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OPTIMIZATION PROBLEM WITH APPLICATION

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Abstract. Optimization problem arises because the resources that are available have varying degree of efficiency for performing different activities. The main objective of this paper is to improve the performance of first year students in their various courses at Lagos State University (LASU). We set to achieve this by assigning each of the first year courses to final year students (as Tutorial Teachers) on the basis of one course per Tutorial Teacher in order to obtain maximum performance. The scores obtained in each of the courses by the Tutorial Teachers would be used as the effectiveness of assigning the courses to them.

Keywords: Optimization; assignment problem; linear programming; hungarian method

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1. Introduction

Optimization is everywhere. It is human nature to seek the best option among all that are available. Allocation processes involve the allocation of resources to activities in such a way that some measure of effectiveness is optimized. These processes arises when

- There are a number of activities to be performed and there are alternative ways of doing them.

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- Resource or facilities are not available for performing each activities in the most effective and efficient way.

The problem then, is to combine activities and resources in such a way as to maximize overall effectiveness. Assignment problem is the special type of Linear Programming, which is one of the standard tools of Operations Research and concern largely with how to allocate limited resources among the various activities of an organization. Linear Programming has been used successfully in the solution of problems concerned with assignment of personnel, blending of materials, distribution and transportation, and investment port-folios [3]. In section 2, we started by given critical analysis of optimization theory. Section 3, is devoted to development and description with detailed algorithm of heuristic method, the Hungarian method for solving assignment problem which provides solution to the problem much faster in most cases than numerical methods. In section 4, we discussed how we developed a computer program based on the algorithm in the previous section and use it to find the maximum effectiveness of allocating year one courses to each of the final year students as Tutorial Teachers whose Cumulative Grade Point Average (CGPA) is not less than 2.5 in each semester of their 100 level results. The scores obtained in each of the of the courses by the Tutorial Teachers would be used as the effectiveness of assigning the courses to them. Section 5 gives the result and conclusions.

2. Analysis of Optimization

Let X be a normed space and f a real valued function defined on a non-empty closed convex subset $\digamma$ of X. The general optimization problem denoted by (P) is to find an element $u \in \digamma$ such that $f(u) \leq f(v)$ for all $v \in \digamma$. If such an element u exists, we then write

$$f(u) = \inf_{v \in \digamma} f(v).$$

When this occurs, we say that f has a minimum at u[4, 6]. If $\digamma \neq X$, we referred to this problem as the *constrained optimization problem*, however, the case $\digamma = X$ is called the *unconstrained optimization problem*.

We now define a set in which a solution exist.

Definition 2.1. Let A be a subset of a normed space X and f a real valued function on A; f is said to have a local or relative minimum, (respectively, relative maximum) at a point $x_0 \in A$ if there is an open sphere $S_r(x_0)$ of X such that $f(x_0) \leq f(x)$ (respectively $f(x) \leq f(x_0)$) holds for all $x \in S_r(x_0) \cap A$. If f has either a relative minimum or relative maximum at x_0, then f is said to have a relative extremum. The set A on which an extremum problem is defined is called the **admissible set**.

We give definition of some useful derivatives as follows:

Definition 2.2. Let X and Y denote Banach spaces over $\mathbb{R}$, and T denotes an operator on X into Y, if x and t are elements of X and

$$\lim_{\epsilon \to \infty} \parallel \frac{T(x+\epsilon) - T(x)}{\epsilon} - DT(x)t \parallel = 0$$

for every $t \in X$, where $\epsilon \to 0$ in $\mathbb{R}$, $DT(x)t \in Y$ is called the value of the Gâteaux derivative of T at x in the direction t, and T is said to be Gâteaux differentiable at x in the direction t. Therefore, the Gâteaux operator T is itself an operator often denoted by $DT(x)$.

Remark 2.3. (a) If T is a linear operator, then $DT(x)t = T(t)$, that is $DT(x) = T$ for all $x \in X$. This is from

$$\lim_{\eta \to 0} \parallel \frac{T(x) + \eta T(t) - T(x)}{\eta} - DT(x)t \parallel = 0 \tag{1}$$

$$\Longrightarrow \lim_{\eta \to 0} \parallel T(t) - DT(x)t \parallel = 0$$

(b) If $T = F$ is a real-valued functional on X; i.e $T : X \to \mathbb{R}$, and F is Gâteaux differentiable at some $x \in X$, then

$$DT(x) = \left[\frac{d}{d\eta} F(x + \eta t) \right]_{\eta = 0}$$

and, for each fixed $x \in X$, $DF(x)t$ is a linear functional of $t \in X$

(c) The Gâteaux derivative is a generalization of the idea of the directional derivative well known in finite dimensions.

