Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7105-7125
https://doi.org/10.28919/jmcs/5995
ISSN: 1927-5307

ANALYSIS OF FRACTIONAL KAWAHARA AND MODIFIED KAWAHARA EQUATIONS BASED ON CAPUTO-FABRIZIO DERIVATIVE OPERATOR

BHAUSAHEB R. SONTAKKE[1], SHANKAR R. RAUT[2,*]

[1]Department of Mathematics, Pratishthan Mahavidyalaya, Paithan - 431107, Aurangabad, India
[2]Department of Mathematics, Mrs. K.S.K. College, Beed, India

Abstract. In this paper, nonlinear time fractional Kawahara and modified Kawahara equations based on Caputo-Fabrizio derivative operator is analysed using iterative Laplace transform method to obtain approximate solutions. The substantive features of the manuscript is to offer the stability conditions of solution for proposed technique. The acquired approximate solutions are in comparison with the precise solutions to confirm the applicability, performance and accuracy of the method. Moreover, the 3D plots of obtained numerical solution of the concerned equations for various specific cases are presented.

Keywords: fractional Kawahara and modified Kawahara equations; Caputo-Fabrizio derivative operator; stability analysis; Laplace transform; new iterative method.

2010 AMS Subject Classification: 35R11.

1. Introduction

From the past three decades, the most captivating rise in scientific and engineering applications have been found within the framework of Fractional calculus. It has fascinated the attention of many scholars due to its usefulness in various fields of science and engineering,

*Corresponding author
E-mail address: shankar.kskm123@gmail.com
Received May 10, 2021

such as fluid mechanics, diffusive transport, electrical networks, electromagnetic theory, different branches of physics, biological sciences and groundwater problems , [1, 2, 3, 4, 5]. In recent times, many scholars have tried to model various physical or biological processes using fractional differential equations. Moreover, obtaining numerical solutions of these equations is turn out to be wide area of research and interest for researchers. Some of the most used and efficient analytical or numerical methods for solving these fractional differential equations are given as the Finite difference method [8], Adomian decomposition method ADM[6, 7], Homotopy-perturbation method HPM [9], Homotopy analysis method [10], Adams-Bashforth- Moulton method [11], Variational iteration method VIM [12, 13], monotone iterative method [14, 15], etc. Recently, Daftardar-Gejji and Jafari [16] suggested an iterative method which is known as new iterative method (NIM). Furthermore, applying Laplace transform utilizing NIM [17] is turned out to be most efficiect and reliable method in fractional calculus for solving linear and nonlinear fractional partial differential equations.

These FDEs involves several fractional differential operators like Riemann-Liouville operator [18], Caputo operator [19], Hilfer operator [20], Katugampola operator [21], etc. However these operators possesses a power law kernel and has singularity which leads to some limitations in modelling physical problems. To overcome this difficulty, in recent times Caputo and Fabrizio have proposed a reliable operator having nonlocal and nonsingular kernel in the form of exponential function known as Caputo-Fabrizio operator [22, 23].

Nonlinear wave phenomena has significant importance in various parts of mathematical physics and engineering such as dispersion, reaction, diffusion and convection. Moreover, one of the well-known nonlinear evolution equation is the fifth order Kawahara equation which appeares in the study of shallow water waves having magneto-acoustic waves in a plasma, surface tension and capillary-gravity waves. This equation has attarcted several authors in recent times [24, 25]. To describe solitary-wave propagation in media, in 1972, Kawahara [26] suggested the kawahara equation. Moreover, the modified Kawahara equation has some useful applications in physics such as,capillary-gravity water waves, plasma waves, water waves with surface tension, etc. [27, 28, 29].

Inspired by above literature, in this paper we have applied iterative Laplace transform with to find approximate solutions of time fractional Kawahara and modified Kawahara equations having Caputo-Fabrizio operator. These equations are given below as follows:

$$\frac{{}^{CF}\partial^{\beta} v}{\partial t^{\beta}} + v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \frac{\partial^5 v}{\partial x^5} = 0 \; 0 < \beta \leq 1 \tag{1.1}$$

with initial condition

$$v(x,0) = \frac{105}{169} sech^4\left(\frac{x}{2\sqrt{13}}\right) \tag{1.2}$$

and

$$\frac{{}^{CF}\partial^{\beta} v}{\partial t^{\beta}} + v^2\frac{\partial v}{\partial x} + h\frac{\partial^3 v}{\partial x^3} + l\frac{\partial^5 v}{\partial x^5} = 0 \; 0 < \beta \leq 1 \tag{1.3}$$

where h,l are nonzero real constants and initial condition is

$$v(x,0) = \frac{3h}{\sqrt{-10l}} sech^2(Mx),\; M = \frac{1}{2}\sqrt{\frac{-h}{5l}} \tag{1.4}$$

Equations (1.1) and (1.3) becomes the original Kawahara and modified Kawahara equations for $\beta = 1$ [26]

The remaining of this manuscript is arranged as below. Section 2 is presentation of some basic definitions and lemmas of fractional calculus. Preliminary idea of iterative Laplace transform method is illustrated in section 3. In Section 4, stability criteria for obtained approximate solutions of considered equations are displayed. The numerical simulations, plots and tables for the obtained solutions are demonstrated in section 5. In section 6, we give our conclusions.

2. Basics of Fractional Calculus

In this section, we present some useful definitions and lemmas of fractional calculus.

Definition 2.1. ([30]) *The Caputo-Fabrizio fractional integral operator with order* $0 < \beta < 1$ *is given by*

$$ {}^{CF}I_t^{\beta} v(x,t) = \frac{2(1-\beta)}{(2-\beta)\mathbb{M}(\beta)} u(t) + \frac{2\beta}{(2-\beta)\mathbb{M}(\beta)} \int_0^t v(x,\zeta)d\zeta, \tag{2.1}$$

Definition 2.2. *([31]) Let $v \in H^1(0,a),\ a > 0, 0 < \beta < 1$, then the time fractional Caputo-Fabrizio differential operator is given as*

$$
{}^{CF}D_t^{\beta} v(x,t) = \frac{(2-\beta)\mathbb{M}(\beta)}{2(1-\beta)} \int_0^t exp\Big[-\frac{\beta(t-s)}{1-\beta}\Big] v'(\zeta)d\zeta,\ t \geq 0,\ 0<\beta<1, \tag{2.2}
$$

where $\mathbb{M}(\beta)$ is a normalisation function depending on β such that $\mathbb{M}(0) = \mathbb{M}(1) = 1$.

Similar to Caputo derivative operator, the CF operator gives ${}^{CF}D_t^{\beta} v(x,t) = 0$, if v is a constant function.

