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THE MULTI-FUZZY GROUP SPACES ON MULTI-FUZZY SPACE

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Abstract: In this paper the concept of multi-fuzzy space and multi-fuzzy binary operation are presented and developed. This concept generalizes the concept of fuzzy space ([15] and [32]) to multi-membership function. This generalization leads us to introduce and study the approach of multi-fuzzy group theory. A new theory of multi-fuzzy group is introduced and developed.

Keywords: multi-fuzzy space; multi-fuzzy subspaces; multi-fuzzy binary operations; multi-fuzzy group; multi-fuzzy subgroup.

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1. INTRODUCTION

Fuzzy set theory, which was first initiated by Zadeh [22], has become a very important tool to solve many complicated problems arising in the fields of economics, social sciences, engineering,

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medical sciences. A new type of fuzzy set (multi-fuzzy set) was introduced in a paper by Sebastian and Ramakrishnan (Sebastian and Ramakrishnan, [24]) using the ordered sequences of membership function. The notion of multi-fuzzy sets provides a new method to represent some problems that are difficult to explain using other extensions of fuzzy set theory, such as the colour of pixels. Sebastian and Ramakrishnan. (Sebastian and Ramakrishnan, [24], [25]) discussed multi-fuzzy extensions of functions and multi-fuzzy subgroups. The theory of multi-fuzzy sets in terms of multi-dimensional membership functions, which is an extension of theories of fuzzy sets, Atanassov intuitionistic fuzzy sets and L-fuzzy sets. After introducing, multi fuzzy subsets of a crisp set, they have also introduced and studied some elementary properties of multi fuzzy subgroups in Sabu Sebastian and T. V. Ramakrishnan [23].

In the last four decades, the fuzzy group theory was developed as follows. Rosenfeld [17] introduced the concept of fuzzy subgroup of a group. In [35] Negoita and Ralescu considered a generalization of Rosenfeld's definition in which the unit interval [0, l] was replaced by an appropriate lattice structure. In [8] Anthony and Sherwood redefined fuzzy subgroup of a group using the concept of triangular norm previously defined by Schweizer and Sklar [18]. Several mathematicians followed the Rosenfeld-Anthony-Sherwood approach in investigating fuzzy group theory (Anthony & Sherwood [9]; Das [12]; Sessa [19]). Youssef and Dib [14] introduced a new approach to define fuzzy groupoid and fuzzy subgroupoid. In the absence of the concept of fuzzy universal set, formulation of the intrinsic definition for fuzzy group and fuzzy subgroup is not evident. This is considered as a purpose to introduce fuzzy universal set and fuzzy binary operation by Youssef and Dib [14] and Dib [15].

In the case of intuitionistic fuzzy sets Zhan and Tan [31] introduced the concept of intuitionistic fuzzy subgroup as a generalization of Rosenfeld's approach by adding constrains over the non-membership functions. We introduce the notion of intuitionistic fuzzy group based on the notion of intuitionistic fuzzy space and intuitionistic fuzzy function defined by Fathi and Salleh, which will serve as a universal set in the classical case. Fathi and Salleh [32] introduced the notion of intuitionistic fuzzy space which replaces the concept of intuitionistic fuzzy universal set in the ordinary case. The notion of intuitionistic fuzzy group as a generalization of fuzzy group based on fuzzy space by Dib in [15] was introduced by Fathi and Salleh [32].

In particular, the concept of fuzzy set spreads in many mathematical fields, such as multi topology, multi-fuzzy topology fuzzy algebra and multi-fuzzy sets (Lowen. [34], Hungerford. [30], Sebastian and Ramakrishnan, [24]. In this paper, we incorporate two mathematical fields on fuzzy set, which are fuzzy algebra and multi-fuzzy set theory to achieve a new algebraic system, called multi- fuzzy group based on multi-fuzzy space. We will use the approach of Youssef and Dib ([14], [15]) to generalization multi-fuzzy group and multi-fuzzy subgroup on based multi-fuzzy spaces.

Our aim in this study is to incorporate two mathematical fields on multi-fuzzy set, which are multi-fuzzy algebra theory to achieve a new algebraic system, called multi-fuzzy group based on multi-fuzzy space. Having defined multi-fuzzy space and multi-fuzzy binary operation, the concepts of multi-fuzzy group and multi-fuzzy subgroup are introduced and the theory of multi-fuzzy group is developed. The main difference between the Rosenfeld, (Sebastian and Ramakrishnan) approach and that of Youssef and Dib ([14], [15]) is the replacement of the t-norm f with a family of co-membership functions $\{f_{xy} : x, y \in X\}$. Also in this work using Dib's approach. The introduced approach can be considered as a new formulation of the classical theory of multi-fuzzy subgroups, which gives us a powerful tool to develop the multi-fuzzy group theory.