The following is the uniqueness theorem of the Gâteaux derivative.

Theorem 2.4. If the Gâteaux derivative of an operator T exist, then it is unique.

Proof: Let two operators $T_1(t)$ and $T_2(t)$ satisfy (1). Then, for every $t \in X$ and every $\eta > 0$ we have

$$\|T_1(t) - T_2(t)\| = \left\| \left(\frac{T(x+\eta t) - T(x)}{\eta} - T_1(t) \right) - \left(\frac{T(x+\eta t) - T(x)}{\eta} - T_2(t) \right) \right\|$$

$$< \left\| \frac{T(x+\eta t) - T_1(t)}{\eta} \right\| + \left\| \frac{T(x+\eta t) - T_2(t)}{\eta} \right\| \to 0 \text{ as } \eta \to 0$$

So that we have $\|T_1(t) - T_2(t)\| = 0$ for all $t \in X$ and this implies $T_1(t) = T_2(t)$ ■

Definition 2.5.(Frêchet Derivative): Let x be a fixed point in a Banach space X and Y be another Banach space. A continuous linear operator $S : X \to Y$ is called the Frêchet derivative of the operator $T : X \to Y$ at x if

$$T(x+t) - T(x) = S(t) + \varphi(x,t) \tag{2}$$

and

$$\lim_{\|t\| \to 0} \frac{\|\varphi(x,t)\|}{\|t\|} = 0 \tag{3}$$

or, equivalently

$$\lim_{\|t\| \to 0} \frac{\|T(x+t) - T(x) - S(t)\|}{\|t\|} = 0 \tag{4}$$

The Frêchet derivative of T at x is usually denoted by $dT(x)$ or $T'(x)$. We say T is Frêchet differentiable on its domain if $dT(x)$ exists at every point of the domain.

Remark 2.6. (a) If $X = \mathbb{R}$, $Y = \mathbb{R}$, then the classical derivative $f'(x)$ of real function $f : \mathbb{R} \to \mathbb{R}$ at x which is defined by

$$f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t} \tag{5}$$

is a number representing the slope of the graph of the function f at x. The Frêchet derivative of f is not a number, but a linear operator on $\mathbb{R}$ into $\mathbb{R}$. The existence of the classical derivative $f'(x)$ implies the existence of the Frêchet derivative at x, and from equations (2) and (5) we can write

$$f(x+t) - f(x) = f'(x)t + tg(t)$$

and we find that S is the operator which multiplies every t by the number $f'(x)$.

(b) The Frêchet derivative gives the best linear approximation of T near x.

(c) Clearly, from equations(2) and (4), we have that if T is linear, then $dT(x) = T(x)$, that is, if T is a linear operator, then the Frêchet derivative (linear approximation) of T is T itself.

Theorem 2.7. If an operator has the Frêchet derivative at a point, then it has the Gâteaux derivative at that point and both derivative have equal values.

Proof: Let $T : X \to Y$, and let T have the Frêchet derivative at x, then

$$\lim_{\|t\|\to 0} \frac{\|T(x+t) - T(x) - S(t)\|}{\|t\|} = 0$$

for some bounded linear operator $S : X \to Y$. In particular for any fixed nonzero $t \in X$, we have

$$\lim_{\eta\to 0} \frac{\|T(x+\eta t) - T(x)}{\eta} - S(\eta t)\|$$

$$= \lim_{\eta\to 0} \frac{\|T(x+\eta t) - T(x) - S(\eta t)\|}{\|\eta t\|}\|t\|$$

This implies that S is the Gâteaux derivative of T at x ■

We now link the derivatives defined above to the optimization problem with the following theorem.

Theorem 2.8. Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable functional at $x_0 \in X$ (where X is normed space) and f have a local extremum at x_0, then $Df(x_0)t = 0$ for all $t \in X$.

Proof: For every $t \in X$, the function $f(x_0 + \alpha t)$ (where α is a real variable) has a local extremum at $\alpha = 0$. Since it is differentiable at 0, it follows from ordinary calculus that

$$\left[\frac{d}{dx} f(x_0 + \alpha t)\right]_{\alpha=0} = 0$$

This means that $Df(x_0)t = 0$ for all $t \in X$, and this concludes the prove.

