1\}\subseteq\mathscr{F}_1\Theta_0$. Thus from the assumptions, for all $t>0$

$$\alpha(\Xi_0,\Xi_1,t)\leq\eta(\Xi_0,\Xi_1,t).$$

Again by proposition 2.1, for the point Ξ_1, $\exists\ \Xi_2=\{\mathscr{F}_2\Theta_1\}_1$. But, by (ii)(b), $\exists\Theta_2\in\mathscr{X}$ s.t. $\{\Xi_2\}=\{\mathscr{I}\Theta_2\}\subseteq\mathscr{F}_2\Theta_1$. Then $\alpha(\Xi_1,\Xi_2,t)\leq\eta(\Xi_1,\Xi_2,t)$ for each $t>0$. By using (iv) and proposition 2.2, we obtain

$$\mathscr{M}(\Xi_2,\Xi_3,t)\geq\mathscr{M}(\Xi_1,\Xi_2,t)\geq\mathscr{H}_{\mathscr{M}(D_1)}(\mathscr{F}_1\Theta_1,\mathscr{F}_2\Theta_2,t)\geq\mathscr{H}_{\mathscr{M}(D)}(\mathscr{F}_1\Theta_1,\mathscr{F}_2\Theta_2,t)\geq$$
$$\psi(m(\Theta_1,\Theta_2,t))$$

$$\geq \psi \begin{pmatrix} \mathscr{H}_{\mathscr{M}d)}(\mathscr{I}\Theta_1, \mathscr{J}\Theta_2, t), \mathscr{H}_{\mathscr{M}(_p)}(\mathscr{I}\Theta_1, \mathscr{F}_1\Theta_1, t), \mathscr{H}_{\mathscr{M}(_p)}(\mathscr{J}\Theta_2, \mathscr{F}_2\Theta_2, t), \\ \mathscr{H}_{\mathscr{M}(_p)}(\mathscr{I}\Theta_1, \mathscr{F}_2\Theta_2, t), \mathscr{H}_{\mathscr{M}(_p)}(\mathscr{J}\Theta_2, \mathscr{F}_1\Theta_1, t) \end{pmatrix}$$

$$\geq \psi \left(\mathscr{H}_{\mathscr{M}d)}(\Xi_1, \{\Xi_2\}, t), \mathscr{H}_{\mathscr{M}d)}(\Xi_1, \{\Xi_2\}, t), \mathscr{H}_{\mathscr{M}d)}(\{\Xi_2\}, \{\Xi_3\}, t), \mathscr{H}_{\mathscr{M}d)}(\Xi_1, \{\Xi_3\}, t), \right.$$
$$\left. \mathscr{H}_{\mathscr{M}d)}(\{\Xi_2\}, \{\Xi_2\}, t) \right)$$
$$\geq \psi (\mathscr{M}(\Xi_1, \Xi_2, t), \mathscr{M}(\Xi_1, \Xi_2, t), \mathscr{M}(\Xi_2, \Xi_3, t), \mathscr{M}(\Xi_1, \Xi_2, t) + \mathscr{M}(\Xi_2, \Xi_3, t), 1)$$

which, by (ii) gives $\mathscr{M}(\Xi_2, \Xi_3, t) \geq h.\mathscr{M}(\Xi_1, \Xi_2, t)$. By ongoing action and conditions of proposition 2.2 and (a) & (b) of (ii), a sequence $\{\Xi_k\} \in \mathscr{X}$ developed for each $k = 0, 1, 2, ...$ Thus

$$\{\Xi_{2k+1}\} = \{\mathscr{J}\Theta_{2k+1}\} \subseteq \mathscr{F}_1(\Theta_{2k}) \ , \ \{\Xi_{2k+2}\} = \{\mathscr{I}\Theta_{2k+2}\} \subseteq \mathscr{F}_2(\Theta_{2k+1})$$

we get

$$\mathscr{M}(\Xi_{k+1}, \Xi_{k+2}, t) \geq h.\mathscr{M}(\Xi_k, \Xi_{k+1}, t)$$

and

$$\alpha(\Xi_{k+1}, \Xi_{k+2}, t) \leq \eta(\Xi_{k+1}, \Xi_{k+2}, t)$$

From lemma 2.1, $\lim_{n\to\infty} \bigodot_{i=n}^{\infty} \mathscr{M}(\Xi_1, \Xi_2, ts^i) = 1$, implies that $\{\Xi_k\} \in \mathscr{X}$ is a CS.
Now since $\mathscr{J}(\mathscr{X})$ is complete. Then $\mathscr{J}\Theta_{2k+1} \to \Sigma = \mathscr{J}v$ for some $v \in \mathscr{X}$.

