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BOOTSTRAPPING ESTIMATORS OF THE SDE PARAMETERS

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Abstract. In this paper, we consider an appropriate bootstrapped version of the estimators of the stochastic differential equation. The theoretical aspects are studied and some examples are given.

Keywords: bootstrapping; continuous autoregressive; stochastic differential equation

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1. Introduction.

The stochastic differential equation (SDE) is a necessary tool for analyzing random phenomena occurring in engineering, finance and physics. The SDE's, however, suffer from existence of some unknown parameters, in practice. There are several perfect methods to estimate these parameters and the limiting distributions of estimators are well studied see Iacus (2008) and references therein. It is known that the bootstrap is a very good method to derive the finite sample approximations of distributions of estimators. In this paper, we consider the bootstrapping the estimators of SDE's parameters.

Consider the one-dimensional SDE $\{x_t\}_{0\leq t\leq 1}$ defined by

$$dx_t = a(t, x_t; \underline{\theta})dt + b(t, x_t; \underline{\theta})dw_t,$$

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with non-random initial value $x_0 = a$, where w_t is standard Brownian motion on (0,1) and $\underline{\theta}$ is the vector of unknown parameters. We suppose that global Lipschitz and linear growth assumptions are hold for $a(t, x; \underline{\theta})$, $b(t, x; \underline{\theta})$, as functions of x, when $\underline{\theta}$ is assumed to be fixed. These assumption guarantee that the solution of above SDE exists and it is unique. Suppose that, using the Euler scheme at $t_i = i/n$, $i = 1, ..., n$ (for some positive integer n), samples $y_i = x_{t_i}$ are generated, i.e.,

$$y_i - y_{i-1} = a(t_{i-1}, y_{i-1}; \underline{\theta})\Delta + \sqrt{\Delta} b(t_{i-1}, y_{i-1}; \underline{\theta}) z_i,$$

where $\Delta = 1/n$ and $z_i = \frac{w_{t_i} - w_{t_{i-1}}}{\sqrt{\Delta}} \overset{iid}{\sim} N(0, 1)$. One can see that the $y_{[nt]+1}$ approximates well, as $n \to \infty$, the distribution of x_t, the solution of SDE. The following theorem states this fact.

Theorem 1. Given SDE $dx_t = a(t, x_t; \underline{\theta})dt + b(t, x_t; \underline{\theta})dw_t$, the Euler solution $y_{[nt]+1}$ converges in distribution to x_t.

Proof. The $y_{[nt]+1}$ plays the role of $\varepsilon(t_i)$ defined in Amano (2005) page 4. Therefore, following Amano (2005), we conclude that $y_{[n\cdot]+1} \Rightarrow x.$ (notation $\Rightarrow$ stands for convergence in distribution).

At the first glance, this theorem is obvious. Since we generate samples y_i from the x_t and we say the distribution of $y_{[nt]+1}$ is close to the distribution of x_t. However, this theorem helps one to understand that from which SDE the discrete samples are generated. We use this fact in the bootstrap cases.

To handle the bootstrap method here, based on estimated residuals, the following three steps are done. The alternative method for bootstrapping is the blockwise bootstrap method (Lahiri, 2003) which isn't considered here.

(a) Estimate the vector of unknown parameters $\underline{\theta}$ using a suitable method and derive the estimated errors $\widehat{z}_i = \frac{y_i - y_{i-1} - a(t_{i-1}, y_{i-1}; \widehat{\underline{\theta}})\Delta}{\sqrt{\Delta} b(t_{i-1}, y_{i-1}; \widehat{\underline{\theta}})}$.

(b) Generate a bootstrapped sample $\{z_1^*, ..., z_n^*\}$ form $\{\widehat{z}_1 - \overline{\widehat{z}}, ..., \widehat{z}_n - \overline{\widehat{z}}\}$ where $\overline{\widehat{z}} = (1/n)\sum_{i=1}^{n} \widehat{z}_i$. Then generate $y_i^*, i = 1, ..., n$ with $y_0^* = a$ by recursive equation

$$y_i^* = y_{i-1}^* + a(t_{i-1}, y_{i-1}^*; \underline{\widehat{\theta}})\Delta + \sqrt{\Delta} b(t_{i-1}, y_{i-1}^*; \underline{\widehat{\theta}}) z_i^*.$$

Then using the estimation method proposed in (a) and samples $y_i^*, i = 1, ..., n$ calculate $\underline{\widehat{\theta}}^*$.

(c) Repeat steps (a) and (b), R times. Then, sample properties of $\underline{\widehat{\theta}}_r^*$, $r = 1, 2, ..., R$ approximates the finite sample behavior of $\underline{\widehat{\theta}}$, see Efron and Tibshirani (1993).

