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A COMPUTATIONAL METHOD FOR SOLVING A CLASS OF NON-LINEAR SINGULARLY PERTURBED VOLTERRA INTEGRO-DIFFERENTIAL BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper, a computational method is presented for solving a class of singularly perturbed Volterra integro-differential boundary-value problems with a boundary layer at one end. The implemented technique consists of solving two problems which are a reduced problem and a boundary layer correction problem. The Pade' approximation technique is used to satisfy the conditions at infinity. Theoretical and numerical results are presented.

Keywords: Volterra integro-differential equation, singularly perturbed boundary-value problems, boundary layer, boundary layer correction, Pade' approximation.

2000 AMS Subject Classification: 65R20

1. Introduction

Many physical phenomena and engineering problems are governed by mathematical models involving ordinary differential equations with a very small positive parameter multiplying the highest order derivative thus leading to singularly perturbed boundary-value problems.

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For examples; applications related to geophysical fluid dynamics problems, fluid dynamics in particular boundary layer types, studies of edge effect in elastic shells, modeling oceanic and atmospheric circulation, chemical reactors theory, convection diffusion processes, and optimal control; among many other areas of applied mathematics and engineering. For many years; singularly perturbed boundary value problems have drawn the attention of many researchers and practitioners who devised various techniques for their numerical solutions; among them the works in [1]-[5].

The numerical solution of these problems is very challenging as the dependence on the small positive parameter causes the solution to vary very fast over parts of the domain and slowly over others. This creates narrow layers where the solutions of the given problems exhibit abrupt jumps followed by layers where normal behavior of the solutions dominates. This phenomenon requires careful examination of the technique to be used to tackle such problems. Hence; it is important to seek robust and efficient computational methods which will accurately produce the solutions and inherit the properties of the exact solutions of the original problems.

Interests in accurately approximating the solutions of singularly perturbed boundary value problems have been the focus of attention of many scientists. There are numerous special purpose techniques to adequately deal with singularly perturbed boundary value problems; for example the work in [2], [3], [6], [7] and [8]. Most of the attention was given to problems governed by second order differential equations. However efforts have been made to devised special numerical techniques for higher order differential equations for example [9]-[13].

In this paper the numerical solution of a class of non-linear Volterra integro-differential type of singularly perturbed problems is considered; namely

$$-\epsilon y''(x) + u(x,y)y'(x) + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), \tag{1}$$

$$y(0) = \alpha, y(1) = \beta, \tag{2}$$

where $\epsilon > 0$ is a small positive parameter, α, β are given constants, and $u(x,y)$, $v(x,y)$, $f(x)$ are sufficiently smooth functions with respect to the corresponding variables. Additional conditions on the functions $u(x,y)$, $v(x,y)$, and $f(x)$ will be added later in the paper. Motivated by the work in [14], [15], and [16] and the proposed method therein; an extension is presented in this paper with the aim to address a class of nonlinear Volterra integro-differential singularly perturbed equation.

In section 2 of this paper; some relevant analytical results are presented and the maximum principle for the problem in hand is discussed. Section 3; presents the concept underlying the proposed method; namely the concept of reduced problem and boundary layer correction. In section 4; numerical results for a number of examples are presented and discussed.

2. Analytical Results

In this section, three important theorems are presented which are the maximum principle, the stability result, and the uniqueness result. Firstly problem (1)-(2) is transformed into an equivalent problem as follows:

$$Py \quad : \quad = -\epsilon y''(x) + u(x,y)y'(x) + \int_0^x K(x,t)v(t,y)dt = f(x), x \in (0,1), \tag{3}$$

$$y(0) \quad = \quad \alpha, y(1) = \beta. \tag{4}$$

The following conditions are needed in order to guarantee that problem (3)-(4) does not have turning-point problem;

$$-k_2 \quad \geq \quad u(x,y) \geq -k_1 \tag{5}$$

$$0 \quad \geq \quad v(x,y) \geq -k3, \tag{6}$$

$$K(x,t) \quad \geq \quad k_4 \geq 0, \tag{7}$$

for all $x \in [0,1]$, where k_1, k_2, k_3, and k_4 are positive constants and $y \in C^2(0,1) \cup C[0,1]$.

Theorem 2.1. (Maximum Principle). Consider the boundary value problem (3)-(4) with the conditions (5)-(7). Assume that $P\varphi \geq 0$ in $(0,1)$,$\varphi(0) \geq 0$, and $\varphi(1) \geq 0$. Then, $\varphi(x) \geq 0$ in [0,1].

