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ON SOME COMMON FIXED POINT RESULTS IN MENGER SPACES

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Abstract: In this paper, we establish common fixed point theorems for four self-maps and five self maps of Menger space using different contractive conditions and also deduce some consequences.

Keywords: common fixed point; Menger space; weakly compatible mappings; occasionally weakly compatible mappings.

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1. INTRODUCTION

Menger space is a generalization of metric space in which distribution functions are used instead of nonnegative real numbers as value of metric. K. Menger [6] introduced the notion of probabilistic metric space and studied some properties of it. A Menger space is a space in which the concept of distance is considered to be a probabilistic, rather than deterministic. For more details of Menger spaces, we refer to [9,10]. The theory of Menger space is fundamental importance in probabilistic functional analysis.

In 1986, Jungck [2] introduced the concept of compatible mappings in metric spaces. Later, Mishra [7] extended this concept to probabilistic metric spaces. This concept was further weakened by Jungck and Rhoades [3, 4] by introducing weakly compatible mappings. Recently,

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Al-Thagafi and Shahzad [1] weakened the notion of weakly compatible maps by introducing occasionally weakly compatible maps and proved some fixed point results for these mappings. The purpose of this paper is to prove common fixed point theorems for four maps and five maps using compatibility and occasionally weakly compatibility in Menger spaces.

2. PRELIMINARIES

We begin with

Definition 2.1[10]: A probabilistic metric space, shortly PM-Space, is an ordered pair (X, M) consists of a non empty set X and a map M from $X \times X$ to L, where L is the collection of all distribution functions .The value of M at $(u, v) \in X \times X$ is represented by $M_{u,v}$. The function $\mathrm{M}_{u,v}$ is assumed to satisfy the following conditions;

(a) $\mathrm{M}_{u,v}(x) = 1$ for all $x > 0$, iff $u = v$;

(b) $M_{u,v}$ (x) $= 0$, if $x = 0$;

(c) $\mathrm{M}_{u,v}(x) = M_{v,u}(x)$;

(d) if $\mathrm{M}_{u,v}(x) = 1$ and $\mathrm{M}_{v,w}(y) = 1$ then $\mathrm{M}_{u,w}(x + y) = 1$.

Definition 2.2[10]: A mapping $*$ or $t: [0,1] \times [0,1] \to [0,1]$ is a t-norm, if it satisfies the following conditions:

(a) $t(a, 1) = a$ for every $a \in [0,1]$;

(b) $t(0, 0) = 0$,

(c) $t(a, b) = t(b, a)$ for every $a, b \in [0,1]$;

(d) $t(c, d) \geq t(a, b)$for $c \geq a$ and $d \geq b$

(e) $t(t(a, b), c) = t(a, t(b, c))$ where $a, b, c, d \in [0,1]$.

Example 2.3: t(a, b) = min{a, b} is a t-norm.

Definition 2.4[10] : A Menger space is a triplet (X, M, t), where (X, M) is a PM-Space, X is a non-empty set and a t-norm satisfying $M_{u,w}(x + y) \geq t\left(M_{u,v}(x), M_{v,w}(y)\right)$ for all $x, y \geq 0$.

Example 2.5[10]: Let $X = [0, \infty)$ and d be the usual metric on X.For each t ∈ [0, 1],define

$M_{u,v}(t) = \frac{t}{t+|u-v|}$ if $t > 0$ and $M_{u,v}(t) = 0$ if $t = 0$. Then (X, M, t) is a Menger space.

Definition 2.6[7]: Two self mappings A and B of a Menger space (X, M, $*$) are said to be compatible if $M_{ABx_m,BAx_n,}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to x, Bx_n \to x$ for some x in X as $n \to \infty$.

Definition 2.7 [5]: Let (X, M,$*$) be a Menger space and A, B be self maps of X. A point x ∈ X is called a coincidence point of A and B if and only if Ax = Bx. In this case w = Ax = Bx is called a point of coincidence of A and B.

Definition 2.7[5]: Two self mappings A and B of a Menger space (X, M, $*$) are said to be weakly compatible if they commute at coincidence point.

Definition 2.8 [5,8]: Two self mappings A and B of a Menger space (X, M, $*$) are said to be occasionally weakly compatible if there is a point x ∈ X , a coincidence point of A and B at which A and B commute.