2. PRELIMINARIES

In this section we recall the definitions and related results which are needed in this work.

Definition 2.1 ([22]) A fuzzy set *A* in a universe of discourse *U* is characterized by a membership function $\mu_A(x)$ that takes values in the interval $[0,1]$.

Definition 2.2 ([27]) An intuitionistic fuzzy set *A* in a non-empty set *U* (a universe of discourse) is an object having the form:

$$A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle : x\in U\},$$

where the functions $\mu_A(x):U\to[0,1]$ and $\gamma_A(x):U\to[0,1]$, denote the degree of membership and degree of non-membership of each element $x\in U$ to the set A respectively, and $0\le\mu_A(x)+\gamma_A(x)\le 1$ for all $x\in U$.

Definition 2.3 ([27] the complement, union, and intersection of two IFSs $A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle : x\in U\}$ and $B=\{\langle x,\mu_B(x),\gamma_B(x)\rangle : x\in U\}$ in a universe of discourse U, are defined as follows:

i. $\overline{A}=\{\langle x,\gamma_A(x),\mu_A(x)\rangle : x\in U\}$,

ii. $A\cup B=\{\langle x,\max(\mu_A(x),\mu_B(x)),\min(\gamma_A(x),\gamma_B(x))\rangle : x\in U\}$,

iii. $A\cap B=\{\langle x,\min(\mu_A(x),\mu_B(x)),\max(\gamma_A(x),\gamma_B(x))\rangle : x\in U\}$.

Definition 2.4 ([25] Let k be a positive integer, A multi-fuzzy set A in U is a set of ordered sequences $A=\{u/(\mu_1(u),\mu_2(u),...,\mu_i(u),...,\mu_k(u)) : u\in U\}$, where $\mu_i\in P(U)$, $i=1,2,3,...,k$. The function $\mu_A=(\mu_1,\mu_2,...,\mu_k)$ is called the multi-membership function of multi-fuzzy set A, k is called the dimension of A. The set of all multi-fuzzy sets of dimension k in U is denoted by $M^kFS(U)$.

Definition 2.5 ([25]) Let $A=\{u/(\mu_1(u),\mu_2(u),...,\mu_k(u)) : u\in U\}$ and $B=\{u/(\gamma_1(u),\gamma_2(u),...,\gamma_k(u)) : u\in U\}$ be two multi-fuzzy sets of dimension k in U.

We define the following relations and operations

(1) $A\subseteq B$ iff $\mu_i(u)\le\gamma_i(u),\forall u\in U$ and $1\le i\le k$.

(2) $A=B$ iff $\mu_i(u)=\gamma_i(u),\forall u\in U$ and $1\le i\le k$.

(3) $A\cup B=\{u/(\mu_1(u)\vee\gamma_1(u),\mu_2(u)\vee\gamma_2(u),...,\mu_k(u)\vee\gamma_k(u)) : u\in U\}$.

(4) $A \cap B = \left\{ u / \left(\mu_1(u) \wedge \gamma_1(u), \mu_2(u) \wedge \gamma_2(u), ..., \mu_k(u) \wedge \gamma_k(u) \right) : u \in U \right\}.$

(5) $A^c = \left\{ u / \left(\mu^c{}_1(u), \mu^c{}_2(u), ..., \mu^c{}_k(u) \right) : u \in U \right\}.$

Remark 1 ([25]) A multi-fuzzy set of dimension 1 is a Zadeh's fuzzy set and a multi-fuzzy set of dimension 2 with $\mu_1(u) + \mu_2(u) \leq 1$ is an Atanassov's intuitionistic fuzzy set.

Remark 2 ([25]) If $\sum_{i=1}^{k} \mu_i(u) \leq 1,\ u \in U,$ then the multi-fuzzy set of dimension k is called a normalized multi-fuzzy set. If $\sum_{i=1}^{k} \mu_i(u) = l > 1,$ for some $u \in U.$

Definition 2.6 ([15]) A *fuzzy space* $(X\ ,\ I = [0,1])$ is the set of all ordered pairs $(x\ , I\ ),\ x\ \in X$

$$(X\ , I\ ) = \{(x\ , I\ ) :\ x\ \in X\ \},\ \text{ where }\ (x\ , I\ ) = \{(x\ , r\ ) : r\ \in I\ \}.$$

The ordered pair $(x\ , I\ )$ is called a fuzzy element in the fuzzy space (X, I).

Theorem 2.1 ([15]) *Associated to each fuzzy group* $((X\ , I\ ); \underline{F})$ *there is an ordinary group* $(X\ , F\ )$, *and they are isomorphic to the each other by correspondence* $(x\ , I\ ) \leftrightarrow x$.