Proof: Assume that the conclusion is false, then $\varphi(x) < 0$ for some $x \in [0,1]$. Thus, $\varphi(x)$ has a local minimum at x_0 for some $x_0 \in (0,1)$. Thus, $\varphi'(x_0) = 0$ and $\varphi''(x_0) > 0$. Simple calculations implies that

$$\begin{aligned} P\varphi(x_0) &= -\epsilon\varphi''(x_0) + u(x_0,\varphi)\varphi'(x_0) + \int_0^{x_0} K(x_0,t)v(t,\varphi)dt \\ &< \int_0^{x_0} K(x_0,t)v(t,\varphi)dt \leq 0. \end{aligned}$$

This is a contradiction. Therefore, $\varphi(x) \geq 0$ in [0,1].

In the next theorem, the stability result is presented.

Theorem 2.2. (Stability Result). Consider the problem (3)-(4) under the conditions (5)-(7) with $u = u(x)$ and $v = v(x)$. If $y(x)$ is a smooth function, then

$$\|y\| = \max\{|y(x)| : x \in [0,1]\} \leq 2a \ \max\left\{|\alpha|, |\beta|, \max_{x\in[0,1]} |Py|\right\}. \tag{8}$$

where $a = 1 + \frac{1}{k_2}$.

Proof: Following [10] and [11], let

$$K_0 = \max\left\{|\alpha|, |\beta|, \max_{x\in[0,1]} |Py|\right\} = \max\left\{|\alpha|, |\beta|, \max_{x\in[0,1]} |f(x)|\right\},$$

and let

$$s^{\pm}(x) = 2a \ K_0(1 - \frac{x}{2}) \pm y(x), x \in [0,1].$$

Simple calculations implies that

$$\begin{aligned} Ps^{\pm} &= -\epsilon(2a \ K_0(1 - \frac{x}{2}) \pm y(x))'' + u(x)(\ 2a \ K_0(1 - \frac{x}{2}) \pm y(x))' + \int_0^{x} K(x,t)v(t)dt \\ &= -a \ K_0 u(x) \pm Py > K_0 \pm Py \geq 0, \forall x \in [0,1]. \end{aligned}$$

Similarly,

$$s_1^{\pm}(0) = 2a \ K_0 \pm \alpha > K_0 \pm \alpha \geq 0$$

and

$$s_1^{\pm}(1) = a\ K_0 \pm \beta > K_0 \pm \beta \geq 0.$$

From Theorem 2.1, one can conclude that $s^{\pm}(x) \geq 0$ for all $x \in [0,1]$. Thus,

$$\|y\| \leq \max_{x\in[0,1]} \left\{2a\ K_0(1-\frac{x}{2})\right\} \leq 2a\ K_0 = 2a\ \max\left\{|\alpha|, |\beta|, \max_{x\in[0,1]} |Py|\right\}, x \in [0,1].$$

Theorem 2.3. (Uniqueness Result). Consider the problem (3)-(4) under the conditions (5)-(7) with $u = u(x)$ and $v = v(x)$. If y_1 and y_2 are two solutions to problem (3)-(4), then $y_1(x) = y_2(x)$ for all $x \in [0,1]$.

Proof: Let $z(x) = y_1(x) - y_2(x)$. Then,

$$\begin{aligned} Pz &= 0, z(0) = 0, z(1) = 0, \\ P(-z) &= 0, -z(0) = 0, -z(1) = 0 \end{aligned}$$

Using Theorem 2.1it follows that $z(x) \geq 0$ and $z(x) \leq 0$ for all $x \in [0,1]$ which means that $y_1(x) = y_2(x)$ for all $x \in [0,1]$, hence; uniqueness os solution.

3. Reduced and Boundary Layer Correction method

In this section, the implemented approach used in this paper to solve problem (1)-(2) is presented. This approach consists of two steps. In the first step, a reduced problem is obtained by setting $\epsilon = 0$ in equation (1) to get

$$u(x, y_1)y_1'(x) + \int_0^x K(x,t)v(t, y_1)dt = f(x), x \in (0,1). \tag{9}$$

Equation (9) is solved with the the following boundary conditions

$$y_1(1) = \beta. \tag{10}$$

On most of the interval, the solution of problem (9)-(10) behaves like the solution of problem (1)-(2). However, there is small interval around $x = 0$ in which the solution of problem (9)-(10) does not agree with the solution of problem (1)-(2). To handle this situation, the boundary layer correction problem is introduced. The stretching transformation