$$\mathscr{M}(\mathscr{I}\Theta_{2k}, \mathscr{J}v, t) \geq \mathscr{M}(\mathscr{I}\Theta_{2k}, \mathscr{J}\Theta_{2k+1}, t) + \mathscr{M}(\mathscr{J}\Theta_{2k+1}, \mathscr{J}v, t) \to 1$$

, as $k \to \infty$.

Hence $\mathscr{I}\Theta_{2k} \to \mathscr{J}v$, as $k \to \infty$. By condition (iv), lemma 2.7 and lemma 2.8, we have

$$\alpha(\mathscr{I}\Theta_{2k}, \mathscr{J}v, t) \leq \eta(\mathscr{I}\Theta_{2k}, \mathscr{J}v, t), \forall k \in \mathbb{N}, t > 0.$$

also

$$\mathscr{H}_{\mathscr{M}(p)}(\Sigma, \mathscr{F}_2 v, t) \geq \mathscr{H}_{\mathscr{M}(d)}(\Sigma, \mathscr{J}\Theta_{2k+1}, t) + \mathscr{H}_{\mathscr{M}(D)}(\mathscr{F}_1\Theta_{2k}, \mathscr{F}_2 v, t)$$

$\geq \mathscr{H}_{\mathscr{M}(d)}(\Sigma, \mathscr{J}\Theta_{2k+1},t) + \psi\Big(\mathscr{H}_{\mathscr{M}(d)}(\mathscr{I}\Theta_{2k}, \mathscr{J}v,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta_{2k}, \mathscr{F}_1\Theta_{2k},t),$

$\mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}v, \mathscr{F}_2v,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta_{2k}, \mathscr{F}_2v,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}v, \mathscr{F}_1\Theta_{2k},t)\Big)$

$\geq \mathscr{H}_{\mathscr{M}(d)}(\Sigma, \mathscr{J}\Theta_{2k+1},t) + \psi\Big(\mathscr{H}_{\mathscr{M}(d)}(\mathscr{I}\Theta_{2k}, \Sigma,t), \mathscr{H}_{\mathscr{M}(p)}(\Xi_{2k}, \Xi_{2k+1},t),$

$\mathscr{H}_{\mathscr{M}(p)}(\Sigma, \mathscr{F}_2v,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta_{2k}, \mathscr{F}_2v,t), \mathscr{H}_{\mathscr{M}(p)}(\Sigma, \Xi_{2k+1},t)\Big)$

letting $k \to \infty$ it implies,

$$\mathscr{H}_{\mathscr{M}(p)}(\Sigma, \mathscr{F}_2v,t) \geq \psi\big(1,1,\mathscr{H}_{\mathscr{M}(p)}(\Sigma, \mathscr{F}_2v,t), \mathscr{H}_{\mathscr{M}(p)}(\Sigma, \mathscr{F}_2v,t),1\big)$$

which, by (iii), yields that $\mathscr{H}_{\mathscr{M}(p)}(\Sigma, \mathscr{F}_2v,t) = 1$. So by lemma 2.6, we get $\Sigma \subseteq \mathscr{F}v$ i.e. Jv$\in$ $\{\mathscr{F}_2v\}_1$. Since by (ii)(b), $\{\mathscr{F}_2(\mathscr{X})\}_1 \subseteq I(\mathscr{X})$ and $\mathscr{J}v \in \{\mathscr{F}_2v\}_1$, therefore $\exists u \in \mathscr{X}$, s.t.$\mathscr{I}u = \mathscr{J}vu = \Sigma \in \{\mathscr{F}_2v\}_1$.