Example 2.9. Let (X, M, $*$) be a Menger space, where X = R and $M_{u,v}(t) = \frac{t}{t+|u-v|}$ if $t > 0$ and $M_{u,v}(t) = 0$ if $t = 0$. Define A, B : R → R by Av = 2v and Bv = v^2 for all v ∈ R. Then '0' is a coincidence point of A and B and also AB(0) = BA(0). Hence A and B are OWC maps.

Lemma 2.10[5]. Let A and B be occasionally weakly compatible self maps of a Menger space (X, M, $*$) . Suppose A and B have a unique point of coincidence, then w is the unique common fixed point of A and B.

3. MAIN RESULTS

We first prove a common fixed point theorem for five self mappings in a complete Menger space.

Theorem 3.1: Let A, B, S, T and P be self maps on a complete Menger space $(X, \mathrm{M}, *)$ with continuous t norm $*$ and $t * t \geq t$ for all $t \in [0,1]$, such that

3.1.1 $P(X) \subseteq AB(X), P(X) \subseteq ST(X)$;

3.1.2 there exists a constant $k \in (0,1)$ such that

$M_{Px,Py}(kt) \geq M_{ABx,Px}(t) * M_{Px,STy}(t) * M_{ABx,STy}(t) * \frac{M_{Px,ABx}(t)*M_{Px,STy}(t)}{M_{STy,ABx}(t)} *$

$M_{ABx,Py}(3-\alpha)t$ for all $x, y \in X, \alpha \in (0,3)$ and $t > 0$,

3.1.3 $PB = BP, PT = TP, AB = BA$ and $ST = TS$,

3.1.4 A and B are continuous, and

3.1.5 the pair (P, AB) is compatible. Then A, B, S, T and P have a unique common fixed point in X.

Proof: Since $P(X) \subset AB(X)$, for $x_0 \in X$, we can choose a point $x_0 \in X$ such that $Px_0 = ABx_1$. Since $P(X) \subset ST(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Px_1 = STx_2$. Thus by induction, we can define a sequence $y_n \in X$ as follows; $y_{2n} = Px_{2n} = ABx_{2n+1}$ and $y_{2n+1} = Px_{2n+1} = STx_{2n+1}$ for $n = 1,2, ...$from 3.1.2 ,

Now for $t > 0$ and $\alpha = 2 - q$ with $q \in (0,2)$, we have $M_{y_{2n+1},y_{2n+2}}(kt) =$

$M_{Px_{2n+1},Px_{2n+2}}(kt) \geq M_{y_{2n+1},y_{2n+1}}(t) * M_{y_{2n},y_{2n+1}}(t)$

$$* M_{y_{2n},y_{2n+1}}(t) * \frac{M_{y_{2n+1},y_{2n}}(t)*M_{y_{2n+1},y_{2n+1}}(t)}{M_{y_{2n+1},y_{2n}}(t)} * M_{y_{2n},y_{2n+2}}(1+q)t$$

$M_{y_{2n+1},y_{2n+2}}(kt) \geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n},y_{2n+2}}(1+q)t$

$\geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n+1},y_{2n+2}}(qt)$

$\geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n+1},y_{2n+2}}$(t) as $q \to 1$.

Since $*$ is continuous and $M_{x,y}(*)$ is continuous, letting $q \to 1$ in above equation, we get

$M_{y_{2n+1},y_{2n+2}}(kt) \geq M_{y_{2n},y_{2n+1}}(t) * M_{y_{2n+1},y_{2n+2}}(t)$ (1)

Similarly, we have

$M_{y_{2n+2},y_{2n+3}}(kt) \geq M_{y_{2n+1},y_{2n+2}}(t) * M_{y_{2n+2},y_{2n+2}}(t)$ (2)

Thus from (1) and (2), it follows that

$$M_{y_{n+1},y_{n+2}}(kt) \geq M_{y_n,y_{n+1}}(t) * M_{y_{n+1},y_{n+2}}(t) \text{ for } n = 1,2,...$$

and then for any positive integers n and p,we have

$$M_{y_{n+1},y_{n+2}}(kt) \geq M_{y_n,y_{n+1}}(t) * M_{y_{n+1},y_{n+2}}\left(\frac{t}{k^p}\right).$$

Thus, since $M_{y_{n+1},y_{n+1}}\left(\frac{t}{k^p}\right) \to 1$ as $p \to \infty$ we have $M_{y_{n+1},y_{n+2}}(kt) \geq M_{y_n,y_{n+1}}(t)$.