$x = \epsilon s$ is introduced which leads to

$$\frac{dy}{dx} = \frac{1}{\epsilon}\frac{dy}{ds} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{\epsilon^2}\frac{d^2y}{ds^2}.$$

Thus, equation (1) becomes

$$-\epsilon\frac{1}{\epsilon^2}\frac{d^2y}{ds^2} + \frac{1}{\epsilon}u(\epsilon s, y)\frac{dy}{ds} + \int_0^{\epsilon s} K(\epsilon s, t)v(t, y)dt = f(\epsilon s)$$

hence;

$$-\frac{d^2y}{ds^2} + u(\epsilon s, y)\frac{dy}{ds} + \epsilon\int_0^{\epsilon s} K(\epsilon s, t)v(t, y)dt = \epsilon f(\epsilon s) \tag{11}$$

Setting $\epsilon = 0$ in equation (11) implies that

$$-\frac{d^2y}{ds^2} + u(0, y)\frac{dy}{ds} = 0.$$

Since the solution of the reduced problem (9)-(10) does not satisfy the boundary condition at $x = 0$, then solution of the above equation should satisfy it. This means, its solution has the form $y_1(0) + y_2(x)$. Substitute $y(x) = y_1(0) + y_2(x)$ in the above equation to get boundary layer correction equation

$$-\frac{d^2y_2}{ds^2} + u(0, y_1(0) + y_2)\frac{dy_2}{ds} = 0. \tag{12}$$

The solution of equation (1) will be expressed in the form

$$y(x) = y_1(x) + y_2(\frac{x}{\epsilon}), \tag{13}$$

and the boundary conditions (2) must be satisfied by expression (13). When $x = 0$, the condition will be

$$\alpha = y(0) = y_1(0) + y_2(0) \text{ which implies that } y_2(0) = \alpha - y_1(0). \tag{14}$$

When $x = 1$, the condition will be

$$\beta = y(1) = y_1(1) + y_2(\frac{1}{\epsilon}) \text{ which implies } y_2(\frac{1}{\epsilon}) = 0. \tag{15}$$

Since $0 < \epsilon << 1$, condition (15) can be replaced by;

$$\underset{s\to\infty}{Lim} y_2(s) = 0. \tag{16}$$

The condition at infinity above can be replaced by $y_2'(0) = \theta$. To obtain the values of θ, we approximate the solution $y_2(s)$ using the Pade' approximation as a rational function of the form

$$y_2(s) \approx \frac{p(s,\theta)}{q(s,\theta)},$$

where p and q are two polynomials in s. Then equation (15) is solved for θ. In the Pade' approximation, the degree of p is selected to be equal to the degree of q.

4. Numerical results

First, we describe the numerical procedure employed to obtain approximate solution to the reduced problem (9)-(10). The solution of problem (12)-(16) can be found using Mathematica through the following steps:

(1) Approximate the solution of y_2 by the series solution of the form $y_2(s) = \sum_{k=0}^{\infty} a_k s^k$ with the $a_0 = \alpha - y_1(0)$ and $a_1 = \theta$.

(2) Substituting the series solution into equation (12).

(3) Approximating $y_2(s)$ using the Pade' approximation of order $[m, m]$ to obtain $\frac{p(s,\theta)}{q(s,\theta)}$.

(4) To find θ,solve the equation $\underset{s\to\infty}{Lim}\frac{p(s,\theta)}{q(s,\theta)} = 0$.

Next, to find the solution of problem (9)-(10), the interval $[0, 1]$ is discretized with the nodes $x_i = ih,\ h = \frac{1}{n},\ n \in \mathbb{N}$. Let $y_{1,k} \approx y_1(x_k)$ and $u_k = u(x_k, y_{1,k})$ for $k = 0 : n$. Using the backward finite difference method to approximate $y_1'(x_k)$ and using the trapezoidal quadrature rule to approximate the integral $\int_0^{x_k} K(x_k, t)v(t, y_1)dt$, we obtain the following nonlinear system of equations

$$\begin{aligned} &u_k \frac{y_{1,k} - y_{1,k-1}}{h} + \frac{h}{2}\sum_{j=0}^{k-1}\left[K(x_k, x_j)v(x_j, y_j) + K(x_k, x_{j+1})v(x_{j+1}, y_{j+1})\right] \\ &= f(x_k), 1 \le k \le n. \end{aligned} \tag{17}$$

Then Mathematica code is used to solve the nonlinear system (17). In order to assess the accuracy of the technique, several examples are tested. Some of these examples are presented below:

Example 1: Consider the nonlinear singular Volterra integro-differential boundary-value problem

$$-\epsilon y''(x) - 2y'(x) - \int_0^x K(x,t)e^{y(t)}dt = f(x), x \in (0,1),$$

subject to

$$y(0) = y(1) = 0.$$

where $K(x,t) = 1$ and $f(x) = -2\ln(x+1) + \frac{2}{x+1}$. Following the above discussion, we set $\epsilon = 0$ to obtain the following reduced problem

$$-2h_1'(x) - \int_0^x e^{h_1(t)}dt = -2\ln(x+1) + \frac{2}{x+1}, \quad h_1(1) = 0.$$

It can be easily verified that system (17) has the form

$$AY_1 + Be^{Y_1} = F$$

where

$$A = \frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & 0 & -1 & 1 & 0 \\ \vdots & \dots & \dots & 0 & -1 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 \end{bmatrix}, B = \frac{h}{4}\begin{bmatrix} 1 & 1 & 0 & \dots & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & 0 & \dots & 0 \\ 1 & 2 & 2 & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \dots & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \dots & 2 & 2 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 & 2 \end{bmatrix},$$

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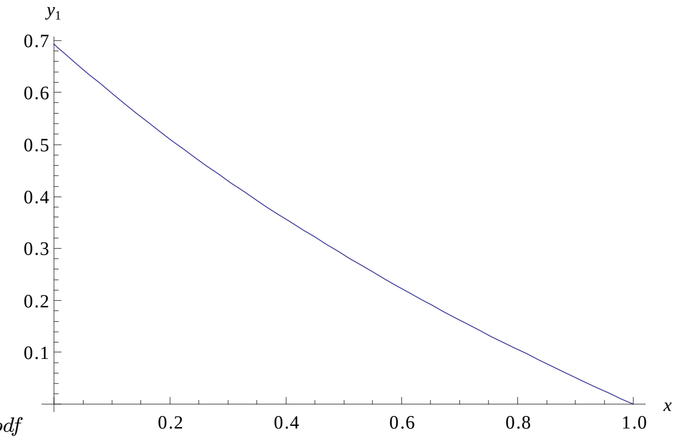

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FIGURE 1. Graph of $y_1(x)$

$$F = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) - \frac{h}{4} \end{bmatrix}, Y_1 = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}.$$

Using Mathematica code, one can see that the solution for the above system is given by Figure 1. It is worth mention that $y_1(0) \approx y_0 = 0.693147$. However, using the stretching transformation $x = \epsilon s$, we have the following boundary layer correction problem

$$-y_2''(s) - 2y_2'(s) = 0, \quad y_2(0) = -y_1(0) = -0.693147, \quad y_2'(0) = \theta.$$

Simple calculations implies that

$$y_2(s) = \frac{\theta - \theta e^{-2s} - 1.38629}{2} \tag{18}$$

Approximating (18) using the Pade' approximation of order $[5, 5]$, we have

$$y_2(w) \approx \bar{y}_2(s) = \frac{p(s,\theta)}{q(s,\theta)},$$

where

$$p(s,\theta) = -0.693147+(\theta-0.693147)x-0.308065x^2+\frac{1}{9}(\theta-0.693147)x^3-0.0110023x^4+\frac{1}{945}(\theta-0.693147)x^5$$

and

$$q(s,\theta) = 1 + x + \frac{4}{9}x^2 + \frac{1}{9}x^3 + \frac{1}{63}x^4 + \frac{1}{945}x^5.$$