To show that $\mathscr{I}u \in \{\mathscr{F}_1u\}_1$. By lemma 2.8 and condition (iv), we have

$$\alpha(\mathscr{I}u, \mathscr{F}_1u,t) \leq \eta(\mathscr{I}u, \mathscr{F}_1u,t), \forall u \in \mathscr{X}, t > 0.$$

Also

$$\begin{aligned}\mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_1u,t) =& \mathscr{H}_{\mathscr{M}(p)}(\mathscr{F}_1u, \mathscr{I}u,t)\\ \geq& \mathscr{H}_{\mathscr{M}(D_1)}(\mathscr{F}_1u, \mathscr{F}_2v,t) \geq \mathscr{H}_{\mathscr{M}(D)}(\mathscr{F}_1u, \mathscr{F}_2v,t).\\ \geq& \psi\big(\mathscr{H}_{\mathscr{M}(d)}(\mathscr{I}u, \mathscr{J}v,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_1u,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}v, \mathscr{F}_2v,t),\\ & \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_2v,t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}v, \mathscr{F}_1u,t)\big)\end{aligned}$$

yielding thereby

$$\mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_1u,t) \geq \psi\big(1, \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_1u,t),1,1, \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_1u,t)\big)$$

which, by (iii), gives $\mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}u, \mathscr{F}_1u,t) = 1$. Thus, by lemma 2.6, $\mathscr{I}u \subseteq \mathscr{F}_1u i.e. \mathscr{I}u \in \{\mathscr{F}_1u\}_1$.

Thus by the above assumption of $\alpha^* - \eta_*-$admissibility and $\alpha^* - \eta_* - \psi-$hybrid contraction in the pairs $\{\mathscr{F}_1, \mathscr{I}\}$ and $\{\mathscr{F}_2, \mathscr{J}\}$, we have

$$\alpha(\mathscr{I}u, \mathscr{F}_1 u, t) \leq \eta(\mathscr{I}u, \mathscr{F}_1 u, t) \Rightarrow$$
$$\mathscr{H}_{\mathscr{M}}(\mathscr{I}\{\mathscr{F}_1 u\}_1, \{\mathscr{F}_1 \mathscr{I}u\}_1, t) \geq \psi(m(\mathscr{I}u, \{\mathscr{F}_1 u\}_1, t)) = 1$$

and

$$\alpha(\mathscr{J}v, \mathscr{F}_2 v, t) \leq \eta(\mathscr{J}v, \mathscr{F}_2 v, t) \Rightarrow$$
$$\mathscr{H}_{\mathscr{M}}\left(\mathscr{J}\{F_2 v\}_1, \{\mathscr{F}_2 \mathscr{J}v\}_1, t\right) \geq \psi(m(\mathscr{J}v, \{\mathscr{F}_2 v\}_1, t)) = 1$$

which gives $\mathscr{I}\{\mathscr{F}_1 u\}_1 = \{\mathscr{F}_1 \mathscr{I}u\}_1 = \{\mathscr{F}_1 \Sigma\}_1$ and $\mathscr{J}\{\mathscr{F}_2 v\}_1 = \{\mathscr{F}_2 \mathscr{J}v\}_1 = \{\mathscr{F}_2 \Sigma\}_1$ respectively.

But $\mathscr{I}u \in \{\mathscr{F}_1 u\}_1$ and $\mathscr{J}v = \{\mathscr{F}_2 v\}_1$ implies $\mathscr{I}\Sigma = \mathscr{I}\mathscr{I}u \in \mathscr{I}\{\mathscr{F}_1 u\}_1 = \{\mathscr{F}_1 \Sigma\}_1$ and $\mathscr{J}\Sigma = \mathscr{J}\mathscr{J}v \in \mathscr{J}\{\mathscr{F}_2 v\}_1 = \{\mathscr{F}_2 \Sigma\}_1$.