This shows that $\{y_n\}$ is Cauchy sequence in X and since X is complete, the sequence converges to a point $z \in X$. Since Px_n, ABx_{2n+1} and STx_{2n+2} are subsequences of $\{y_n\}$, they also converge to the point z. Now since A, B are continuous and pair $\{P, AB\}$ is compatible and also weak compatible, we have $lim_{n\to\infty} PABx_{2n+1} = ABz$ and $lim_{n\to\infty}(AB)^2 x_{2n+1} = ABz$. From 3.1.2 with $\alpha = 2$, we get

$$M_{PABx_{2n+1},\ Px_{2n+2}}(kt) \geq M_{(AB)^2x_{2n+1},\ STx_{2n+2}}(t) * M_{PABx_{2n+1},\ STx_{2n+2}}(t)$$

$$* M_{(AB)^2x_{2n+1},\ STx_{2n+2}}(t) * \frac{M_{PABx_{2n+1},(AB)^2 x_{2n+1}}(t) * M_{PABx_{2n+1},\ STx_{2n+2}}(t)}{M_{STx_{2n+2},\ (AB)^2 x_{2n+1}}(t)}$$

$$* M_{(AB)^2x_{2n+1},\ Px_{2n+2}}(t)$$ which implies that

$$M_{ABz,z}(kt) = lim_{n\to\infty} M_{PABx_{2n+2}}(kt)$$

$$\geq 1 * M_{ABz,z}(t) * M_{ABz,z}(t) * \frac{1 * M_{ABz,z}(t)}{M_{z,ABz}(t)} * M_{ABz,z,z}(t).$$

Thus, we have $ABz = z$, and $STz = z$, since $M_{z,STz}(t) \geq M_{z,ABz}(t) = 1$ for all $t > 0$. Again by 3.1.2 with $\alpha = 2$, we have

$$M_{PABx_{2n+1},\ Pz}(kt) \geq M_{(AB)^2x_{2n+1},\ PABx_{2n+1}}(t) * M_{PABx_{2n+1},\ STz}(t)$$

$$* M_{(AB)^2x_{2n+1},\ STz}(t) * \frac{M_{PABx_{2n+1},(AB)^2 x_{2n+1}}(t) * M_{PABx_{2n+1},\ STz}(t)}{M_{STz,(AB)^2 x_{2n+1}}(t)} * M_{(AB)^2x_{2n+1}\ Pz}(t)$$

which implies that $M_{ABz,Pz,Pz}(kt) = lim_{n\to\infty} M_{PABx_{2n+1},\ Pz}(kt)$

$$\geq 1 * 1 * 1 * 1 * M_{ABz,Pz}(t)$$

$$\geq M_{ABz,Pz}(t).$$

Thus,we have $ABz = Pz$. Now, we show that $Bz = z$.Infact, from 3.1.2 with $\alpha = 2$, and 3.1.3 we get, $M_{Bz,z}(kt) = M_{BPz,Pz}(kt)$

$$= M_{PBz,Pz}(kt)$$

$$\geq M_{PBz,STz}(t) * M_{ABBz,STz}(t)$$

$$* \frac{M_{PBz,\ ABBz}(t) * M_{PBz,z,z}(t)}{M_{z,PBz}(t)} * M_{PBz,z}(t)$$

$$= 1 * M_{Bz,z}(t) * M_{Bz,z}(t) * 1 * M_{Bz,z}(t)$$

$= M_{Bz,z}(t)$,which implies that $Bz = z$.

Since $ABz = z$, we have $Az = z$. Next, we show that $Tz = z$. Indeed from 3.1.2 with $\alpha = 2$, and 3.1.3 we get $M_{Tz,z}\ (kt) = M_{TPz,Pz}\,(kt)\ = M_{Pz,Pz}\ (kt)$

$$\geq 1 * M_{z,Tz}\ (t) * \ M_{z,Tz}\ (t) * 1 * M_{z,Tz}\ (t)$$

$$\geq M_{Tz,z}\ (t).$$

which implies that $Tz = z$. Since $STz = z$, we have $Sz = STz = z$.Therefore, by combining the above results we obtain,$Az = Bz = Sz = Tz = Pz$ showing that z is a common fixed point of A, B, S, T and P. Finally, the uniqueness of the fixed point of A, B, S, T and P follows from 3.1.2 .