The benefit of Caputo-Fabrizio operator is that there is no singularity for $t = s$ in the new kernel as compared to Caputo operator

Definition 2.3. *([31]) The Laplace transform for the Caputo-fabrizio fractional operator of order $0 < \beta \leq 1$ and $m \in \mathbb{N}$ is given by*

$$
\begin{aligned}
L\Big({}^{CF}D_t^{m+\beta} v(x,t)\Big)(s) &= \frac{1}{1-\beta} L(v^{(m+1)}(x,t)) L\bigg(exp\bigg(-\frac{\beta}{1-\beta}t\bigg)\bigg) \\
&= \frac{s^{m+1}L(v(x,t)) - s^m v(x,0) - s^{m-1}v'(x,0) - \cdots - v^{(m)}(x,0)}{s+\beta(1-s)}.
\end{aligned} \tag{2.3}
$$

In particular, we have

$$
L\Big({}^{CF}D_t^{\beta} v(x,t)\Big)(s) = \frac{sL(v(\bar{x},t))}{s+\beta(1-s)},\quad m = 0.
$$

$$
L\Big({}^{CF}D_t^{\beta+1} v(x,t)\Big)(s) = \frac{s^2L(v(x,t)) - sv(x,0) - v'(x,0)}{s+\beta(1-s)},\quad m = 1.
$$

3. Iterative Laplace Transform Method

In this section, a general nonhomogeneous Caputo-Fabrizio fractional differential equation is considered which is given as below

$$
{}^{CF}D_t^{\beta} u(x,t) + \mathscr{R}u(x,t) + \mathscr{N}u(x,t) = g(x,t) \tag{3.1}
$$

with initial condition

$$
u(x,0) = \psi(x,t) \tag{3.2}
$$

Where $g(x,t)$ denotes source term, $\mathscr{R}$ and $\mathscr{N}$ are given linear and non-linear operator respectively. Applying Laplace transform on (3.1) we get

$$L\{v(x,t)\} - \frac{1}{s}v(x,0) + \left(\frac{s+\beta(1-s)}{s}\right)\Big(L\{\mathscr{R}v(x,t)\} + L\{\mathscr{N}v(x,t)\} - L\{g(x,t)\}\Big) = 0. \tag{3.3}$$

Rearranging terms we get

$$L\{v(x,t)\} = \frac{1}{s}\psi(x,t) - \left(\frac{s+\beta(1-s)}{s}\right)\Big(L\{\mathscr{R}v(x,t)\} + L\{\mathscr{N}v(x,t)\} - L\{g(x,t)\}\Big) \tag{3.4}$$

$$L(v(x,t)) = \psi(x,s) - \left(\frac{s+\beta(1-s)}{s}\right)L\big(\mathscr{R}(v(x,t)) + \mathscr{N}(v(x,t))\big), \tag{3.5}$$

where

$$\psi(x,s) = \frac{1}{s}\psi(x,t) - \frac{s+\beta(1-s)}{s}\widetilde{g}(x,s).$$

Next, we apply inverse laplace transform on (3.5) then we get

$$v(x,t) = \psi(x,t) - L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\big(\mathscr{R}(v(x,t)) + \mathscr{N}(v(x,t))\big)\right], \tag{3.6}$$

where $\psi(x,t)$ is the term derived from source term.

Further, we use new iterative method to obtain infinite series solution. This method is introduced by Daftardar-Gejji and Jafari [16].

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \tag{3.7}$$

since $\mathscr{R}$ is linear,

$$\mathscr{R}\left(\sum_{n=0}^{\infty} v_n(x,t)\right) = \sum_{n=0}^{\infty} \mathscr{R}\big(v_n(x,t)\big). \tag{3.8}$$

The decomposion of nonlinear operator $\mathcal{N}$ is given as

$$\mathcal{N}\left(\sum_{n=0}^{\infty} v_n\right) = \mathcal{N}(v_0(x,t)) + \sum_{n=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} v_j(x,t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} v_j(x,t)\right)\right\}. \tag{3.9}$$

In view of (3.7), (3.8) and (3.9), the equation (3.6) is equivalent to

$$\begin{aligned}\sum_{i=0}^{\infty} v_i(x,t) = \psi(x,t) - L^{-1}&\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(\sum_{i=0}^{\infty}\mathcal{R}\big(v_i(\bar{x},t)\big)\right)\right] \\ &- L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(\mathcal{N}\big(v_0(x,t)\big) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} v_j(x,t)\right)\right.\right.\right. \\ &\left.\left.\left.- \mathcal{N}\left(\sum_{j=0}^{i-1} v_j(\bar{x},t)\right)\right\}\right)\right],\end{aligned} \tag{3.10}$$

further, consider the recurrence relation as follows

$$\begin{aligned} v_0(x,t) &= \psi(x,t) \\ v_1(x,t) &= L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(\mathcal{R}\big(v_0(x,t)\big) + \mathcal{N}\big(v_0(x,t)\big)\right)\right] \\ &\vdots \end{aligned} \tag{3.11}$$

$$\begin{aligned} v_{p+1}(x,t) = L^{-1}&\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(\mathcal{R}\big(v_p(x,t)\big) + \left\{\mathcal{N}\left(\sum_{j=0}^{p} v_j(x,t)\right)\right.\right.\right. \\ &\left.\left.\left.- \mathcal{N}\left(\sum_{j=0}^{p-1} v_j(x,t)\right)\right\}\right)\right] \end{aligned} \tag{3.12}$$

The approximate solution with p$-$term is given as

$$v = v_0 + v_1 + v_2 + \cdots + v_{p-1}. \tag{3.13}$$

The convergence condition of the above approximate solution is obtained in [32]

4. Stability Analysis

4.1. Stability analysis of the fractional Kawahara equation. Let $(\mathbb{B}, \|\cdot\|)$ as a Banach space. Further, define Γ as self-map of $\mathbb{B}$. and $v_{m+1} = f(\Gamma, v_m)$ shows exact recurring process. The fixed-point set on Γ is denoted by $\mathbb{F}(\Gamma)$. Moreover, Γ has atleast one element such that v_m converges to $k \in \mathbb{F}(\Gamma)$. Let $\{\omega_m\} \subseteq \mathbb{B}$ and define $y_m = \|\omega_{m+1} - f(\Gamma, \omega_m)\|$. If $\lim_{m\to\infty} y^m =$

0 implies that $\lim_{m\to\infty} \omega^m = k$, then the iteration method $v_{m+1} = f(\Gamma, v_m)$ is called as Γ-stable. Comparably, we think about that, this sequence $\{\omega_m\}$ has an upper bound. This iteration is called as Picard's iteration and it is $\Gamma-$ stable, if all these criterias are fulfilled for $v_{m+1} = \Gamma v_m$.