Hence $\mathscr{I}\Sigma \subseteq \mathscr{F}_1 \Sigma$ and $\mathscr{J}\Sigma \subseteq \mathscr{F}_2 \Sigma$. This completes the theorem. □

Corollary 4.1. *Let $(\mathscr{W}(\mathscr{X}), \mathscr{H}_{\mathscr{M}}, \odot)$ be a HFMS with $a \odot b = ab$ and $\psi \in \Psi$. Let $\mathscr{I}, \mathscr{J} : \mathscr{X} \to \mathscr{X}$ be self mappings and $\mathscr{F}_1, \mathscr{F}_2 : \mathscr{X} \to W(\mathscr{X})$ are MVFM's. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are $\alpha - \psi-$hybrid contraction with SVM's $\mathscr{I}, \mathscr{J}$ satisfying conditions (i)-(iii), (3) and the following assertions:,*

(v) $\mathscr{F}_1, \mathscr{F}_2$ are $\alpha^* - \eta_*-$admissibel mapping's with $\eta = 1$,

(vi) (a) $\cup\{\mathscr{F}_1(\mathscr{X})\}_\lambda \subset \mathscr{J}(\mathscr{X})$ (b) $\cup\{\mathscr{F}_2(\mathscr{X})\}_\lambda \subset \mathscr{I}(\mathscr{X})$ for each $\lambda \in [0,1]$

(vii) there is a $\psi \in \Psi$ s.t. $\forall\ \Theta, \Xi \in \mathscr{X}$, $t > 0$,

$$\mathscr{H}_{\mathscr{M}(D)}(\mathscr{F}_1\Theta, \mathscr{F}_2\Xi, t) \geq \Big(\mathscr{H}_{\mathscr{M}d)}(\mathscr{I}\Theta, \mathscr{J}\Xi, t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta, \mathscr{F}_1\Theta, t),$$
$$\mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}\Xi, \mathscr{F}_2\Xi, t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{I}\Theta, \mathscr{F}_2\Xi, t), \mathscr{H}_{\mathscr{M}(p)}(\mathscr{J}\Xi, \mathscr{F}_1\Theta, t)\Big)$$

(viii) for any sequence $\{\Xi_n\} in \mathscr{X}$, converging to $\Theta \in \mathscr{X}$ and $\alpha(\Xi_n, \Xi_{n+1}, t) \leq 1$, $\forall n \in \mathbb{N} \cup \{0\}, \Sigma \in \mathscr{X}$ and $t > 0$, s.t. $\mathscr{I}\Sigma \subseteq \mathscr{F}_1\Sigma$ and $\mathscr{J}\Sigma \subseteq \mathscr{F}_2\Sigma$ and satisfy the condition (2), i.e. $\lim\limits_{n\to\infty} \bigodot\limits_{i=n}^{\infty} \mathscr{M}(\Xi_1, \Xi_2, ts^i) = 1$. If $\mathscr{I}(\mathscr{X})$or $\mathscr{J}(\mathscr{X})$ is complete, then Σ is a common fixed point of $\mathscr{I}, \mathscr{J}, \mathscr{F}_1and\mathscr{F}_2$.

If $\mathscr{I} = \mathscr{J} : \mathscr{X} \to \mathscr{X}$ be self mapping and $\mathscr{F}_1 = \mathscr{F}_2 : X \to \mathscr{W}(\mathscr{X})$ is multi-valued fuzzy mapping's. Then we have following corollary.

Corollary 4.2. *Let $(\mathscr{W}(\mathscr{X}),\mathscr{H}_{\mathscr{M}},\odot)$ be a HFMS with $a\odot b=ab$ and $\psi\in\Psi$. Let $\mathscr{I}:\mathscr{X}\to\mathscr{X}$ be self mapping and $\mathscr{F}_1:\mathscr{X}\to W(\mathscr{X})$ is multi-valued fuzzy mapping. Suppose $\mathscr{F}_1$ is $\alpha-\psi-$hybrid contraction with single valued mapping $\mathscr{I}$, satisfying conditions (i)-(iii), (2) and the following assertions:*

(ix) $\mathscr{F}_1$ is $\alpha-\psi-$admissible mapping,

(x) $\cup\{\mathscr{F}_1(\mathscr{X})\}_\lambda\subset\mathscr{I}(\mathscr{X})$ for each $\lambda\in[0,1]$