Definition 2.7 ([32] Let X be a nonempty set. An *intuitionistic fuzzy space* denoted by $(X\ ,\ I = [0,1]\ ,\ I = [0,1])$ is the set of all ordered triples (x, I, I), where (x, I, I) = {(x, r, s): r, $s \in I$} with $r + s \leq 1$ and $x \in X$. The ordered triple $(x,\ I,\ I)$ is called an *intuitionistic fuzzy element of the intuitionistic fuzzy space* $(X,\ I,\ I)$ and the condition $r, s \in I$ with $r + s \leq 1$.

Definition 2.8 ([32] An *intuitionistic fuzzy groupoid,* denoted by $((X\ , I\ , I\ ), \mathbf{F})$ is an intuitionistic fuzzy space $(X,\ I,\ I)$ together with an intuitionistic fuzzy binary operation $\mathbf{F}$ defined over it.

Theorem 2.2 ([32]) Associated *to each* intuitionistic *fuzzy group* $((X, I, I), \mathbf{F})$ *where* $\mathbf{F} = (F, \underline{f}_{xy}, \overline{f_{xy}})$, *is a fuzzy group* $((X, I), \underline{\mathbf{F}})$ *where* $\underline{\mathbf{F}} = (F, f_{xy})$ *which is isomorphic to the* intuitionistic *fuzzy group* $((X, I, I), \mathbf{F})$ *by the correspondence* $(x, I, I) \leftrightarrow (x, I)$.

Theorem 2.3 ([32]**)** *To each intuitionistic fuzzy group* $((X, I, I), \mathbf{F})$ *there is an associated (ordinary) group* (X, F) *which is isomorphic to the fuzzy group by the correspondence* $(x, I, I) \leftrightarrow x$.

Definition 2.9 ([31] An intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ in a group G having a binary operation $\circ$ is called an *intuitionistic fuzzy subgroup of the group* G if the following conditions hold:

(1) $\mu_A(x \circ y) \geq \min\{A(x), A(y)\}$ and $\nu_A(x \circ y) \leq \max\{A(x), A(y)\}, \forall x, y \in A$,

(2) $\mu(x^{-1}) \geq \mu(x)$ and $\nu(x^{-1}) \leq \nu(x)$ for all $x \in A$.

Definition 2.10 ([24] if G is an ordinary group having the ordinary binary operation $\circ$ and A is a multi-fuzzy subset of G, then A is called a multi-*fuzzy subgroup* of G if and only if

(1) $A(x \circ y) \geq \min\{A(x), A(y)\}$,

(2) $A(x^{-1}) \geq A(x)$, for all $x, y \in G$.

Note: A multi-fuzzy subgroup of dimension 1 is a Rosenfeld's fuzzy subgroup and a multi-fuzzy subgroup of dimension 2 is the Zhan's intuitionistic fuzzy subgroup.

3. MAIN RESULTS

In this section, we generalize the notion of a fuzzy space of ([15] and [32]) to the case of a multi-fuzzy space and discuss its properties. We also discuss some related results.

Definition 3.1 Let X be a non-empty set. A multi-fuzzy space $(X, I_i^X = [0,1])$ is the set of all ordered sequences $(x, I_i^X), x \in X$; where $i \in \mathbb{N}$ that is, $(X, I_i^X) = \{(x, I_i^X) : x \in X\}$, where

$(x, I_i^X) = \{(x, r_i) : r_i \in I_i^X\}$. The ordered sequences (x, I_i^X) is called a multi-fuzzy element in the multi-fuzzy space $(X,\ I_i^X)$.

Therefore, the multi-fuzzy space is an (ordinary) set of ordered sequences. In each ordered sequences the first component indicates the (ordinary) element and the second component indicates a set of possible multi-membership values.

Note: A multi-fuzzy space of dimension 1 is a Dib's fuzzy group and a multi-fuzzy space of dimension 2 is the Fathi's intuitionistic fuzzy group.

Definition 3.2 Let U_0 denote the *support* of U, that is $U_0 = \{x \in X : A_i(x) > 0, \text{where } i \in \mathbb{N}\}$. A multi-*fuzzy subspace* U of the multi-fuzzy space $(X,\ I_i^X)$ is the collection of all ordered sequences $(x,\ u_{(x)_i})$, where $x \in U_0$ for some $U_0 \subset X$ and $u_{(x)_i}$ is a subset of I_i^X which contains at least one element besides the zero element. If it happens that $x \notin U_0$, then $u_{(x)_i} = 0$. An empty multi-fuzzy subspace is defined as $\{(x,\ \varnothing_i(x):\ x \notin U_0\}$.