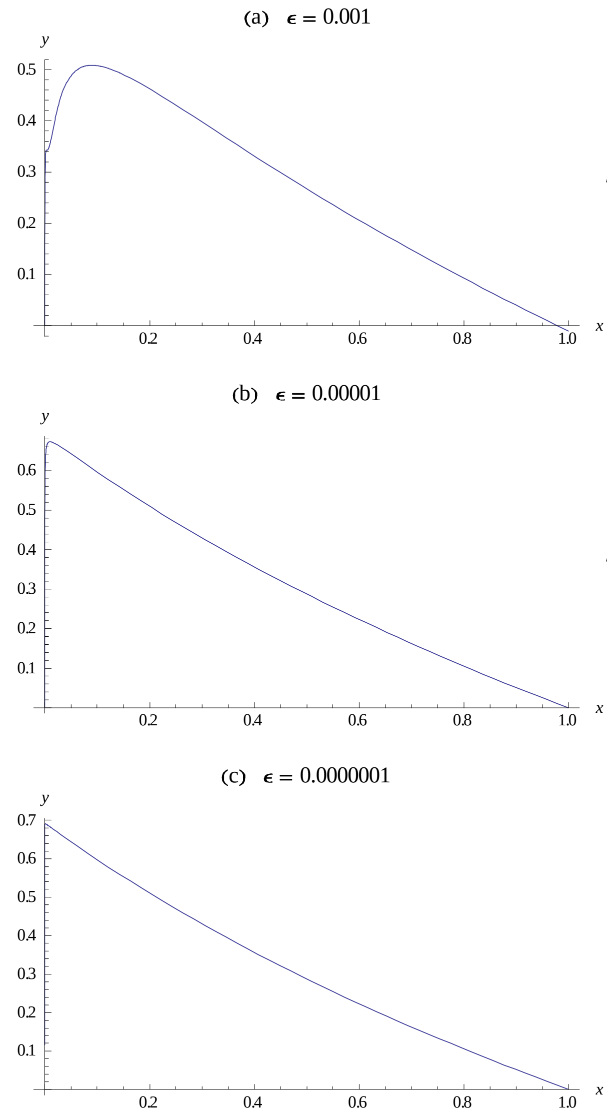

FIGURE 2. $y(x)$ with $\epsilon = 0.001, 0.00001, 0.0000001$

Solving the equation

$$\lim_{s\to\infty} \frac{p(s,\theta)}{q(s,\theta)} = \theta - 0.693147 = 0,$$

we obtain $\theta = 0.693147$.

The graphs of the approximate solutions y for several values of ϵ are displayed in Figure (2). Obviously, the singularity of the solution at $x = 0$ is accurately captured by the present technique.

Example 2. Consider the nonlinear singular Volterra integro-differential boundary-value problem

$$-\epsilon y''(x) - y^2 y'(x) - \int_0^x K(x,t)\,(y(t))^2\,dt = f(x), x \in (0,1),$$

subject to

$$y(0) = -1, y(1) = 6.$$

where $K(x,t) = (x-t)^2$ and $f(x) = -25 - 100x - -x^2 - \frac{25}{3}x^3 - \frac{5}{6}x^4 - \frac{x^5}{30}$.

The reduced problem is

$$-y_1^2 y_1'(x) - \int_0^x K(x,t)\,(y_1(t))^2\,dt = f(x), \quad y_1(1) = 6.$$

It can be easily verified that system (17) has the form

$$y_{1,k}^2 \frac{y_{1,k} - y_{1,k-1}}{h} + \frac{h}{2}\sum_{j=0}^{k-1}\left[r_{k,j}\ y_{1,j}^2 + r_{k,j+1}\ y_{1,j+1}^2\right] = -f(x_k), 1 \le k \le n.$$

where

$$r_{k,j} = (x_k - x_j)^2 = (k-j)^2 h^2.$$

Using a Mathematica code, the solution for the above system is displayed in Figure 3. It is worth mention that $y_1(0) \approx y_0 = 5$. From Figure (4), one can deduce that

$$y_1(x) = x + 5.$$

The boundary layer correction problem is

$$-y_2''(t) - (5+y_2)^2 y_2'(t) = 0, \;\; y_2(0) = -1 - y_1(0) = -6, \;\; y_2'(0) = \theta. \tag{19}$$

To solve problem (19), we approximate the solution of y_2 by the series solution of the form $y_2(x) = \sum_{k=0}^{\infty} a_k t^k$ with the $a_0 = -6$ and $a_1 = \theta$. Substituting the series solution into equation (19) we get

$$\begin{aligned} y_2(s) \;\; = \;\; & -6 + \theta s - \frac{\theta}{2}s^2 + \frac{1}{6}(\theta 2\theta^2)s^3 + \frac{1}{24}(-\theta - 8\theta^2 + 2\theta^3)s^4 + \frac{1}{120}\theta(1 + 22\theta + 30\theta^2)s^5 \\ & + \frac{1}{720}\theta(-1 - 52\theta - 200\theta^2 - 60\theta^3)s^6 + \frac{\theta}{5040}(1 + 114\theta + 964\theta^2 + 1040\theta^3 + 60\theta^4)s^7 + ... \end{aligned}$$