(xi) there is a $\psi\in\Psi$ s.t. $\forall\ \Theta,\Xi\in\mathscr{X},\ t>0$,

$$\mathscr{H}_{\mathscr{M}(_D)}(\mathscr{F}_1\Theta,\mathscr{F}_1\Xi,t)\geq\Big(\mathscr{H}_{\mathscr{M}d)}(\mathscr{I}\Theta,\mathscr{I}\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\mathscr{I}\Theta,\mathscr{F}_1\Theta,t),$$
$$\mathscr{H}_{\mathscr{M}(_p)}(\mathscr{I}\Xi,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\mathscr{I}\Theta,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\mathscr{I}\Xi,\mathscr{F}_1\Theta,t)\Big)$$

(xii) for any sequence $\{\Xi_n\}\in\mathscr{X}$, converging to $\Theta\in\mathscr{X}$ and $\alpha(\Xi_n,\Xi_{n+1},t)\leq 1$, for all $n\in\mathbb{N}\cup\{0\},\Sigma\in\mathscr{X}$ and $t>0$, we have $\alpha(\Xi_n,\Sigma,t)\leq 1,\forall n\in\mathbb{N}\cup\{0\},\Sigma\in\mathscr{X}$ andt >0, s.t. $\mathscr{I}\Sigma\subseteq\mathscr{F}_1\Sigma$ and satisfy the condition (2) i.e. $\lim_{n\to\infty}\bigodot_{i=n}^{\infty}\mathscr{M}(\Xi_1,\Xi_2,ts^i)=1$. If $\mathscr{I}(\mathscr{X})$ is complete, then Σ is a common fixed point of $\mathscr{I}$ and$\mathscr{F}_1$.

If $\mathscr{I}=\mathscr{J}=1$ and $\mathscr{F}_1=\mathscr{F}_2:X\to\mathscr{W}(\mathscr{X})$ is MVFM. Then we have following corollary.

Corollary 4.3. *Let $(\mathscr{W}(\mathscr{X}),\mathscr{H}_{\mathscr{M}},\odot)$ be a Hausdorff fuzzy metric space with $a\odot b=ab$ and $\psi\in\Psi$. Let $\mathscr{F}_1:\mathscr{X}\to W(\mathscr{X})$ is MVFM. Suppose $\mathscr{F}_1$ is $\alpha-\psi-$contractive and satisfying conditions (i)-(iii), (2) and the following assertions:*

(xiii) F_1 is $\alpha-\psi-$admissible mapping,

(xiv) there is a $\psi\in\Psi$ s.t. $\forall\ \Theta,\Xi\in\mathscr{X},\ t>0$,

$$\mathscr{H}_{\mathscr{M}(_D)}(\mathscr{F}_1\Theta,\mathscr{F}_1\Xi,t)\geq\psi\Big(\mathscr{H}_{\mathscr{M}d)}(\Theta,\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Theta,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Xi,t),$$
$$\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Theta,t)\Big)$$

(xv) for any sequence $\{\Xi_n\}\in\mathscr{X}$, converging to $\Sigma\in\mathscr{X}$ and $\alpha(\Xi_n,\Xi_{n+1},t)\leq 1$, $\forall n\in\mathbb{N}\cup\{0\},\Sigma\in\mathscr{X}$ and $t>0$, we have $\alpha(\Xi_n,\Sigma,t)\leq 1,\forall n\in\mathbb{N}\cup\{0\},\Sigma\in\mathscr{X}$ and $t>0$, then $\mathscr{F}_1$ has a FP.

Corollary 4.4. *Let all hypothesis of corollary 4.3 hold except (xv) changed into the following one as (xvi) for a* $\psi \in \Psi$ *s.t.* $\forall\ \Theta, \Xi \in \mathscr{X}$, $t > 0$,

$$\alpha(\Theta,\Xi,t).\mathscr{H}_{\mathscr{M}(_D)}(\mathscr{F}_1\Theta,\mathscr{F}_1\Xi,t) \geq \psi\Big(\mathscr{H}_{\mathscr{M}d)}(\Theta,\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Theta,t),$$
$$\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Theta,t)\Big)$$