Remark 2.9. (i) It follows immediately from the above theorem that if a functional $f : X \to \mathbb{R}$ is Frêchet differentiable at $x_0 \in X$ and has a relative extremum at x_0, then $dT(x_0) = 0$.

(ii) Let f be a real-valued functional on a normed space X and x_0 a solution of (P) on a convex set K. If f is Gâteaux differentiable at x_0, then

$$Df(x_0)(x - x_0) > 0 \text{ for all } x \in K.$$

To verify this, we have that since K is a convex set, $x_0 + \alpha(x - x_0) \in K$ for all $\alpha \in (0, 1)$ and $x \in K$. Hence,

$$Df(x_0)(x - x_0) = \left[\frac{d}{d\alpha} f(x_0 + \alpha(x - x_0))\right]_{\alpha=0} \geq 0.$$

We give the statement only of the following theorem:

Theorem 2.10.[4, 5] Let K be a convex subset of a normed space X,
(1) If $J : K \to \mathbb{R}$ is a convex function, then (P) has a solution u whenever J has a local minimum at u.
(2) If $J : O \subset X \to \mathbb{R}$ is a convex function defined over an open subset of X containing K and J if Frêchet differentiable at a point $u \in K$, then J has a minimum at u (i.e u is a solution of (P) on K) if and only if $J'(u)(v - u) \geq 0$ for every $v \in K$.

3. Heuristic Method

This method is credited to the Hungarian Mathematician D. Konig [2, 3]. The method provides solution to assignment problem much faster in most cases than numerical method that are used to solve an appropriate model of the given problem. The procedure is discussed as follows:

3.1. Procedure

The method successively modifies the rows and columns of the effectiveness matrix until there is at least one zero component in each row and column such that a complete assignment corresponding to these zeros can be made. This complete assignment will be an optimal assignment in that when it is applied to the original effectiveness matrix, it will be a minimum. The method will always converge to an optimal assignment in a finite number of steps.

The basis of this method is the fact that a constant can be added to or subtracted from any row or column without changing the set of optimal assignments. For example, if 5 units are subtracted from the ith row and 4 units are added to the jth column, then the objective function in the linear programming model would be

$$
\begin{aligned}
Minimize: \qquad Z &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - 5\sum_{j=1}^{n} x_{ij} + 4\sum_{i=1}^{n} x_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - 5 + 4 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - 1
\end{aligned}
$$

Since $\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ij} = 1$ (from the constraints). Adding a constant to or subtracting a constant from the objective function does not change the optimal solution, since every basic feasible solution would have the same amount added to or subtracted from the objective function.

Generally, if a_i is subtracted from each element in the ith row of the effectiveness matrix and b_j is subtracted from each element in the jth column for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. The new objective function would then be

$$
\begin{aligned}
Minimize: \qquad Z &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{n} a_i \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} b_j \sum_{i=1}^{n} x_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{n} a_i - \sum_{j=1}^{n} b_j
\end{aligned}
$$

where again subtracting the constants $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} b_j$ from the original objective function does not change the set of basic feasible solutions.

Now, we develop an algorithm for the method as follows:

3.2. Algorithm

The following is a step-by-step algorithm that uses the Hungarian method to solve the general n-resources, n-activity assignment problem for the optimum total effectiveness [1].

Step 1: If the total effectiveness is to be maximized, change the sign of each element in the effectiveness matrix and go to step 2; otherwise, go directly to step 2.

Step 2: If the minimum element in row i is not zero, then subtract this minimum element from each element in row i $(i = 1, 2, \ldots, n)$.

Step 3: If the minimum element in column j is not zero, then subtract this minimum element from each element in column j $(j = 1, 2, \ldots, n)$.

Step 4: Examine rows successively, beginning with row 1, for a row with exactly one unbold zero. If at least one exists, bold this zero to denote an assignment. Cross out (×) the 0 other zeros in the same column so that additional assignments will not be made to that column (activity). Repeat the process until each row has no unbold zeros or at least two unbold zeros.

Step 5: Examine the columns successively for single, unbold zeros and bold them to denote an assignment. Cross out (×) the other zeros in the same row so that corresponding resource will not be assigned to other activities. Repeat the process until each column has no unbold zeros or has at least two unbold zeros.

Step 6: Repeat steps 4 and 5 successively (if necessary) until one of these three things occurs:

(a) Every row has an assignment.

(b) There are at least two unbold zeros in each row and each column.