Remark 3.1 Let $(X,\ I_i^X)$ be a multi- fuzzy space and let A_i be a multi-fuzzy subset of X. Denote by $A_0 = \{x \in X : A_i(x) \neq 0, \text{where } i \in \mathbb{N}\}$ the subset of X containing all the elements with non-zero multi-membership values in A_i *i.e.*, the multi-fuzzy subset A_i induces the following multi-fuzzy subspaces:

1. The lower multi-fuzzy subspace, induced by A_i is given as:

$$\underline{H}(A) = \{(x, , A_i(x))\ : x \in A_o\}$$

2. The upper multi-fuzzy subspace, induced by A_i is given as:

$$\overline{H}(A) = \{(x, \{(0,..,0)\} \cup [A_i(x),1]:\ x \in A_o\},$$

3. The finite multi-fuzzy subspace, induced by A_i is given as:

$$H_o(A) = \{(x, \{(0,...,0), A_i(x)\}),\ x \in A_o\}.$$

By using the definitions of union and intersection of (Sebastian and Ramakrishnan, 2011), we introduce the following definitions:

Definition 3.3 Let $U=\{(x,\ r_i):\ x\in U_o\}$ and $V=\{(x,\ s_i):\ x\in V_o\}$, where $i\in\mathbb{N}$ be multi-fuzzy subspaces of the multi-fuzzy space $(X,\ I_i^X)$. The union $U\cup V$ and the intersection $U\cap V$ of multi-fuzzy subspaces are defined as follows:

$$U\cup V=\{(x,\ r_i\cup s_i):\ x\in U_o\cup V_0\}$$
$$U\cap V=\{(x,\ r_i\cap s_i):\ x\in U_o\cap V_0\}.$$

The support of these multi-fuzzy subspaces satisfies the following:

$$S(U\cup V)=S(U)\cup S(V)=U_0\cup V_0$$
$$S(U\cap V)=S(U)\cap S(V)=U_0\cap V_0.$$

Definition 3.4 The *difference* $U-V$ between the multi-fuzzy subspaces $U=\{(x,\ r_i):\ x\in U_o,\text{where } i\in\mathbb{N}\}$ and $V=\{(x,\ s_i):\ x\in V_o,\text{where } i\in\mathbb{N}\}$ is defined by $U-V=(r_i-s_i)\cup\{0,...,0\}$.

Notice that $S(U-V)\supset U_0-V_0$ and equality holds if $r_i\subset s_i$, $\forall x\in U_0\cup V_0$.

Definition 3.5 The multi-fuzzy subspace $V=\{(x,\ s_i):\ x\in V_o,\text{where } i\in\mathbb{N}\}$ is *contained* in the multi-fuzzy subspace $U=\{(x,\ r_i):\ x\in U_o,\text{where } i\in\mathbb{N}\}$ and denoted by $V\subset U$, if $V_0\subset U_0$, $r_i<s_i$ for all $x\in V_0$, where $i\in\mathbb{N}$.

Notice. It is clear that the empty multi-fuzzy subspace is contained in any multi-fuzzy subspace $U=\{(x,\ r_i):\ x\in U_o,\text{where } i\in\mathbb{N}\}$.

Now, we introduce the Cartesian product of the multi-fuzzy spaces and multi-fuzzy binary operation.

Definition 3.6 The *Cartesian product of the* multi- *fuzzy spaces* $(X,\ I_i^X)$ and $(Y,\ I_i^Y)$ is a multi-fuzzy space, denoted by $(X,\ I_i^X)\boxtimes(Y,\ I_i^Y)$, defined by

$$(X,\ I_i^X)\boxtimes(Y,\ I_i^Y)=(X\times Y,\ I_i^X\times I_i^Y)$$
$$=\{((x,\ y),\ I_i^X\times I_i^Y):\ x\in X\wedge y\in Y\}.$$

Every $I_i^X\times I_i^Y$-multi-fuzzy subset of $X\times Y,\ A_i: X\times Y\to I_i^X\times I_i^Y$, belongs to the multi-fuzzy space $(X\times Y,\ I_i^X\times I_i^Y)$ and $((x,y),\ I_i^X\times I_i^Y)$ is the multi-fuzzy element of this space.