(xvii) for a $\psi \in \Psi$ *s.t.* $\forall\ \Theta, \Xi \in \mathscr{X}$, $t > 0$,

$$(\alpha(\Theta,\Xi,t)+\lambda)^{\mathscr{H}_{\mathscr{M}(_D)}(\mathscr{F}_1\Theta,\mathscr{F}_1\Xi,t)} \geq (1+\lambda)^{\psi\mathscr{H}_{\mathscr{M}d)}(\Theta,\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Theta,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Xi,t),}$$
$$\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Theta,t),\lambda > 0$$

(xviii) for a $\psi \in \Psi$ *s.t.* $\forall\ \Theta, \Xi \in \mathscr{X}$, $t > 0$,

$$\Big(\mathscr{H}_{\mathscr{M}(_D)}(\mathscr{F}_1\Theta,\mathscr{F}_1\Xi,t)+\lambda\Big)^{\alpha(\Theta,\Xi,t)} \geq \psi\Big(\mathscr{H}_{\mathscr{M}d)}(\Theta,\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Theta,t),$$
$$\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Theta,t)\Big)+\lambda,\lambda > 0$$

Then F_1 *has a Fixed Point.*

Corollary 4.5. *Let* $(\mathscr{W}(\mathscr{X}),\mathscr{H}_{\mathscr{M}},\odot)$ *be a Hausdorff fuzzy metric space with* $a \odot b = ab$ *and* $\psi \in \Psi$. *Let* $\mathscr{F}_1 : \mathscr{X} \to W(\mathscr{X})$ *is an* $\eta - admissiable\ MVFM$ *with* $\alpha = 1$. *Suppose* $\mathscr{F}_1$ *is* $\eta - \psi - contractive and satisfying conditions (i) - (iii), (2) and the following assertions :$
$(xix) there is a \psi \in \Psi$ *s.t.* $\forall\ \Theta, \Xi \in \mathscr{X}$, $t > 0$,

$$\eta(\Theta,\Xi,t) \geq 1 \Rightarrow \mathscr{H}_{\mathscr{M}(_D)}(\mathscr{F}_1\Theta,\mathscr{F}_1\Xi,t) \geq \psi\Big(\mathscr{H}_{\mathscr{M}d)}(\Theta,\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Theta,t),$$
$$\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Theta,\mathscr{F}_1\Xi,t),\mathscr{H}_{\mathscr{M}(_p)}(\Xi,\mathscr{F}_1\Theta,t)\Big)$$

(xx) for any sequence $\{\Xi_n\} \in \mathscr{X}$, *converging to* $\Sigma \in \mathscr{X}$ *and* $\eta(\Xi_n,\Xi_{n+1},t) \geq 1$, $\forall n \in \mathbb{N} \cup \{0\}, t > 0$, *and* $t > 0$, *we have* $\eta(\Xi_n,\Sigma,t) \geq 1, \forall n \in \mathbb{N} \cup \{0\}, \Sigma \in \mathscr{X} and t > 0$, *then* $\mathscr{F}_1$ *has a fixed point.*

Example 4.1. *Let $\mathscr{X} = [1,\infty)$ be endowed with the usual fuzzy metric $\mathscr{M}(\Theta,\Xi,t) = \frac{t}{t+|\Theta-\Xi|}$, $\forall \Theta,\Xi \in \mathscr{X}$, $t>0$. Define $\alpha,\eta : \mathscr{X}\times\mathscr{X} \to [0,\infty)$ by*

$$\alpha(\Theta,\Xi,,t) = \begin{cases} 1, & if\, \Theta,\Xi \in \{0,1\} \\ 1, & if\, \Theta,\Xi > 1 \\ 0, & if\, otherwise \end{cases} \quad and \quad \eta(\Theta,\Xi,t) = \begin{cases} 1, & if\, \Theta,\Xi \in \{0,1\} \\ 2, & if\, \Theta,\Xi > 1 \\ 0, & if\, otherwise \end{cases}$$