Theorem 4.1. *Consider a Banach space $(\mathscr{B}, \|\cdot\|)$ and define Γ as self-map on $\mathscr{B}$ fulfilling*

$$\|\Gamma_m - \Gamma_r\| \leq \Delta\|m - \Gamma_m\| + \xi\|m - r\|$$

for all $m,\ r \in \mathscr{B}$ where $0 \leq \Delta,\ 0 \leq \xi < 1$. Assume that Γ is Picard Γ-stable. Let the following equation related to (1.1)

$$v_{m+1}(x,t) = v_m(x,t) + L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(-v_m\frac{\partial v_m}{\partial x} - \frac{\partial^3 v_m}{\partial x^3} + \frac{\partial^5 v_m}{\partial x^5}\right)\right] \tag{4.1}$$

where $\dfrac{s+\beta(1-s)}{s}$ is a fractional Lagrange multiplier.

Theorem 4.2. *Consider a self-map Γ defined as*

$$\Gamma(v_m(x,t)) = v_{m+1}(x,t) = v_m(x,t) + L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(-v_m\frac{\partial v_m}{\partial x} - \frac{\partial^3 v_m}{\partial x^3} + \frac{\partial^5 v_m}{\partial x^5}\right)\right].$$

is $\Gamma-$stable in $L^2(m,r)$ if

$$\left\{1 + F_1(\beta)\sigma_2 + F_2(\beta)\sigma_3 + \frac{F_3(\beta)\sigma_1}{2}(\delta_1+\delta_2)\right\} < 1. \tag{4.2}$$

Proof. Here, we will show that Γ consists a fixed point. Hence, for all $(m,\ r) \in \mathbb{N}\times\mathbb{N}$, we consider the following.

$$\begin{aligned}\Gamma(v_m(x,t)) - \Gamma(v_r(x,t)) = v_m(x,t) - v_r(x,t) + L^{-1}&\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(-v_m\frac{\partial v_m}{\partial x}\right.\right.\\ &\left.\left.-\frac{\partial^3 v_m}{\partial x^3} + \frac{\partial^5 v_m}{\partial x^5}\right)\right] - L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\left(-v_r\frac{\partial v_r}{\partial x}\right.\right.\\ &\left.\left.-\frac{\partial^3 v_r}{\partial x^3} + \frac{\partial^5 v_r}{\partial x^5}\right)\right].\end{aligned} \tag{4.3}$$

By applying norm on both sides of (4.3) and without loss of generality, we obtain

$$
\begin{aligned}
\left\|\Gamma(v_m(x,t))-\Gamma(v_r(x,t))\right\| = \Bigg\| & v_m(x,t)-v_r(x,t)+L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_m\frac{\partial v_m}{\partial x}\right. \\
& \left.-\frac{\partial^3 v_m}{\partial x^3}+\frac{\partial^5 v_m}{\partial x^5}\Big)\right]-L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_r\frac{\partial v_r}{\partial x}\right. \\
& \left.-\frac{\partial^3 v_r}{\partial x^3}+\frac{\partial^5 v_r}{\partial x^5}\Big)\right]\Bigg\|. \qquad (4.4)
\end{aligned}
$$

Next, utilizing triangular inequality and simplifying further (4.4) we get,

$$
\begin{aligned}
\left\|\Gamma(v_m(x,t))-\Gamma(v_r(x,t))\right\| \le \|v_m(x,t)-v_r(x,t)\| & +L^{-1}\Bigg\{\left(\frac{s+\beta(1-s)}{s}\right)L\Bigg[\Bigg\|-\frac{\partial^3 v_m}{\partial x^3} \\
& +\frac{\partial^5 v_m}{\partial x^5}+\frac{\partial^3 v_r}{\partial x^3}-\frac{\partial^5 v_r}{\partial x^5}\Bigg\|\Bigg] \\
& +L\left[\left\|-v_m\frac{\partial v_m}{\partial x}+v_r\frac{\partial v_r}{\partial x}\right\|\right]\Bigg\}. \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
\left\|\Gamma(v_m(x,t))-\Gamma(v_r(x,t))\right\| \le \|v_m(x,t)-v_r(x,t)\|+L^{-1}\Bigg\{ & \left(\frac{s+\beta(1-s)}{s}\right) \\
& L\left[\left\|\left(\frac{\partial^3 v_m}{\partial x^3}-\frac{\partial^3 v_r}{\partial x^3}\right)+\left(\frac{\partial^5 v_m}{\partial x^5}-\frac{\partial^5 v_r}{\partial x^5}\right)\right\|\right] \\
& +L\left[\left\|v_m\frac{\partial v_m}{\partial x}-v_r\frac{\partial v_r}{\partial x}\right\|\right]\Bigg\}. \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
\le \|v_m(x,t)-v_r(x,t)\|+L^{-1}\Bigg\{ & \left(\frac{s+\beta(1-s)}{s}\right) \\
& L\left[\left\|\frac{\partial^3 v_m}{\partial x^3}-\frac{\partial^3 v_r}{\partial x^3}\right\|\right]+L\left[\left\|\frac{\partial^5 v_m}{\partial x^5}-\frac{\partial^5 v_r}{\partial x^5}\right\|\right] \\
& +L\left[\frac{1}{2}\left\|\frac{\partial v_m^2}{\partial x}-\frac{\partial v_r^2}{\partial x}\right\|\right]\Bigg\}. \qquad (4.7)
\end{aligned}
$$

Now substituting differential operators $\frac{\partial}{\partial x} = \sigma_1,\ \frac{\partial^3}{\partial x^3} = \sigma_2$ and $\frac{\partial^5}{\partial x^5} = \sigma_3$ we get

$$\begin{aligned} \Big\|\Gamma(v_m(x,t)) - \Gamma(v_r(x,t))\Big\| \leq \|v_m(x,t) - v_r(x,t)\| + L^{-1}\Bigg\{\Bigg(\frac{s+\beta(1-s)}{s}\Bigg) \\ L\Bigg[\sigma_2\Bigg\|v_m(x,t) - v_r(x,t)\Bigg\|\Bigg] + L\Bigg[\sigma_3\Bigg\|v_m(x,t) - v_r(x,t)\Bigg\|\Bigg] \\ + L\Bigg[\frac{\sigma_1}{2}\Bigg\|\big(v_m(x,t) + v_r(x,t)\big)\big(v_m(x,t) - v_r(x,t)\big)\Bigg\|\Bigg]\Bigg\}. \end{aligned} \tag{4.8}$$

Since, u_m and u_r are bounded functions, we have $\|u_m\| \leq \delta_1$ and $\|u_r\| \leq \delta_2$. Therefore, simplifying (4.8), we obtain

$$\begin{aligned} \Big\|\Gamma(v_m(x,t)) - \Gamma(v_r(x,t))\Big\| \leq \Big\{1 + F_1(\beta)\sigma_2 + F_2(\beta)\sigma_3 \\ + \frac{F_3(\beta)\sigma_1}{2}(\delta_1 + \delta_2)\Big\}\|v_m(x,t) + v_r(x,t)\|. \end{aligned} \tag{4.9}$$

where $F_1,\ F_2$ and F_3 are functions of $L^{-1}\Big\{\Big(\frac{s+\beta(1-s)}{s}\Big)L\Big\}$.