The Cartesian product of the two multi-fuzzy elements $(x,\ I_i^X)$ and $(y,\ I_i^X)$ is defined by

$$(x,\ I_i^X)\boxtimes(y,\ I_i^Y)=((x,\ y),\ I_i^X\times I_i^Y).$$

By using this relation, we can write

$$(x,I_i^X)\times(y,I_i^y)=\{(x,I_i^X)\times(y,I_i^y)\ \ x\in X\ \text{ and }\ y\in Y\}.$$

Definition 3.7 The *Cartesian product of the* multi-*fuzzy subspaces* $U=\{(x,\ r_i):\ x\in U_o, \text{where } i\in\mathbb{N}\}$ and $V=\{(x,\ s_i):\ x\in V_o, \text{where } i\in\mathbb{N}\}$ of the multi-fuzzy spaces $(X,\ I_i^X)$ and $(Y,\ I_i^Y)$, respectively, is a multi-fuzzy subspace of the multi- fuzzy Cartesian product $(X\times Y,\ I_i^X\times I_i^Y)$ which is denoted by $U\boxtimes V$:

$$U\boxtimes V=\{((x,y),\ r_i\times s_i\ ):\ (x,y)\in U_o\times V_o, \text{where i}\in\mathbb{N}\}.$$

Definition 3.8 Let $(X,\ I_i^X)$, $(Y,\ I_i^Y)$ and $(Z,\ I_i^Z)$ be multi-fuzzy spaces**.** The multi-fuzzy function $\underline{F}$ from $(X,\ I_i^X)\boxtimes(Y,\ I_i^Y)=(X\times Y, I_i^X\times I_i^Y)$ into $(Z,\ I_i^Z)$ is defined by the ordered sequences $(F,\{f_{(xy)_i}\}_{(x,y)\in X\times Y})$ where $F:X\times Y\to Z$ is a function and $\{f_i(xy)\}_{(x,y)\in X\times Y}$ is a family of functions $f_{(xy)_i}:I_i^X\times I_i^Y\to I_i^Z$ where $i\in\mathbb{N}$ satisfying the following conditions:

(i) $f_{(xy)_i}$ is non-decreasing on $I_i^X\times I_i^Y$,

(ii) $f_i(0,\dots,0)=(0,\dots,0)$ and $f_i(1,\dots,1)=(1,\dots,1)$.

Such that the image of any $I_i^X\times I_i^Y$ - multi-fuzzy subset C of $X\times Y$ under $\underline{F}$ is the multi-fuzzy subset $\underline{F}(C)$ of Z defined by

$$\underline{F}(C)z = \begin{cases} \bigvee_{(x,y)\in F^{-1}(z)} f_i\big(C(x,y)\big), & \text{if } F^{-1}(z) \neq \varnothing, \\ 0, & \text{if } F^{-1}(z) = \varnothing, \end{cases}$$

for every $z \in Z$.

We write $\underline{F} = (F,\ f_{(xy)_i}) : (X \times Y,\ I_i^X \times I_i^Y) \to (Z,\ I_i^Z)$ and $f_{(xy)_i}$ are called the multi-co-membership functions associated to $\underline{F}$.

$\underline{F} = (F,\ f_{(xy)_i}) : (X,\ I_i^X) \to (Y,\ I_i^Y)$ is written to denote the multi- *fuzzy function* from $(X,\ I_i^X)$ to $(Y,\ I_i^Y)$ and we call the functions $f_{(xy)_i}$, $x, y \in X$ (where $i \in \mathbb{N}$) the multi-co-membership functions associated to $\underline{F}$.

A multi-fuzzy function $\underline{F} = \left(F, f_{(xy)_i}\right)$ is said to be *uniform* if the multi-co-membership functions f_i are identical for all $y, x \in X$.

Now, introducing the notion of multi-fuzzy binary operation and then introduce the concept of multi-fuzzy group.

Definition 3.9 A multi-*fuzzy binary operation* $\underline{F} = (F(x,y),\ f_{(xy)_i})$ on the multi-fuzzy space $(X,\ I_i^X)$ is a multi-fuzzy function from $(X, I_i^X) \times (X,\ I_i^X) \to (X,\ I_i^X)$ with multi-co-membership functions $f_{(xy)_i}$ that satisfy:

(i) $f_{(xy)_i}(r_i, s_i) \neq 0$ if $r_i \neq 0,\ s_i \neq 0$, where $i \in \mathbb{N}$

(ii) $f_{(xy)_i}(r_i,\ s_i)$ are onto, that is, $f_{(xy)_i}(I_i^X \times I_i^X) = I_i^X$, $x, y \in X$.

The multi-fuzzy binary operation $\underline{F} = (F, f_{(xy)_i})$ on a set X is a multi-fuzzy function from $X \times X$ to X and is said to be *uniform* if $\underline{F}$ is a uniform multi-fuzzy function. From now on, the multi-fuzzy space $(X,\ I_i^X)$ is considered.

Definition 3.10 A multi-fuzzy space $(X,\ I_i^X)$ with multi-fuzzy binary operation $\underline{F}=(F,\ f_{(xy)_i})$ is called a multi-*fuzzy groupoid* and is denoted by $((X,\ I_i^X),\ \underline{F}=(F,\ f_{(xy)_i}))$.