Let us consider the sequence $\{\Theta_n\}_{n=1}^{\infty}$ where $\Theta_n = 1-\frac{1}{n}$,$\forall n \in \mathbb{N}$, and $\mathscr{X} = \{\Theta_n : n \in \mathbb{N}\}$. We also define $\psi : \mathscr{R}^+ \to \mathscr{R}$ by $\psi_\Gamma = \left(1-exp(1-\Theta_n)^{-a}\right), a>0$. Let $\mathscr{F}_1\Theta_n = 1-\frac{1}{n-1}$, and $\mathscr{F}_2\Theta_n = 1-\frac{1}{n-2}$ with $\lim_{n\to\infty} \Theta_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$, and
$\lim_{n\to\infty} \psi_\Gamma(\Theta_n) = \lim_{n\to\infty}\left(1-exp(1-\Theta_n)^{-a}\right) = \lim_{n\to\infty}\left(1-exp\left(1-1+\frac{1}{n}\right)^{-a}\right) = 1-exp(0)^{-a} < 1.$
Also let $\mathscr{I}\Theta = 1+\frac{\Theta}{3}$ and $\mathscr{J}\Theta = 1+\frac{\Theta}{2}$, then $\mathscr{I}\Theta_n = 4-\frac{1}{3n}$ and $\mathscr{J}\Theta_n = 3-\frac{1}{2n}$, so that the condition of theorem 4.1 satisfied

5. Conclusions

In this chapter we introduced the concept of $\alpha^* - \eta_* - \psi-$hybrid contraction for single and multi-valued fuzzy mappings and prove the common fixed point results in Hausdorff fuzzy metric space. Our result helps in the applications of integral equation, which is significantly contributed to the existing literature for fixed point theorem.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] B. Ali, M. Abbas, Suzuki type fixed point theorem for Fuzzy mappings in ordered metric spaces, Fixed Point Theory Appl. 2013 (2013), 9.

[2] J. H. Asl, S. Rezapour, and N. Shahzad, On fixed points of α-ψ-contractive multi-functions, Fixed Point Theory Appl. 2012 (2012), 212.

[3] S. C. Arora, V. Sharma., Fixed point theorems for fuzzy mappings, Fuzzy Sets Syst. 110 (2000), 127-130.

[4] S. Banach, Sur les operations dansles ensembles, absraites et leurs applications, Fund. Math. 3 (1922), 133-181.

[5] V. Berinde, Approximating fixed points of implicit almost contractions, Hacettepe J. Math. Stat. 41 (2012), 93-102.

[6] R. K. Bose, D. Sahani, Fuzzy mappings and fixed point theorems, Fuzzy Sets Syst. 21 (1987), 53-58.

[7] D. Butnariu., Fixed points for fuzzy mappings, Fuzzy Sets Syst. 7 (1982), 191-207.

[8] S. S. Chang, Fixed point theorems for fuzzy mappings, Fuzzy Sets Syst. 17 (1985), 181-187.

[9] A. Chitra, A note on the fixed point of fuzzy maps on partially ordered topological spaces, Fuzzy Sets Syst. 19 (1986), 305-308.

[10] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy set systems, 64, 3 (24 June 1994), 395-439.

[11] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets Syst. 27 (1988), 385–389.

[12] V. Gregori, S. Morillas, A. Sapena, Examples of fuzzy metrics and applications, Fuzzy Sets Syst. 170 (2011), 95–111.

[13] V. Gregori, A. Sapena., On fixed point theorems in fuzzy metric spaces, Fuzzy Sets Syst. 125 (2002), 245-253..

[14] J. Gutierrez Garcia, S. Romaguera, M. Sanchi, An identification theorem for the completion of the Hausdorff fuzzy metric, Fuzzy Sets Syst. 227 (2013), 96–106

[15] O. Hadžic, E. Pap, Fixed Point Theory in Probabilistic Metric Space. Kluwer Academic, Dordrecht (2001).

[16] S. Heilpern, Fuzzy mappings and fixed point theorem, J. Math. Anal. Appl. 83 (1981), 566-569.

[17] N. Hussain, A. Latif, I. Iqbal, Fixed point results for generalized F-contractions in moduler metric and fuzzy metric spaces, Fixed Point Theory Appl. 2015 (2015), 158.

[18] S.H. Hong, Y. Peng, Fixed points of fuzzy contractive set-valued mappings and fuzzy metric completeness. Fixed Point Theory Appl. 2013 (2013), 276.