It is seen that for $0 \leq t \leq 1$

$$y_{[nt]}^* = a + (1/n)\sum_{i=1}^{[nt]} a(t_{i-1}, y_{i-1}^*; \underline{\widehat{\theta}}) + n^{-1/2}\sum_{i=1}^{[nt]} b(t_{i-1}, y_{i-1}^*; \underline{\widehat{\theta}}) z_i^*.$$

Let $x_n^*(t) = y_{[nt]}^*$ and $s_n^*(t) = n^{-1/2}\sum_{i=1}^{[nt]} z_i^*$. We have

$$x_n^*(t) = x_n^*(0) + \int_0^t a(s, x_n^*(s); \underline{\widehat{\theta}}) ds + \int_0^t b(s, x_n^*(s); \underline{\widehat{\theta}}) ds_n^*(s).$$

In the following theorem, we obtain the SDE which generates y_i^*.

Theorem 2. Given on observations, as $n \to \infty$, $x_n^*(\cdot) \Rightarrow x^*(\cdot)$, the solution of the following SDE defined by

$$dx_t^* = a(t, x_t^*; \underline{\zeta}) dt + b(t, x_t^*; \underline{\zeta}) dw_t^*,$$

where w_t^* is standard Wiener process on (0,1).

Proof. Following Bickel and freedman (1981), given on observations, $s_n^*(\cdot) \Rightarrow w^*(\cdot)$. Given observations, $\underline{\widehat{\theta}} = \underline{\zeta}$ is non-random. Then, following Amano (2005), we conclude that $x_n^*(\cdot) \Rightarrow x^*(\cdot)$.

We call the above differential equation for x_t^* as the bootstrapped SDE. This equation says that if we replace the $\underline{\theta}$ with $\underline{\widehat{\theta}}$ the bootstrapped SDE is made and using the Euler discretization scheme (or the other methods such as Milstein approach), the bootstrapped samples $x^*_{t_i} = y^*_i, i = 1, 2, ..., n$ are generated. The above argument also says that this approach is equivalent to the usual bootstrap method proposed by steps (a), (b) and (c).

Remark 1. Suppose that $\widehat{\underline{\theta}}$ is consistent for $\underline{\theta}$, then it can be shown that $x^*_n(.) \Rightarrow x_t$, the solution of original SDE. This shows that the proposed bootstrap is asymptotically valid in the scene that $x^*_n(\cdot)$ is a consistent estimate (in probability) of x_t.

Remark 2. Note that our bootstrap method is very similar to parametric bootstrap. For example, to handle the bootstrap method for $N(\theta, \sigma^2)$, we first replace θ and σ^2 with their estimates and then, given data, we generate re-samples form $N(\widehat{\theta}, \widehat{\sigma}^2)$. A same procedure is done in our problem as described in above.

2. Examples

In this section, we consider three examples to show that how our method is done, in applications.

Example 1. Consider the first order continuous autoregressive CAR(1) model as follows, for $0 \leq t \leq 1$,

$$dx_t = \alpha x_t dt + \sigma dw_t,$$

where α is a real negative number and σ is positive. Both of them are unknown. The least square estimation of α is given by

$$\widehat{\alpha} = (\int_0^1 x_t dx_t)/(\int_0^1 x_t^2 dt).$$

For n discrete observations, for instance, $x_{t_1}, ..., x_{t_n}$; $t_i = i/n$; the $\widehat{\alpha}$ is approximated by $(n \sum_{i=2}^n x_{t_{i-1}}(x_{t_i} - x_{t_{i-1}}))/\sum_{i=2}^n x^2_{t_{i-1}}$. An estimate for σ^2 is given by

$$\widehat{\sigma}^2 = (1/n) \sum_{i=1}^n e^2_{t_i},$$

where $e_{t_i} = x_{t_i} - x_{t_{i-1}} - (\widehat{\alpha}/n)x_{t_{i-1}}$ (see Brockwell *et al.* (2007)). By substituting $\widehat{\alpha}$ and $\widehat{\sigma}$ in the original CAR(1) the following boostrapped CAR(1) is obtained

$$dx^*_t = \widehat{\alpha} x^*_t dt + \widehat{\sigma} dw^*_t,$$

where w_t^* is standard Brownian motion on (0,1). Given data, $\widehat{\alpha}$ and $\widehat{\sigma}$ are fixed and by applying the discretization method to the bootstrapped SDE, the bootstrapped samples are extracted. Therefore, all the known applications of bootstrap can be performed here.