Hence, the self-mapping Γ has a fixed point. This completes the proof.

Further, we prove that Γ satisfies all the criterias in Theorem 4.1. Let (4.9) holds then using

$$\xi = 0, \quad \Delta = \Big\{1 + F_1(\beta)\sigma_2 + F_2(\beta)\sigma_3 + \frac{F_3(\beta)\sigma_1}{2}(\delta_1 + \delta_2)\Big\}. \tag{4.10}$$

Thus, all the conditions in Theorem 4.2 are satisfied by Γ. Therefore, Γ is Picard $\Gamma-$ stable.

4.2. Stability analysis of the fractional modified Kawahara equation. Let $(\mathbb{B}, \|\cdot\|)$ as a Banach space. Further, define Θ as self-map of $\mathbb{B}$. and $\zeta_{m+1} = g(\Theta, \zeta_m)$ shows exact recurring process. The fixed-point set on Θ is denoted by $\mathbb{G}(\Theta)$. Moreover, Θ has atleast one element such that ζ_m converges to $k \in \mathbb{G}(\Theta)$. Let $\{\mu_m\} \subseteq \mathbb{B}$ and define $y_m = \|\mu_{m+1} - g(\Theta, \mu_m)\|$. If $\lim_{m\to\infty} y^m = 0$ implies that $\lim_{m\to\infty} \mu^m = k$, then the iteration method $\zeta_{m+1} = g(\Theta, \zeta_m)$ is called as Θ-stable. Comparably, we think about that, this sequence $\{\mu_m\}$ has an upper bound. This iteration is called as Picard's iteration and it is $\Theta-$ stable, if all these criterias are fulfilled for $\zeta_{m+1} = \Theta\zeta_m$.

Theorem 4.3. *Consider a Banach space* $(\mathbb{B}, \|\cdot\|)$ *and define* Θ *as self-map on* $\mathbb{B}$ *fulfilling*

$$\|\Theta_m - \Theta_r\| \leq \Delta_1\|m - \Theta_m\| + \xi_1\|m - r\|$$

for all $m,\ r \in \mathscr{B}$ *where* $0 \leq \Delta_1,\ 0 \leq \xi_1 < 1$. *Assume that* Θ *is Picard* Θ*-stable. Let the following equation related to* (1.3)

$$v_{m+1}(x,t) = v_m(x,t) + L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_m^2\frac{\partial v}{\partial x} - h\frac{\partial^3 v_m}{\partial x^3} - l\frac{\partial^5 v_m}{\partial x^5}\Big)\right] \tag{4.11}$$

where $\dfrac{s+\beta(1-s)}{s}$ is a fractional Lagrange multiplier.

Theorem 4.4. *Consider a self-map* Θ *defined as*

$$\Theta(v_m(x,t)) = v_{m+1}(x,t) = v_m(x,t) + L^{-1}\left[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_m^2\frac{\partial v}{\partial x} - h\frac{\partial^3 v_m}{\partial x^3} - l\frac{\partial^5 v_m}{\partial x^5}\Big)\right].$$

is $\Theta-$*stable in* $L^2(m,r)$ *if*

$$\Big\{1 + hF_4(\beta)\sigma_5 + lF_5(\beta)\sigma_6 + \frac{F_6(\beta)\sigma_4}{3}(\delta_3^2 + \delta_3\delta_4 + \delta_4^2)\Big\} < 1. \tag{4.12}$$

Proof. Here, we will show that Θ consists a fixed point. Hence, for all $(m,\ r) \in \mathbb{N}\times\mathbb{N}$, we consider the following.

$$\begin{aligned}\Theta(v_m(x,t)) - \Theta(v_r(x,t)) = v_m(x,t) - v_r(x,t) + L^{-1}\Bigg[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_m^2\frac{\partial v}{\partial x}\\ - h\frac{\partial^3 v_m}{\partial x^3} - l\frac{\partial^5 v_m}{\partial x^5}\Big)\Bigg] - L^{-1}\Bigg[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_r^2\frac{\partial v}{\partial x}\\ - h\frac{\partial^3 v_r}{\partial x^3} - l\frac{\partial^5 v_r}{\partial x^5}\Big)\Bigg].\end{aligned} \tag{4.13}$$

By applying norm on both sides of (4.13) and without loss of generality, we obtain

$$\begin{aligned}\big\|\Theta(v_m(x,t)) - \Theta(v_r(x,t))\big\| = \Bigg\|v_m(x,t) - v_r(x,t) + L^{-1}\Bigg[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_m^2\frac{\partial v}{\partial x}\\ - h\frac{\partial^3 v_m}{\partial x^3} - l\frac{\partial^5 v_m}{\partial x^5}\Big)\Bigg] - L^{-1}\Bigg[\left(\frac{s+\beta(1-s)}{s}\right)L\Big(-v_r^2\frac{\partial v}{\partial x}\\ - h\frac{\partial^3 v_r}{\partial x^3} - l\frac{\partial^5 v_r}{\partial x^5}\Big)\Bigg]\Bigg\|.\end{aligned} \tag{4.14}$$