(c) There are no zeros left unbold and a complete assignment has not been made.

Step 7: If (a) occurs, the assignment is complete and it is an optimal assignment. If (b) occurs, arbitrarily make an assignment to one of the zeros and cross out all of the zeros in the same row and column, and then go to step 4. If (c) occurs, go to step 8.

Step 8: Check all rows for which assignments have not been made.

Step 9: Check columns not already checked which have a zero in checked rows.

Step 10: Check rows not already checked which have assignments in checked columns.

Step 11: Repeat steps 9 and 10 until the chain of checking ends.

Step 12: Draw lines through all unchecked rows and through all checked columns. This will necessarily give the minimum number of lines needed to cover each zero at least one time.

Step 13: Examine the elements that do not have at least one line through them. Select the smallest of these and subtract it from every element in each row that contains at least one uncovered element. Add the same element to every element in each column that has a vertical line through it. Return to step 4.

We now illustrate the method with the following example:

3.3. Example

Suppose four equal partners (namely; Ayo, Biodun, Curew and David) operate a consulting business. They currently have four projects (denoted by E, F, G, H) in progress which they are considering completing. The partners estimate their profit (in million dollars) from completing the projects as follows:

Projects

	E	F	G	H
Ayo	40	48	42	44
Biodun	38	43	36	36
Curew	47	49	46	48
David	41	45	39	41

We want to schedule the projects such that the profit to the partnership is maximized. Since we want to maximize the total profit of the partnership, we change the sign of element of the effectiveness matrix and then minimize the resulting matrix. So, we have

Projects

	E	F	G	H
Ayo	-40	-48	-42	-44
Biodun	-38	-43	-36	-36
Curew	-47	-49	-46	-48
David	-41	-45	-39	-41

We now subtract the smallest entry in each row from the other elements in that row to obtained the Table (3.5).

Projects	E	F	G	H
Ayo	8	0	6	4
Biodun	5	0	7	7
Curew	2	0	3	1
David	4	0	6	4

Table 3.5 : Effectiveness Matrix after row reduction.

Now, subtract the smallest entry in each column from the other elements in that column to obtain the effectiveness matrix in Table (3.6).

Projects	E	F	G	H
Ayo	6	0	3	3
Biodun	3	0	4	6
Curew	0	0	0	0
David	2	0	3	3

Table 3.6 : Effectiveness Matrix after column reduction.

Next, we test for the number of independent zeroes.

$$\begin{bmatrix} 6 & 0 & 3 & 3 \\ 3 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 3 \end{bmatrix}$$

As the minimum number lines needed to cover all the zeros is 2, which is two less than the size of the matrix, the smallest uncovered element is located (2), and subtracted from each uncovered element and added to each double covered element.

$$\begin{bmatrix} 4 & 0 & 1 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Since the minimum number of lines needed to cover all the zeros is still less than the size of the matrix, we located the smallest uncovered element (1) and repeat the above procedure, to obtain

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 0 & 1 & 3 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the effectiveness matrix is now optimal since the minimum number of lines equal to the matrix size. But, there are two alternative solutions that allow for zeros to be allocated as solutions. These are

$$\begin{bmatrix} 4 & 0 & \mathbf{0} & 0 \\ 1 & \mathbf{0} & 1 & 3 \\ 1 & 3 & 0 & \mathbf{0} \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix} \texttt{ and } \begin{bmatrix} 4 & 0 & 0 & \mathbf{0} \\ 1 & \mathbf{0} & 1 & 3 \\ 1 & 3 & \mathbf{0} & 0 \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix}$$

Hence, we have the following assignment in Tables (3.7) and (3.8)

	Project	Profit
Ayo	G	42
Biodun	F	43
Curew	H	48
David	E	41

Table 3.7: First assignment option of Project to Partner

	Project	Profit
Ayo	H	44
Biodun	F	43
Curew	G	46
David	E	41

Table 3.8: Second assignment option of Project to Partner

with a maximum profit of 174 million dollars each in both assignments.

4. The Case Study

In this section, we wish to allocate effectively and efficiently each of the fourteen 100 level courses taken by the students to each final year students (as Tutorial Teachers) whose Cumulative Grade Point Average (CGPA) is not less than 2.5 in each semester of their 100 level results at Lagos State University (LASU). In order to achieve our aim, we developed a computer program based on the algorithm of the method discussed in section 3. The code of the program is available from the authors.