Example 2. It is seen that $\max_{0<t<1} w_t \overset{d}{=} |N|$ where N has standard normal distribution $N(0,1)$ and the $\text{argmax}_{0<t<1} w_t$ is distributed as arcsine law. The argmax of a stochastic process is a time point at which the mentioned stochastic process attains its maximum. The question is what are the distributions of $\max_t x_t$ and $\text{argmax}_t x_t$ where x_t is solution of a SDE? Some results can be found in Abundo (2008). Specially, when the SDE involves some unknown parameters, the problem is more difficult. In this case, our method can be applied. Here, we report results of some simulation study. We consider the SDE $dx_t = a(t, x_t; \underline{\theta})dt + b(t, x_t; \underline{\theta})dw_t$ on $t \in [0, 1]$ with deterministic initial value $x_0 = 1$ and $\underline{\theta} = (\theta_1, \theta_2)' = (-0.2, 0.5)$, $(2, 2)$, $(0.2, 0.75)$ for illustration purpose. We let $n = 100$ and the mean (m) and standard deviation (std) of $\max_t x_t$ and $\text{argmax}_t x_t$ is simulated by bootstrap method based on $B = 1000$ replications. The discretization method is Euler scheme. The drift and diffusion coefficients are $a(t, x; \underline{\theta}) = \theta_1 \sqrt{x}$, $b(t, x; \underline{\theta}) = \theta_2 x$ for Table 1 and they are $a(t, x; \underline{\theta}) = \theta_1 x$, $b(t, x; \underline{\theta}) = \theta_2 x$ in Table 2. The parameters are estimated using maximum likelihood procedure.

Table 1: Simulation results, $a(t,x;\underline{\theta})=\theta_1\sqrt{x}$, $b(t,x;\underline{\theta})=\theta_2 x$

(θ_1, θ_2)	$(\widehat{\theta}_1, \widehat{\theta}_2)$	m(arg)	std(arg)	m(max)	std(max)
(-0.2, 0.5)	(-0.18, 0.45)	0.3433	0.3323	1.308	0.3543
(2, 2)	(1.89, 2.05)	0.5482	0.3327	7.192	12.762
(0.2, 0.75)	(0.19, 0.78)	0.4856	0.3419	1.872	1.0916

Table 2: Simulation results, $a(t,x;\underline{\theta})=\theta_1 x$, $b(t,x;\underline{\theta})=\theta_2 x$

(θ_1, θ_2)	$(\widehat{\theta}_1, \widehat{\theta}_2)$	m(arg)	std(arg)	m(max)	std(max)
(-0.2, 0.5)	(-0.21, 0.48)	0.3308	0.3191	1.286	0.3423
(2, 2)	(1.93, 2.15)	0.2974	0.3462	10.249	26.875
(0.2, 0.75)	(0.2, 0.81)	0.4711	0.3533	1.9142	1.1532

Example 3. In Example 2, we mentioned that $\max_{0<t<1} w_t \overset{d}{=} |N|$. Therefore, $(\max_t w_t)^2$ has chi-square distribution with one degree of freedom, $\chi^2_{(1)}$. In this example, using the mentioned bootstrap method, we approximate tail probabilities of cU at which U =$(\max_{0<t<1} x_t)^2$ by tail of a $\chi^2_{(r)}$ distribution where

$$dx_t = \theta_1 x dt + \theta_2 dw_t.$$

Here, c is a positive parameter and the degree of freedom r is a non-integer positive unknown parameter. We first replace (θ_1, θ_2) by their estimates $(\widehat{\theta}_1, \widehat{\theta}_2)$ and dx^*_t is constructed. Then using a bootstrap based on $B = 1000$ replications the mean (μ) and variance (v) of U are approximated. The moment estimates of c and r are $2\mu/v$ and $2\mu^2/v$, respectively. Let $\alpha = 0.85(0.01)0.99$ and q_α denotes the α-th empirical quantile of cU and F_r denotes the cumulative distribution function of $\chi^2_{(r)}$ random variable. If the chi-square approximation fits well then the errors $e(\alpha) = |F_r(q_\alpha) - \alpha|$ should be negligible. The following table gives the values of maximum (max) and median (med) of $e(\alpha)$ over $\alpha = 0.85(0.01)0.99$. It also reports the values of c and r as well as $(\widehat{\theta}_1, \widehat{\theta}_2)$. It is seen that our approximation distribution works well for tail probabilities of cU.

Table 3: Simulation results of chi-square distribution fitting to cU

(θ_1, θ_2)	$(\widehat{\theta}_1, \widehat{\theta}_2)$	max	med	c	r
$(0.2, 0.75)$	$(0.18, 0.83)$	0.00384	0.00155	1.5142	4.753
$(2, 2)$	$(1.93, 2.15)$	0.00801	0.00192	0.0122	1.284

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