Next, utilizing triangular inequality and simplifying further (4.14) we get,

$$\begin{aligned}\left\|\Theta(v_m(x,t))-\Theta(v_r(x,t))\right\| \le \|v_m(x,t)-v_r(x,t)\| + L^{-1}\Bigg\{ \left(\frac{s+\beta(1-s)}{s}\right) L\Bigg[\Bigg\| -h\frac{\partial^3 v_m}{\partial x^3} \\ -l\frac{\partial^5 v_m}{\partial x^5}+h\frac{\partial^3 v_r}{\partial x^3}+l\frac{\partial^5 v_r}{\partial x^5}\Bigg\|\Bigg] \\ +L\left[\left\| -v_m^2\frac{\partial v_m}{\partial x}+v_r^2\frac{\partial v_r}{\partial x}\right\|\right]\Bigg\}.\end{aligned} \tag{4.15}$$

$$\begin{aligned}\left\|\Theta(v_m(x,t))-\Theta(v_r(x,t))\right\| &\le \|v_m(x,t)-v_r(x,t)\| + L^{-1}\Bigg\{ \left(\frac{s+\beta(1-s)}{s}\right) \\ &L\left[\left\| h\left(\frac{\partial^3 v_m}{\partial x^3}-\frac{\partial^3 v_r}{\partial x^3}\right)+l\left(\frac{\partial^5 v_m}{\partial x^5}-\frac{\partial^5 v_r}{\partial x^5}\right)\right\|\right] \\ &+L\left[\left\| v_m^2\frac{\partial v_m}{\partial x}-v_r^2\frac{\partial v_r}{\partial x}\right\|\right]\Bigg\}.\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\le \|v_m(x,t)-v_r(x,t)\| + L^{-1}\Bigg\{ \left(\frac{s+\beta(1-s)}{s}\right) \\ &hL\left[\left\|\frac{\partial^3 v_m}{\partial x^3}-\frac{\partial^3 v_r}{\partial x^3}\right\|\right]+lL\left[\left\|\frac{\partial^5 v_m}{\partial x^5}-\frac{\partial^5 v_r}{\partial x^5}\right\|\right] \\ &+L\left[\frac{1}{3}\left\|\frac{\partial v_m^3}{\partial x}-\frac{\partial v_r^3}{\partial x}\right\|\right]\Bigg\}.\end{aligned} \tag{4.17}$$

Now substituting differential operators $\frac{\partial}{\partial x}=\sigma_4,\ \frac{\partial^3}{\partial x^3}=\sigma_5$ and $\frac{\partial^5}{\partial x^5}=\sigma_6$ we get

$$\begin{aligned}\left\|\Theta(v_m(x,t))-\Theta(v_r(x,t))\right\| &\le \|v_m(x,t)-v_r(x,t)\| + L^{-1}\Bigg\{ \left(\frac{s+\beta(1-s)}{s}\right) \\ &L\left[h\sigma_5\left\|v_m(x,t)-v_r(x,t)\right\|\right]+L\left[l\sigma_6\left\|v_m(x,t)-v_r(x,t)\right\|\right] \\ &+L\Bigg[\frac{\sigma_4}{3}\Big\|\big(v_m^2(x,t)+v_m(x,t)v_r(x,t) \\ &+v_r^2(x,t)\big)\big(v_m(x,t)-v_r(x,t)\big)\Big\|\Bigg]\Bigg\}.\end{aligned} \tag{4.18}$$

Since, u_m and u_r are bounded functions, we have $\|u_m\| \leq \delta_3$ and $\|u_r\| \leq \delta_4$. Therefore, simplifying (4.18), we obtain

$$\begin{aligned} \left\|\Theta(v_m(x,t)) - \Theta(v_r(x,t))\right\| \leq & \Big\{1 + hF_4(\beta)\sigma_5 + lF_5(\beta)\sigma_6 \\ & + \frac{F_6(\beta)\sigma_4}{3}(\delta_3^2 + \delta_3\delta_4 + \delta_4^2)\Big\}\|v_m(x,t) - v_r(x,t)\|. \end{aligned} \tag{4.19}$$

where $F_4,\ F_5$ and F_6 are functions of $L^{-1}\Big\{\Big(\frac{s+\beta(1-s)}{s}\Big)L\Big\}$.

Hence, the self-mapping Θ has a fixed point. This completes the proof.

Further, we prove that Θ satisfies all the criterias in Theorem 4.3. Let (4.19) holds then using

$$\xi_1 = 0, \quad \Delta_1 = \Big\{1 + hF_4(\beta)\sigma_5 + lF_5(\beta)\sigma_6 + \frac{F_6(\beta)\sigma_4}{3}(\delta_3^2 + \delta_3\delta_4 + \delta_4^2)\Big\}. \tag{4.20}$$

Thus, all the conditions in Theorem 4.4 are satisfied by Θ. Therefore, Θ is Picard $\Theta-$ stable.

5. Numerical Simulations

This section deals with the interpretation of the analytical results for the time fractional Kawahara and modified Kawahara equations with the graphical illustrations. We have used Mathematica software to compute approximate solutions.

5.1 Approximate solution for time fractional Kawahara equation

Consider the time fractional Kawahara equation (1.1) with initial condition (1.2).

The exact solution to (1.1) is given in [33] as

$$v(x,t) = \frac{105}{169} sech^4\left(\frac{1}{2\sqrt{13}}\left(x - \frac{36t}{169}\right)\right) \tag{5.1}$$

The initial condition (1.2) is rewritten as

$$v(x,0) = \frac{1680}{169}\frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4}$$

Applying laplace transform on both side of (1.1) we get

$$L\{v(x,t)\} - \frac{1}{s}u(x,0) + \left(\frac{s+\beta(1-s)}{s}\right)L\Big\{v\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} - \frac{\partial^5 v}{\partial x^5}\Big\} = 0. \tag{5.2}$$

Rearranging terms we obtain

$$L\{v(x,t)\} = \frac{1}{s}\left(\frac{1680}{169}\frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4}\right) - \left(\frac{s+\beta(1-s)}{s}\right)L\left\{v\frac{\partial v}{\partial x}+\frac{\partial^3 v}{\partial x^3}-\frac{\partial^5 v}{\partial x^5}\right\} \tag{5.3}$$

Next, the inverse Laplace transform on (5.3), gives

$$v(x,t) = \frac{1680}{169}\frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4} - L^{-1}\left\{\left(\frac{s+\beta(1-s)}{s}\right)L\left\{v\frac{\partial v}{\partial x}+\frac{\partial^3 v}{\partial x^3}-\frac{\partial^5 v}{\partial x^5}\right\}\right\} \tag{5.4}$$

The series solution is given as follows,

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \tag{5.5}$$

The nonlinear term $v\dfrac{\partial v}{\partial x}$ is written as $v_n\dfrac{\partial v_n}{\partial x} = \sum_{n=0}^{\infty}\mathbb{P}_n$; whereas $\mathbb{P}_n$ is further decomposed as follows

$$\mathbb{P}_n = \sum_{i=0}^{n} v_i.\frac{\partial}{\partial x}\left(\sum_{i=0}^{n} v_i\right) - \sum_{i=0}^{n-1} v_i.\frac{\partial}{\partial x}\left(\sum_{i=0}^{n-1} v_i\right)$$

by using $v_0(x,\ t) = \dfrac{1680}{169}\dfrac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4}$, we get the recursive formula as follows

$$v_{n+1}(x,t) = v_0(x,\ t) - L^{-1}\left\{\left(\frac{s+\beta(1-s)}{s}\right)L\left\{v_n\frac{\partial v_n}{\partial x}+\frac{\partial^3 v_n}{\partial x^3}-\frac{\partial^5 v_n}{\partial x^5}\right\}\right\} \tag{5.6}$$