4.1. Allocation of Courses

First, we develop a computer program for the algorithm discussed in section 3 to solves the N-resources, N-activity optimization problem. The problem can be either maximization or minimization, but the elements in the effectiveness matrix must be integers.

The program is written into parts. Part I inputs and outputs the data. Part II is a subroutine called WANDE that actually carries out the algorithm.

To increase the maximum size of the effectiveness matrix to N resources and N activities, change the DIMENSION statement in the mainline program to DIMENSION MAT(N,N), IASMAT(N) and also change the components of the variables in the DIMENSION statement in the subroutine WANDE to N.

If the original matrix is not square (either more resources than activities or vice versa), the necessary rows or columns of zeros must be supplied before using the program.

Now, the scores obtained by each Tutorial Teacher in each course out of 100 marks are shown in Table (4.1). Since there are more courses than Tutorial Teachers, two dummy Tutorial Teachers are added to the Table (4.1) and the corresponding scores are zero for each course. Table (4.2) shows the new scores. The idea is that each course has an equal opportunity to be assigned to Tutorials Teachers 13 and 14 so that each course can be effectively and efficiently assigned.

	COURSE CODE													
Tutorial Teacher	MAT 101	MAT 111	PHY 101	PHY 103	PHY 105	CHM 101	GNS 101	MAT 112	MAT 162	PHY 102	PHY 104	PHY 106	CHM 102	GNS 102
1	56	62	57	53	34	43	54	37	30	40	42	25	27	58
2	52	88	69	42	41	45	48	65	62	68	48	57	58	46
3	30	72	67	46	20	53	50	64	52	47	38	58	41	48
4	40	63	47	49	53	58	54	43	54	46	23	32	48	57
5	45	67	49	45	51	47	58	46	63	59	30	42	27	54
6	63	64	47	44	30	52	64	51	24	54	40	41	53	44
7	52	94	76	51	32	46	53	66	48	54	43	35	44	63
8	53	85	48	40	46	63	61	68	56	41	21	43	52	54
9	62	73	46	52	52	58	54	67	51	63	41	23	30	49
10	60	74	70	43	56	54	57	72	53	42	38	63	44	47
11	50	82	68	44	42	41	54	58	33	47	22	40	49	43
12	54	77	63	41	36	52	44	53	67	43	30	31	40	47

Table 4.1 : Scores obtained by Tutorial Teachers.

	COURSE CODE													
Tutorial Teacher	MAT 101	MAT 111	PHY 101	PHY 103	PHY 105	CHM 101	GNS 101	MAT 112	MAT 162	PHY 102	PHY 104	PHY 106	CHM 102	GNS 102
1	56	62	57	53	34	43	54	37	30	40	42	25	27	58
2	52	88	69	42	41	45	48	65	62	68	48	57	58	46
3	30	72	67	46	20	53	50	64	52	47	38	58	41	48
4	40	63	47	49	53	58	54	43	54	46	23	32	48	57
5	45	67	49	45	51	47	58	46	63	59	30	42	27	54
6	63	64	47	44	30	52	64	51	24	54	40	41	53	44
7	52	94	76	51	32	46	53	66	48	54	43	35	44	63
8	53	85	48	40	46	63	61	68	56	41	21	43	52	54
9	62	73	46	52	52	58	54	67	51	63	41	23	30	49
10	60	74	70	43	56	54	57	72	53	42	38	63	44	47
11	50	82	68	44	42	41	54	58	33	47	22	40	49	43
12	54	77	63	41	36	52	44	53	67	43	30	31	40	47
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4.2: New Scores

5. Result and Conclusions

By using the computer program on the effectiveness matrix in Table (4.2), we have this optimal assignment:

Tutorial Teacher	Course Code
1	PHY 103
2	PHY 102
3	PHY 106
4	GNS 102
5	PHY 105
6	GNS 101
7	MAT 111
8	CHM 101
9	MAT 101
10	MAT 112
11	PHY 101
12	MAT 162
13	PHY 104
14	CHM 102

with a maximum score of 777.

In conclusion, assignment problem is one of the fundamental optimization problems in mathematics. The Hungarian method of assignment provides us with an efficient and effective way of finding the optimal solution without having to make a direct comparison of every solution. The technique is a vital tool that can be used in solving allocation problems as demonstrated above in the allocation of courses for maximum performance. Hence, in the end, the Tutorial Teachers supplement the available number of Graduate Assistants in the department and thus limit the problems faced by first year (100 Level) students in their various courses and therefore improve their performance.

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