We now prove some common fixed point theorems using occasionally weakly compatible mappings in Menger space.

Theorem 3.2: Let $(X, M,*)$ be a Menger space and let A, B, S and T be self-mappings of X. Let the pairs $(\mathrm{A,S})$ and $(\mathrm{B,T})$ be OWC. If there exists a point $k \in (0,1)$, for all $x, y \in X$ and $t > 0$, such that

3.2.1 $M_{(Ax,By)}(kt) \geq \alpha_1\ min\{ M_{(Sx,Ty)}(t), M_{(Sx,Ax)}(t)\} + \alpha_2\, min\{ M_{(By,Ty)}(t), M_{(Ax,Ty)}(t)\}$

$+\alpha_3\ M_{(By,Sx)}(t)$ where $\alpha_1, \alpha_2, \alpha_3 > 0$, and $(\alpha_1 +\ \alpha_2 +\ \alpha_3) > 1$.

Then there exists a unique point of $w \in\ X$, such that $Aw = Sw = w$ and a unique point $z \in X$, such that $Bz = Tz = z$. Moreover, $z = w$, and there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs $(\mathrm{A,S})$ and $(\mathrm{B,T})$ be OWC. So there exist $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. From 3.2.1 ,we have

$$M_{(Ax,By)}(kt) \geq \alpha_1\ min\{ M_{(Sx,Ty)}(t), M_{(Sx,Ax)}(t)\}\ +\alpha_2\, min\{ M_{(By,Ty)}(t), M_{(Ax,Ty)}(t)\}$$

$$+\ \alpha_3\ M_{(By,Sx)}(t)$$

$$= \alpha_1\, min\{ M_{(Ax,By)}(t), M_{(Ax,Ax)}(t)\}\ +\alpha_2\ min\{M_{(By,By)}(t), M_{(Ax,By)}(t)\}$$

$$+\alpha_3 M_{(By,Ax)}(t)$$

$$= \alpha_1\, min\{ M_{(Ax,By)}(t), 1\} + \alpha_2\ min\{1, M_{(Ax,By)}(t)\}+\alpha_3 M_{(By,Ax)}(t)$$

$$= (\alpha_1 + \alpha_2 + \alpha_3)\ M_{(Ax,By)}(t)$$,a contradiction, since $(\alpha_1 + \alpha_2 + \alpha_3)\ > 1$.

Therefore $Ax = By, i.e.,\ Ax = Sx = By = Ty$. Suppose that there is a another point z such that

$Az = Sz$ .Then by 3.2.1 we have $Az = Sz = By = Ty,$ so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of $A\ and\ S$. Using Lemma 2.10, we get w is the only common fixed point of A and $S.i.e., w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Assume that $w \neq z$. We have

$$M_{(w,z)}(kt) = M_{(Aw,Bz)}(kt)$$
$$\geq \alpha_1 min\{M_{(Sw,Tz)}(t), M_{(Sw,Az)}(t)\} + \alpha_2 min\,\{M_{(Bz,Tz)}(t), M_{(Aw,Tz)}(t)\}$$
$$+\alpha_3 M_{(Bz,Sw)}(t)$$
$$= \alpha_1 min\{M_{(w,z)}(t), M_{(w,z)}(t)\} + \alpha_2\, min\{M_{(z,z)}(t), M_{(w,z)}(t)\} +$$
$$\alpha_3 M_{(z,w)}(t)$$
$$= \alpha_1 min\{M_{(w,z)}(t), 1\} + \alpha_2\, min\{1, M_{(w,z)}(t)\} + \alpha_3 M_{(z,w)}(t)$$
$$= \alpha_1 M_{(w,z)}(t) + \alpha_2\, M_{(w,z)}(t) + \alpha_3\, M_{(z,w)}(t)$$
$$= (\alpha_1 + \alpha_2 + \alpha_3)\, M_{(w,z)}(t), \text{ a contradiction, since } (\alpha_1 + \alpha_2 + \alpha_3) > 1.$$

Therefore we have $z = w$ also z is a common fixed point of A, B, S and T. The uniqueness of the fixed point follows from 3.2.1.