The n−term approximate solution is given by

$$v(x,t) = v_0(x,t)+v_1(x,t)+v_2(x,t)+\cdots+v_{n-1}(x,t). \tag{5.7}$$

Therefore, using (5.6) the first three terms of approximate solution of (1.1) are obtained as follows

$$v_0 = \frac{1680}{169}\frac{e^{\frac{2x}{\sqrt{13}}}}{\left(e^{\frac{x}{\sqrt{13}}}+1\right)^4}$$

$$v_1 = -\frac{120960e^{\frac{2x}{\sqrt{13}}}\left(e^{\frac{x}{\sqrt{13}}}-1\right)(\beta(t-1)+1)}{28561\sqrt{13}\left(e^{\frac{x}{\sqrt{13}}}+1\right)^5}$$

$$v_2 = \frac{4354560e^{\frac{2x}{\sqrt{13}}}\left(-3e^{\frac{x}{\sqrt{13}}}+e^{\frac{2x}{\sqrt{13}}}+1\right)}{137858491849\left(e^{\frac{x}{\sqrt{13}}}+1\right)^{11}}\Big(2197\left(\beta^2((t-4)t+2)+4\beta(t-1)+2\right)$$
$$+10985e^{\frac{x}{\sqrt{13}}}\left(\beta^2((t-4)t+2)+4\beta(t-1)+2\right)+10985e^{\frac{4x}{\sqrt{13}}}\big(\beta^2((t-4)t+2)+4\beta(t-1)$$
$$+2\big)+2197e^{\frac{5x}{\sqrt{13}}}\left(\beta^2((t-4)t+2)+4\beta(t-1)+2\right) +224\sqrt{13}\beta^3\left(\left((t-3)^2t-3\right)\right)\Big)$$

$$v_3 = -\frac{1}{190049637748807994388014\left(e^{\frac{x}{\sqrt{13}}}+1\right)^{23}}17418240\Big(-4962905706564\sqrt{13}e^{\frac{2x}{\sqrt{13}}}$$
$$-47147604212358\sqrt{13}e^{\frac{3x}{\sqrt{13}}}+348305518922790\sqrt{13}e^{\frac{4x}{\sqrt{13}}}+65780525341440e^{\frac{4x}{\sqrt{13}}}$$
$$+4284679410923028\sqrt{13}e^{\frac{5x}{\sqrt{13}}}-25621514620490880e^{\frac{5x}{\sqrt{13}}}+120941967334930560e^{\frac{6x}{\sqrt{13}}}$$
$$+15052992814426944\sqrt{13}e^{\frac{6x}{\sqrt{13}}}+21841020780811128\sqrt{13}e^{\frac{7x}{\sqrt{13}}}+452975230290360960e^{\frac{7x}{\sqrt{13}}}\cdots$$

Continuing in the same way, remaining terms of the iteration formula (5.6) are obtained.

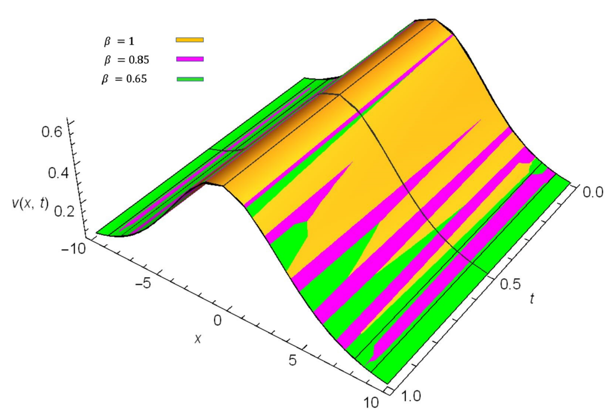

FIGURE 1. Approx. soln of Eq. (1.1), for $\beta = 1,\ 0.85,\ 0.65$

t	x	$\beta = 0.55$	$\beta = 0.75$	$\beta = 0.95$	Absolute error $\lvert v_{\text{exact}} - v_{\text{apprx}} \rvert$ for $\beta = 1$
0.03	-5	0.262901	0.259311	0.255766	0.0010822
	-1	0.602059	0.600445	0.598686	0.000584122
	0	0.620821	0.621140	0.621289	1.27545×10^{-11}
	1	0.592981	0.595140	0.597158	0.000584122
	5	0.246095	0.249485	0.252934	0.0010822
0.07	-5	0.263310	0.259863	0.256455	0.00252512
	-1	0.602234	0.600706	0.599045	0.00136294
	0	0.620775	0.621105	0.621277	3.78065×10^{-10}
	1	0.592729	0.594818	0.596777	0.00136294
	5	0.245714	0.248956	0.252253	0.00252512
0.1	-5	0.263617	0.260277	0.256973	0.00360731
	-1	0.602364	0.600901	0.599313	0.00194704
	0	0.620740	0.621077	0.621266	1.57462×10^{-9}
	1	0.592538	0.594575	0.596490	0.00194704
	5	0.245428	0.248560	0.251743	0.00360731

TABLE 1. The numerical results for various values of β and comparison of absolute error between the exact solution with four term approximations obtained by ILTM of (1.1) for $\beta = 1$

5.2. Approximate solution for modified time fractional Kawahara equation

Here, we consider the modified time fractional Kawahara equation (1.3) with initial condition (1.4).