For every multi-fuzzy elements $(x,\ I_i^X)$ and $(y,\ I_i^Y)$ can be written

$$(x,\ I_i^X)\boxtimes(y,\ I_i^Y)=((x,y),\ I_i^X\,\square\, I_i^Y),$$

which is a multi-fuzzy element of $(X\times Y, I_i^X\,\square\, I_i^Y)$. The action of $\underline{F}=\ (F, f_{(xy)_i})$ on this multi-fuzzy element is given by:

$$\begin{aligned}(x,\ I_i^X)\ \underline{F}\ (y, I_i^X) &= \underline{F}(((x, I_i^X)\times(y, I_i^X)))\\ &= \underline{F}\ (((x,\ y),\ (I_i^X\ \times\ I_i^X)))\\ &= (F(x,y),\ f_{(xy)_i}(I_i^X\times I_i^X))\\ &= (F(x,y), I_i^X).\end{aligned}$$

If we write $F(x,\ y)=xFy,$ then we get: $(x,\ I_i^X)\ \underline{F}\ (y, I_i^X)=(xFy,\ I_i^X).$

The next theorem gives a correspondence relation between multi-fuzzy groupoid and intuitionistic fuzzy groupoid.

Theorem 3.1 Associated *to each* multi-*fuzzy groupoidis* $\underline{F}=(F,\ f_{(xy)_i})$ *where* $((X,\ I_i^X),\ \underline{F})$ *an* intuitionistic fuzzy *groupoid* $((X, I, I),\mathbf{F})$ *which is isomorphic to the* multi-*fuzzy groupoid* $(X,\ I_i^X)\leftrightarrow(x, I, I)$.*by the correspondence* $((X,\ I_i^X),\ \underline{F})$

Proof. Let $((X,\ I_i^X),\ \underline{F}=(F,\ f_{(xy)_i}))$ be a given multi-fuzzy groupoid. Now redefine $\underline{F}=\ (F, f_{(xy)_i})$ to be $\mathbf{F}=(F,\ f_{(xy)_1}, f_{(xy)_2})$ since $f_{(xy)_1}$ and $f_{(xy)_2}$ satisfies the axioms of intuitionistic fuzzy co-membership function and co-non-membership function $\mathbf{F}=(F,\ f_{(xy)_1}, f_{(xy)_2})$ will be an intuitionistic fuzzy binary operation in the sense of Fathi. That is $((X, I),\ \mathbf{F}=(F,\ f_{(xy)_1}, f_{(xy)_2}))$ is will define an intuitionistic fuzzy groupoid in the sense of Fathi. □

Corollary 3.1 *To each* multi-*fuzzy groupoid* $((X,\ I_i^X),\ \underline{F})$ *there is an associated fuzzy groupoid* $((X,\ I),\ \underline{F}=(F,f_{xy}))$ *which is isomorphic to the fuzzy groupoidby the* $((X,\ I_i^X),\ \underline{F})$ *correspondence* $(X,\ I_i^X)\leftrightarrow(x,I)$.

Corollary 3.2 *To each* multi *fuzzy groupoid* $((X,\ I_i^X),\ \underline{F})$ *there is an associated (ordinary) groupoid* $(X\ ,F)$ *which is isomorphic to the fuzzy groupoidby the* $((X,\ I_i^X),\ \underline{F})$ *correspondence* $(X,\ I_i^X)\leftrightarrow x$.

A multi-fuzzy binary operation $\underline{F}=\ (F,\ f_{(xy)_i})$ on multi-fuzzy spaces (X, I_i^X) is said to be *uniform* if the associated multi-co-membership functions $f_{(xy)_i}$ are identical that is, $f_{(xy)_i}=f_i$ for all $x\ ,\ y\in X$. A multi-fuzzy groupoid $((X,\ I_i^X),\ \underline{F})$ is called a *uniform* multi-*fuzzy groupoid* if $\underline{F}=\ (F,f_{(xy)_i}=f_i)$ is uniform.

Definition 3.11 The ordered sequences $(U,\ \underline{F}=(F,\ f_{(xy)_i}))$ is called a multi-*fuzzy subgroupoid* of the multi-fuzzy groupoid $((X,\ I_i^X),\ \underline{F})$ if the multi-fuzzy subspace $U=\{(x,r_i(x):\ x\in U_o\}$ where $i\in\mathbb{N}$ is closed under the multi-fuzzy binary operation $\underline{F}$.

Having defined the concept of multi-fuzzy groupoid and multi-fuzzy subgroupoid, the concept of multi-fuzzy *monoid* is now defined.

Definition 3.12 A multi-*fuzzy semigroup* is a multi-fuzzy groupoid that is associative. A multi-*fuzzy monoid* is a multi-fuzzy semigroup that admits a multi-fuzzy identity.