If we put A = B and S = T in the above Theorem, we get

Corollary 3.3: Let $(X, M,*)$ be a Menger space and let A and S be self-mappings of X. Let the pair (A, S) be OWC. If there exists a point $k \in (0,1)$, for all $x, y \in X$ and $t > 0$, such that

$$M_{(Ax,Ay)}(kt) \geq \alpha_1\ min\{\,M_{(Sx,Sy)}(t), M_{(Sx,Ax)}(t)\} + \alpha_2\, min\{\,M_{(Ay,Sy)}(t), M_{(Ax,Sy)}(t)\}$$
$$+\alpha_3\, M_{(Ay,Sx)}(t) \text{ where } \alpha_1, \alpha_2, \alpha_3 > 0, \text{ and } (\alpha_1 +\ \alpha_2 +\ \alpha_3) > 1.$$

Then the mappings A, S have a unique common fixed point.

Theorem 3.4: Let $(X, M,*)$ be a Menger space and let A, B, S and T be self-mappings of X. Let the pairs $(A, S)$and (B, T) be OWC. If there exists a point $k \in (0,1)$, $\forall\ x, y \in X$ and $t > 0$ such that

$$3.4.1 \quad M_{(Ax,By)}(kt) \geq min \left\{ \begin{matrix} M_{(Sx,Ty)}(t), M_{(Sx,Ax)}(t), M_{(By,Ty)}(t), \\ [M_{(Ax,Ty)}(t) +\ M_{(By,Sx)}\ (t)] \end{matrix} \right\}.$$

Then there exists a unique point of $w \in\ X$, such that $Aw = Sw = w$ and a unique point

$z \in X,$ such that $Bz = Tz = z$. Moreover, $z = w,$ and there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs (A, S) and (B, T) be OWC.So there exist $x, y \in X$ such that $Ax = Sx$ and $By = Ty.$ We claim that $Ax = By$. From inequality 3.4.1 , we have

$$M_{(Ax,By)}(kt) \geq min\left\{\begin{matrix} M_{(Sx,Ty)}(t), M_{(Sx,Ax)}(t), M_{(By,Ty)}(t), \\ \left[M_{(Ax,Ty)}(t) + M_{(By,Sx)}(t)\right] \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} M_{(Ax,By)}(t), M_{(Ax,Ax)}(t), M_{(By,By)}(t), \\ \left[M_{(Ax,By)}(t) + M_{(By,Ax)}(t)\right] \end{matrix}\right\}$$

$$= M_{(Ax,By)}(t).$$

Thus we have $Ax = By, i.e., Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ .Then by 3.4.1 ,we have $Az = Sz = By = Ty,$ so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S.

Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Using Lemma 6.1.9, we get w is the only common fixed point of A and S.

Assume that $w \neq z$.

From 3.4.1 we have $M_{(w,z)}(kt) = M_{(Aw,Bw)}(kt)$

$$\geq min\left\{\begin{matrix} M_{(Sw,Tz)}(t), M_{(Sw,Az)}(t), M_{(Bz,Tz)}(t), \\ \left[M_{(Aw,Tz)}(t) + M_{(Bz,Sw)}(t)\right] \end{matrix}\right\}$$

$$= min\left\{\begin{matrix} M_{(w,z)}(t), M_{(w,z)}(t), M_{(z,z)}(t), \\ \left[M_{(w,z)}(t) + M_{(z,w)}(t)\right] \end{matrix}\right\}$$

$$= M_{(w,z)}(t).$$

Therefore, we have $z = w$ and by Lemma 2.10 we get that z is a common fixed point of A, B, S and T.

The uniqueness of the fixed point holds from 3.4.1.

Corollary 3.5: Let $(X, M, *)$ be a Menger space and let A, B, S and T be self-mappings of X. Let the pairs (A, S) and (B, T) be OWC. If there exists a point $k \in (0,1)$, for all $x, y \in X$ such that

3.5.1 $$M_{(Ax,By)}(kt) \geq \delta\left(min\left\{\begin{matrix} M_{(Sx,Ty)}(t), M_{(Sx,Ax)}(t), M_{(By,Ty)}(t), \\ \left[M_{(Ax,Ty)}(t) + M_{(By,Sx)}(t)\right] \end{matrix}\right\}\right)$$

where $\delta(t) > t$ for all $0 < t < 1$, and $\delta: [0,1] \rightarrow [0,1]$. Then there exists a unique common fixed point of A, B, S and T.

Proof: The proof follows from Theorem 3.4.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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