[19] S. Hong, Fixed points for modified fuzzy ψ-contractive set valued mappings in fuzzy metric spaces, Fixed Point Theory Appl. 2014 (2014), 12.

[20] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces. Kybernetika, 11 (1975), 326-334.

[21] T. Kamran, Common coincidence points of R-weakly commuting map, Int. J. Math. Math. Sci. 3 (2001), 179-182.

[22] T. Kamran, Non-commuting f-contraction mappings, Novi Sad J. Math. 34 (2004), 33-37.

[23] F. Kiany, A. Amini-Harandi, Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces. Fixed Point Theory Appl. 2011 (2011), 94.

[24] D. Klim, D. Wardowaski, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl. 334 (2007), 132–139.

[25] B.S. Lee, S.J. Cho., A fixed point theorem for contractive type fuzzy mappings, Fuzzy Sets Syst. 61 (1994), 309-312.

[26] D. Mihet, Fixed point theorems in fuzzy metric spaces using property E.A., Nonlinear Anal. TMA. 73(7) (2010), 2184-2188.

[27] N. Mizoguchi, W. Takahasi, Fixed point theorems for set valued mappings on complete metric space, J. Math. Anal. Appl. 142 (1989), 177-188.

[28] S. B. Nadler, Multi-valued contraction mappings, Pac. J. Math. 30 (1969), 475-488.

[29] H. K. Pathak, Y. J. Cho, S. M. Kang, Common fixed points of biased maps of type (A) and applications, Int. J. Math. Math. Sci. 21 (1998), 681-694.

[30] H. K. Pathak, Y. J. Cho, S. M. Kang, Remarks on R-weakely commuting mappings and common fixed point theorems, Bull. Korean Math.Soc. 34(2) (1997) 247-257.

[31] Padaliya, R. P. Pant, Common fixed point theorem for R-weakly commuting mappings of type (A), Soochow J. Math. 31 (2005), 155-163.

[32] S. Phiangsungnoen, W. Sintunavarat, P. Kumam, Fuzzy fixed point theorems in Hausdorff fuzzy metric spaces, J. Inequal. Appl. 2014 (2014), 201.

[33] S. Phiangsungnoen, Some New Fuzzy Fixed Point Theorems for Fuzzy Contractive Mappings in Hausdorff Fuzzy Metric Spaces, in: 2018 International Conference on Control, Artificial Intelligence, Robotics & Optimization (ICCAIRO), IEEE, Prague, Czech Republic, 2018: pp. 159–164.

[34] R. A. Rashwan, M. A. Ahmed, Common fixed point theorems for fuzzy mappings, Arch. Math. (Brno), 38 (2002), 219-226.

[35] J. Rodriguez-López, S. Romaguera, The Hausdorff fuzzy metric on compact sets. Fuzzy Sets Syst. 147 (2004), 273-283.

[36] R. K. Saini, S. Kumar, P. Mohamad, Common Fixed Point Theorem For Hybrid Pairs Of R- Weakly Commuting Maps, Surv. Math. Appl. 5 (2010), 191-199.

[37] R. K. Saini, B. Singh, A. Jain, A Fixed Point Theorem with Implicit Relations In Fuzzy Metric Space, Int. Math. Forum, 5 (2010), 2485-2496.

[38] P. Salimi, A. Latif, N. Hussain, Fixed point results for single and set valued α-η-ψ-contractive mappings. Fixed Point Theory Appl. 2013 (2013), 212.

[39] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. TMA. 75(2012) 2154–2165.

[40] B. Schweizer, A. Sklar, Statistical metric spaces, Pac. J. Math. 10 (1960), 313–334.

[41] G. Song, Comments on “A common fixed point theorem in a fuzzy metric spaces”. Fuzzy Sets Sys. 135 (2003), 409-413.

[42] D. Wardowaski, Fixed point for new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 94.

[43] S. Wang, On ψ-contractions in partially ordered fuzzy metric spaces, Fixed Point Theory and Applications, 2015 (2015), 233.

[44] M. D. Weiss, Fixed points, separation and induced fuzzy topologies for fuzzy sets mappings, J. Math. Anal. Appl. 50 (1975), 142-150.

[45] L. A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338-353.