After defining the concept of multi-fuzzy groupiod, multi-fuzzy semigroup and multi-fuzzy monoid we now introduce the concept of multi-fuzzy group.

We call ordered sequences $((X,\ I_i^X),\ \underline{F})$ a multi-fuzzy algebraic system or multi-fuzzy algebraic structure.

Definition 3.13 A multi-fuzzy algebraic system $((X,\ I_i^X), \underline{F})$ is called a multi-*fuzzy group* if and only if for every $(x, I_i^X), (y, I_i^X), (z, I_i^X) \in (X,\ I_i^X)$ the following conditions are satisfied:

(i) Associative:

$$((x, I_i^X)\,\underline{F}\,(y, I_i^X))\,\underline{F}\,(z, I_i^X) = (x, I_i^X)\,\underline{F}\,((y, I_i^X)\,\underline{F}\,(z, I_i^X)),$$

$$\text{that is } ((xFy)Fz, I_i^X) = (xF(yFz), I_i^X).$$

(ii) Existence of a multi-fuzzy identity element (e, I_i^X), for which

$$(x, I_i^X)\,\underline{F}\,(e, I_i^X) = (e, I_i^X)\,\underline{F}\,(x, I_i^X) = (x, I_i^X),$$

$$\text{that is } (xFe, I_i^X) = (eFx, I_i^X) = (x, I_i^X).$$

(iii) Every multi-fuzzy element (x, I_i^X) has an inverse (x^{-1}, I_i^X) such that

$$(x, E^2)\,\underline{F}\,(x^{-1}, E^2) = (x^{-1}, E^2)\,\underline{F}\,(x, E^2) = (e, E^2).$$

From (i), (ii) and (iii), it follows that $((X,\ I_i^X), \underline{F})$ is a multi-fuzzy group over the multi-fuzzy space $(X,\ I_i^X)$.

From the preceding discussion, we have the following example.

Example 3.1 Consider the set $\mathbb{Z}_3 = \{0,\ 1,\ 2\}$. Define the multi-fuzzy binary operation $\underline{F} = (F,\ f_{(xy)_i})$, where $i = 1, 2, \ldots, 4$ $(\mathbb{Z}_3 = \{0,\ 1,\ 2\}$ with four dimensions) over the multi-fuzzy space $(\mathbb{Z}_3,\ I_i^X)$ as follows:

$F(x,\ y) = x_{(+_3)_i}\, y$, where, $(+_3)_i$ refers to addition modulo 3 and

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The multi-co-membership functions $f_{(xy)_i}(r_i, s_i) = r_i.s_i$. Then $((\mathbb{Z}_3, I_i^X), \underline{F} = (F, f_{(xy)_i}))$ is a multi-fuzzy group over the multi-fuzzy space $(\mathbb{Z}_3, I_i^X = [0, 1])$.

Theorem 3.2 Associated *to each* multi-*fuzzy group* $((X, I_i^X), \underline{F})$ *where* $\underline{F} = (F, f_{(xy)_i})$ *is an* intuitionistic fuzzy *group* $((X, I, I), \mathbf{F})$ *which is isomorphic to the* multi-*fuzzy group* $((X, I_i^X), \underline{F})$ *by the correspondence* $(X, I_i^X) \leftrightarrow (x, I, I)$.

Proof. The proof is similar to that of Theorem 3.1.

Corollary 3.3 *To each* multi-*fuzzy group* $((X, I_i^X), \underline{F})$ *there is an associated fuzzy group* $((X, I), \underline{F} = (F, f_{xy}))$ *which is isomorphic to the fuzzy group* $((X, I_i^X), \underline{F})$ *by the correspondence* $(X, I_i^X) \leftrightarrow (x, I)$.

Corollary 3.4 *To each* multi *fuzzy group* $((X, I_i^X), \underline{F})$ *there is an associated (ordinary) group* (X, F) *which is isomorphic to the fuzzy group* $((X, I_i^X), \underline{F})$ *by the correspondence* $(X, I_i^X) \leftrightarrow x$.

Definition 3.14 A multi-fuzzy group $((X, I_i^X), \underline{F})$ is called a commutative or abelian multi-*fuzzy group* if $(x, I_i^X)\, \underline{F}\, (y, I_i^X) = (y, I_i^X)\, \underline{F}\, (x, I_i^X)$, for all multi-fuzzy elements (x, I_i^X) and (y, I_i^X) of the multi-fuzzy space (X, I_i^X). It is clear that $((X, I_i^X), \underline{F})$ is a commutative multi-fuzzy group iff (X, F) is a commutative multi-group.