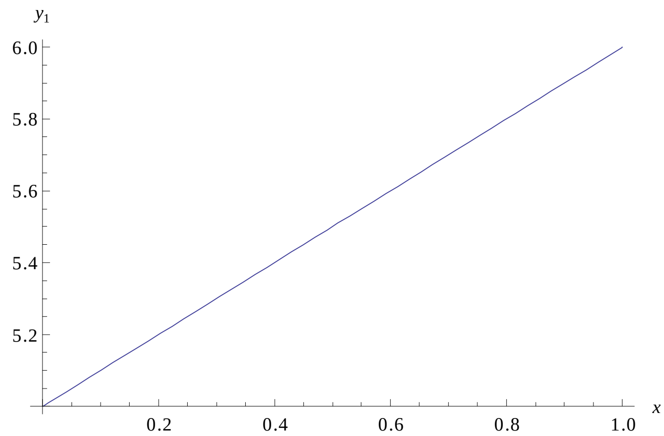

FIGURE 3. Graph of $y_1(x)$

Approximating (19) using the Pade' approximation of order $[3,3]$ and solving the following equation

$$\lim_{s\to\infty}\frac{p(s,\theta)}{q(s,\theta)}=0,$$

we obtain $\theta = 0.405922$. The results for different values of ϵ are given in Figure (4).

Conclusions and remarks:

We will end this section by the following:

(1) The proposed method has accurately depicted the behavior of the solution in the boundary layer.

(2) Numerical results demonstrated satisfactory stability for different small values of the parameter ϵ.

(3) The proposed method is both accurate and efficient.

(4) The proposed method is shown to be an efficient approach in handling the set goal of solving a class of nonlinear singularity perturbed volterra integro-differential boundary value problems.

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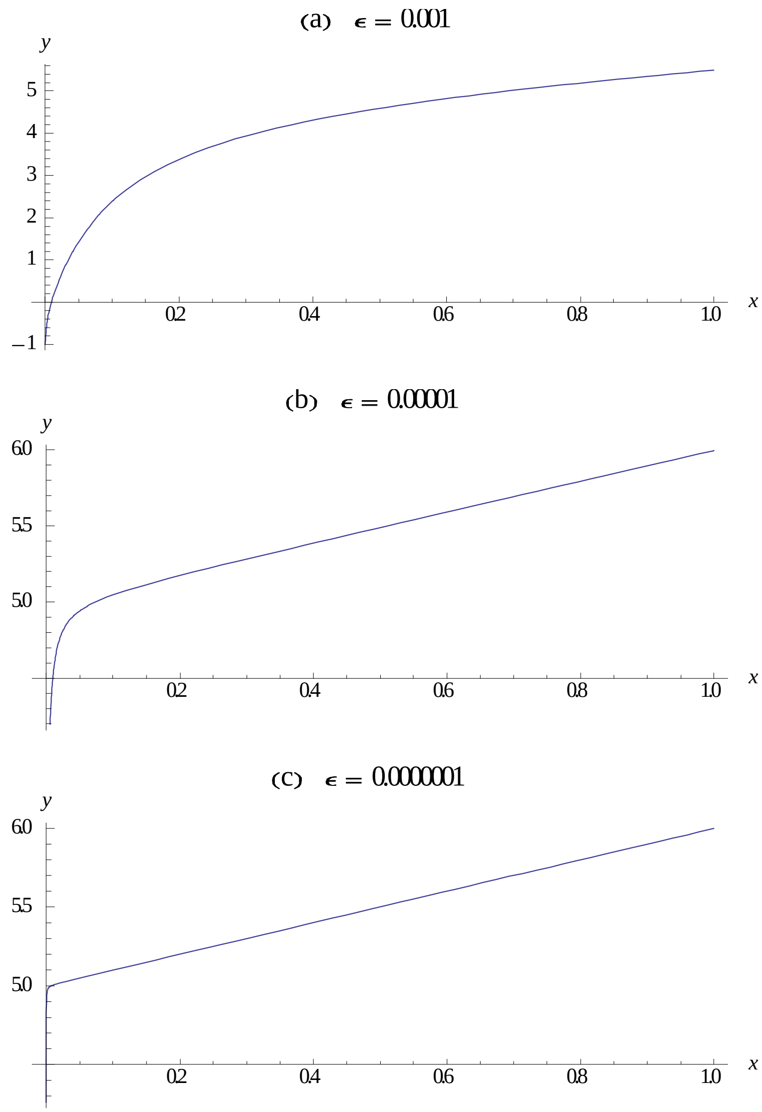

FIGURE 4. $y(x)$ with $\epsilon = 0.001, 0.00001, 0.0000001$

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