The exact solution for the classical modified Kawahara equation is given by [33]

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2[M(x-ct)],\ c = \frac{25q-4p^2}{25q}. \tag{5.8}$$

Applying laplace transform on both side of (1.3) we get,

$$L\{v(x,t)\} - \frac{1}{s}v(x,0) + \left(\frac{s+\beta(1-s)}{s}\right)L\left\{v^2\frac{\partial v}{\partial x} + h\frac{\partial^3 v}{\partial x^3} + l\frac{\partial^5 v}{\partial x^5}\right\} = 0. \tag{5.9}$$

Rearranging terms we obtain,

$$L\{v(x,t)\} = \frac{1}{s}\left(\frac{3h}{\sqrt{-10l}}sech^2(Mx)\right) - \left(\frac{s+\beta(1-s)}{s}\right)L\left\{v^2\frac{\partial v}{\partial x} + h\frac{\partial^3 v}{\partial x^3} + l\frac{\partial^5 v}{\partial x^5}\right\} \tag{5.10}$$

Next, the inverse Laplace transform on (5.10), gives

$$v(x,t) = \frac{3h}{\sqrt{-10l}}sech^2(Mx) - L^{-1}\left\{\left(\frac{s+\beta(1-s)}{s}\right)L\left\{v^2\frac{\partial v}{\partial x} + h\frac{\partial^3 v}{\partial x^3} + l\frac{\partial^5 v}{\partial x^5}\right\}\right\} \tag{5.11}$$

The series solution is given as,

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t), \tag{5.12}$$

The nonlinear term $v^2\frac{\partial v}{\partial x}$ is written as $v_n^2\frac{\partial v_n}{\partial x} = \sum_{n=0}^{\infty}\mathbb{J}_n$; whereas $\mathbb{J}_n$ is further decomposed as follows

$$\mathbb{J}_n = \sum_{i=0}^{n} v_i^2 . \frac{\partial}{\partial x}\left(\sum_{i=0}^{n} v_i\right) - \sum_{i=0}^{n-1} v_i^2 . \frac{\partial}{\partial x}\left(\sum_{i=0}^{n-1} v_i\right)$$

by using $v_0(x,\ t) = \frac{3h}{\sqrt{-10l}}sech^2(Kx)$, we get the recursive formula as follows

$$v_n(x,t) = v_0(x,\ t) + L^{-1}\left\{\left(\frac{s+\beta(1-s)}{s}\right)L\left\{v_n^2\frac{\partial v_n}{\partial x} + h\frac{\partial^3 v_n}{\partial x^3} + l\frac{\partial^5 v_n}{\partial x^5}\right\}\right\} \tag{5.13}$$

The n−term approximate solution is given by

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots + v_{n-1}(x,t). \tag{5.14}$$

$$v_0 = \frac{6\sqrt{\frac{2}{5}}he^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}}{\sqrt{-l}\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^2}$$

$$u_1 = -\frac{36\sqrt{2}h^3\sqrt{-\frac{h}{l}}(-\beta+\beta t+1)e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}\left(17e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}-102e^{\frac{2x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}-1\right)}{125(-l)^{3/2}\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^7}$$

$$-\frac{36\sqrt{2}h^3\sqrt{-\frac{h}{l}}(-\beta+\beta t+1)e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}\left(102e^{\frac{3x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}-17e^{\frac{4x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+e^{\sqrt{5}x\sqrt{-\frac{h}{l}}}\right)}{125(-l)^{3/2}\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^7}$$

$$u_2 = \frac{2317248\sqrt{2}h^8\sqrt{-l}\sqrt{-\frac{h}{l}}e^{\frac{4x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}}{78125l^5\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^{17}} - \frac{46656\sqrt{2}h^8\sqrt{-l}\sqrt{-\frac{h}{l}}e^{\frac{3x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}}{78125l^5\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^{17}}$$

$$+\frac{85396032\sqrt{2}h^8\sqrt{-l}\sqrt{-\frac{h}{l}}e^{\frac{6x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}}{15625l^5\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^{17}} - \frac{2084932224\sqrt{2}h^8\sqrt{-l}\sqrt{-\frac{h}{l}}e^{\frac{7x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}}{78125l^5\left(e^{\frac{x\sqrt{-\frac{h}{l}}}{\sqrt{5}}}+1\right)^{17}}$$

$$-7039036728\sqrt{10}\beta^4h^{11}t^4e^{2\sqrt{5}x\sqrt{-\frac{h}{l}}} - 727056\sqrt{10}\beta^4h^{11}t^4e^{\sqrt{5}x\sqrt{-\frac{h}{l}}}$$

$$-123665616\sqrt{10}\beta^4h^{11}t^4e^{3\sqrt{5}x\sqrt{-\frac{h}{l}}} + \cdots.$$

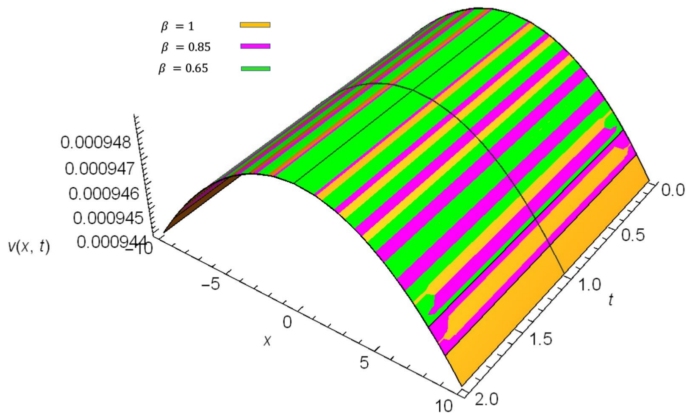

FIGURE 2. Approx. soln of Eq. (1.3), for $\beta = 1, 0.8, 0.6$

t	x	$\beta = 0.55$	$\beta = 0.75$	$\beta = 0.95$	Absolute error $\lvert v_{\text{exact}} - v_{\text{apprx}} \rvert$ for $\beta = 1$
1	-10	0.000943956	0.000943906	0.000943956	9.88185×10^{-7}
	-5	0.000947498	0.000947498	0.000947498	5.20717×10^{-7}
	0	0.000948683	0.000948683	0.000948683	4.74326×10^{-8}
	5	0.000947498	0.000947498	0.000947498	4.26325×10^{-7}
	10	0.000943956	0.000943956	0.000943956	8.95888×10^{-7}
2	-10	0.000941829	0.000941829	0.000941829	2.07022×10^{-6}
	-5	0.000943286	0.000943286	0.000943286	1.13562×10^{-6}
	0	0.000946834	0.000946834	0.000946834	1.89711×10^{-7}
	5	0.000947061	0.000947061	0.000947061	7.58088×10^{-7}
	10	0.000941829	0.000941829	0.000941829	1.69833×10^{-6}

TABLE 2. The numerical results for various values of β and comparison of absolute error between the exact solution with three term approximations obtained by ILTM of (1.3) for $\beta = 1$

Fig. 1 shows surfaces for approximate solution of Eq. (1.1) and exact solution of classical Kawahara equation for $\kappa = 1, 0.85, 0.65$. Fig. 2 shows surfaces for approximate solution of Eq. (1.3) and exact solution of classical modified Kawahara equation for $\kappa = 1, 0.85, 0.65$. It is observed that these surfaces for various values of κ are differes each other and coincides with exact solution as the value of κ approaches towards one. Moreover, from all the plots we can see that the present technique is more accurate and very effective to analyse the considered fractional order differential equations.