Definition 3.15 Let $((X, I_i^X), \underline{F})$ be a multi-fuzzy group and let $U = \{(x, r_{i(x)}) : x \in U_0\}$ be a multi- fuzzy subspace of (X, I_i^X). $(U, \underline{F})$ is called a multi- *fuzzy subgroup* of the multi-fuzzy group $((X, I_i^X); \underline{F})$ if:

(i) $\underline{F}$ is closed on the multi-fuzzy subspace *U*, *i.e.*,

$$(x, r_{i\,(x)})\, \underline{F}\, (y, r_{i\,(y)}) = \left(xFy,\ f_{(xy)_i}(r_{i\,(x)}, r_{i\,(y)})\right)$$
$$= (xFy, r_{i\,(x)} f_{(xy)_i} r_{i\,(y)})$$

(ii) $(U, \underline{F})$ satisfies the conditions of a multi-*fuzzy group*.

Theorem 3.3 *The subspace* $(U; \underline{F})$ *is a* multi-*fuzzy subgroup of the* multi- *fuzzy group* $\left((X, I_i^X), \underline{F}\right)$ *if and only if:*

(i) (U_0, F) *is a multi-fuzzy subgroup of multi-fuzzy group* $\left((X, I_i^X); \underline{F}\right)$

(ii) $f_{(xy)_i}(r_{i\,(x)}, r_{i\,(y)}) = r_{i\,(x)} f_{(xy)_i} r_{i\,(y)}$.

Proof. Suuppose conditions (i) and (ii) are satisfied. Then

(a) $\underline{F}$ is closed on the multi-fuzzy subspace U. Let $(x, r_{i\,(x)}),\ (y, r_{i\,(y)}) \in U$. Then

$$(x, r_i(x))\underline{F}(y, r_i(y)) = \left(xFy,\ f_{(xy)_i}(r_{i\,(x)}, r_{i\,(y)})\right)$$
$$= \left(xFy,\ r_{i\,(x)} f_{(xy)_i} r_{i\,(y)}\right).$$

(b) $(U, \underline{F})$ satisfies the conditions of an multi-fuzzy group. Let $(x, r_{i\,(x)})$, $(y, r_{i\,(y)}),\ (z, r_{i\,(z)}) \in U$. Then

(b1)
$$\left((x, r_{i\,(x)})\, \underline{F}\, ((y, r_{i\,(y)})\right)\underline{F}(z, r_{i\,(z)}) = \left(xFy,\ r_{i\ (xFy)}\right)\underline{F}\,(z, r_{i\,(z)})$$
$$= \left((xFy)Fz,\ r_i\ ((xFy)Fz)\right)$$
$$= \left(xF(yFz), r_i\ (xF(yFz))\right)$$
$$= (x, r_x)\underline{F}\left(yFz,\ r_i\ (yFz)\right)$$
$$= (x, r_{i\,(x)})\ \underline{F}\left((y, r_{i\,(y)})\ \underline{F}\ (z, r_{i\,(z)})\right).$$

(b2)
$$(x, r_{i\,(x)})\ \underline{F}\ (e, r_{i\,(e)}) = (xFe,\ r_{i\,(xFe)})$$
$$= (x, r_{i\,(x)})$$
$$= (eFx,\ r_{i\,(eFx)})$$
$$= (e, r_{i\,(e)})\underline{F}\ (x, r_{i\,(x)}).$$

(b3) Each $(x,\ r_{i(x)})$ has an inverse $(x^{-1},\ r_{i(x^{-1})})$, since

$$\begin{aligned}(x,\ r_{i(x)})\ \underline{F}\ (x^{-1},\ r_{i(x^{-1})}) &= (xFx^{-1},\ r_i(xFx^{-1}))\\ &= (x^{-1}Fx,\ r_i(xFx^{-1}))\\ &= (e, r_{i(e)})\\ &= (x^{-1},\ r_{i(x^{-1})})\ \underline{F}\ (x,\ r_{i(x)}).\end{aligned}$$

From (a) and (b) we conclude that $(U;\ \underline{F} = (F,\ f_{(xy)_i}))$ is a multi-fuzzy subgroup of multi-fuzzy group $((X, I_i^X),\ \underline{F})$. Conversely if $(U\ ;\ \underline{F})$ is a multi-fuzzy subgroup of multi-fuzzy group $((X, I_i^X),\ \underline{F} = (F,\ f_{(xy)_i}))$ then (i) holds by the associativity. Also the following holds $r_{i(x)} f_{(xy)_i} r_{i(y)} = f_{(xy)_i}(r_{i(x)}, r_{i(y)}) = r_i(xFy)$.

4. CONCLUSION

In this study, we have generalized the study by (Dib 1994) about fuzzy groups to multi-fuzzy groups on multi-fuzzy spaces. The present work generalizes a new algebraic structure by combining three branches of mathematics to get a huge structure carrying several properties which could be gained from the properties of fuzzy sets, algebra and multi-fuzzy theory.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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