In table 1 and 2, we have calculated the numerical values of approximate solution of equations (1.1) and (1.3) respectively with $h = 0.001$ and $l = -1$ for various values of $\kappa = 1, 0.95, 0.75$ and 0.55. Moreover, we have obtained comparison of absolute error between the exact solution with obtained approximate solution obtained by ILTM for each equation. It is seen that This technique provides accurate numerical solutions even if lower order approximations are used.

6. CONCLUSIONS

In this work, iterative Laplace transform method is applied lucratively to obtain the approximate solutions of time fractional Kawahara and modified Kawahara equations based on Caputo-Fabrizio fractional derivative. We have also obtained the stability conditions of approximate solution. The present investigation illuminates the effetiveness of the considered derivative operator. It is seen that the results obtained byiterative Laplace transform method are more stimulating as compared to results available in the literature. We can conclude from the numerical results that this is very simple, reliable and powerful technique for finding approximate solutions of many fractional physical models arise in applied sciences.

ACKNOWLEDGEMENT

The authors are thankful to the referees for their valuable comments.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] D. Baleanu, Z.B. Guvenc, J.A. Tenreiro Machado, New trends in nanotechnology and fractional calculus applications, Springer, Dordrecht Heidelberg, London, New York. (2010).

[2] X.J. Yang, Advanced local fractional calculus and its applications, World Science Publisher, New York, USA, 2012.

[3] R. Hilfer, Applications of fractional calculus in physics, World Scientific, Singapore, 2000.

[4] V. Lakshmikantham, J.V. Devi, Theory of Fractional Differential Equations in a Banach Space, Eur. J. Pure Appl. Math. 1(2008), 38-45.

[5] K.B. Oldham, Fractional differential equations in electrochemistry, Adv. Eng. Softw. 41 (2010), 9-12.

[6] G. Adomian, R. Rach, Modified adomian polynomials, Math. Comput. Model. 24(11) (1996), 39-46.

[7] W. Li, Y. Pang, Application of Adomian decomposition method to nonlinear systems, Adv. Differ. Equ. 2020 (2020), 67.

[8] M.Y. Kokurin, S.I. Piskarev, M. Spreafico, Finite-difference methods for fractional differential equations of order 1/2. J. Math. Sci. 230 (2018), 950–960.

[9] J.H. He, Homotopy perturbation technique, Comp. Meth. Appl. Mech. Eng, 178 (1999), 257-262.

[10] S.J. Liao, Homotopy analysis method and its applications in mathematics, J. Basic Sci. Eng. 5(2) (1997), 111–125.

[11] H.M. Baskonus, H. Bulut, On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth- Moulton method, Open Math. 13(1) (2015), 547–556.

[12] J.H. He, Variational iteration method – a kind of non-linear analytical technique: some examples, Int. J. Nonlinear Mech. 34(4)(1999), 699-708.

[13] B.R. Sontakke, A.S. Shelke, A.S. Shaikh, Solution of non-linear fractional differential equations by variational iteration method and applications, Far East J. Math. Sci. 110(1) (2019), 113-129.

[14] D.B. Dhaigude, B.R. Sontakke, Monotone technique for nonlinear time degenerate weakly coupled system of parabolic problems, Commun. Appl. Ana. 15(1) (2011), 13-24.

[15] J.A. Nanaware, D.B. Dhaigude, Monotone technique for finite system of Caputo fractional differential equations with periodic boundary conditions, Dyn. Contin. Discrete Impulsive Syst. 22 (2015), 13-23.

[16] V. Daftardar-Gejji, H. Jafari, An iterative method for solving non linear functional equations, J. Math. Anal. Appl. 316 (2006), 753-763.

[17] A. Shaikh, A. Tassaddiq, K.S. Nisar, D. Baleanu, Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction–diffusion equations, Adv. Differ. Equ. 2019 (2019), 78.

[18] B.R. Sontakke, A. Shaikh, K.S. Nisar, Approximate solutions of a generalized hirota-satsuma coupled kdv and a coupled mkdv systems with time fractional derivatives, Malays. J. Math. Sci. 12(2) (2018), 175-196.

[19] B.R. Sontakke, A.S. Shaikh, Properties of caputo operator and its applications to linear fractional differential equations, Int. J. Eng. Res. Appl. 5(5) (2015), 22-27.

[20] K.M. Furati, M.D. Kassim, N.E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, Comp. Math. Appl. 64 (2012), 1616-1626.

[21] U.N. Katugampola, A new approach to generalized fractional derivatives, Bull. Math. Anal. Appl. 6(4) (2014), 1–15.

[22] N.A. Sheikh, F. Ali, M. Saqib, et al. Comparison and analysis of the Atangana–Baleanu and Caputo–Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction, Results Phys. 7 (2017), 789-800.

[23] F. Ali, M. Saqib, I. Khan, N. A. Sheikh, Application of Caputo-Fabrizio derivatives to MHD free convection flow of generalized Walters-B fluid model, Eur. Phys. J. Plus. 131 (2016), 377.

[24] N.A. Kudryashov, A note on new exact solutions for the Kawahara equation using Exp. function method, J. Comput. Appl. Math. 234 (2012), 3511-3512.

[25] D. Kumar, J. Singh, D. Baleanu, Modified Kawahara equation within a fractional derivative with non-singular kernel, Thermal Sci. 22 (2017), 789-796.

[26] T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan, 33 (1972) 260.

[27] J. Lu, Analytical approach to Kawahara equation using variational iteration method and homotopy perturbation method, Topol. Meth. Nonlinear Anal. 31 (2008), 287-293.

[28] M. Benbachir, A. Saadi. The Kawahara equation with time and space-fractional derivatives solved by the adomian method, J. Interdiscip. Math. 17(3) (2014), 243-253.

[29] S. Bhatter, A. Mathur, D. Kumar, K.S. Nisar, J. Singh, Fractional modified Kawahara equation with Mittag–Leffler law, Chaos Solitons Fractals, 131 (2020), 109508.

[30] T. Bashiri, S.M. Vaezpour, J.J. Nieto, Approximating Solution of Fabrizio-Caputo Volterra's Model for Population Growth in a Closed System by Homotopy Analysis Method, J. Funct. Spaces, 2018 (2018), Article ID 3152502.

[31] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progress Fract. Differ. Appl. 1(2) (2015), 73-85.

[32] S. Bhalekar, V. Daftardar-Gejji, Convergence of the new iterative method, Int. J. Differ. Equ. 2011 (2011), Article ID 989065.

[33] Sirendaoreji, New exact traveling wave solutions for the Kawahara and modified Kawahara equations, Chaos Solitons Fractals, 19 (2004), 147-150.