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IMPACT OF SUBSTITUTION COST ON OPTIMAL ORDER QUANTITY FOR SUBSTITUTABLE AND COMPLEMENTARY PRODUCTS WITH QUANTITY DISCOUNT UNDER QUADRATIC DEMAND

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Abstract: In this article, impact of substitution cost on obtaining optimal ordering quantities for substitutable and complementary products has been studied. Product 1 and 2 are mutually substitutable and each product comprises of two complementary components. The phenomenon of quantity discount is considered in this model. When a product becomes out of stock, partial substitution has been carried out to fulfil the demand of the other product. The demand of both products is assumed to be quadratic. Mathematical model is formulated and a solution a procedure is suggested to obtain the optimal ordering quantities. Optimal total cost for all possible cases has been discussed in this paper. Numerical example is presented and sensitivity analysis is carried out extensively to prove the viability of the proposed inventory model. Substantial improvement in the optimal total cost with substitution over without substitution is observed.

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1. INTRODUCTION

In supermarkets it is common to see, the unavailability of the customer's desired product. In this situation, the behavior of the customers that is very commonly seen is willingness to purchase a substitute product whenever their desired product is out of stock. Some customers may switch to another store to check for the availability of their desired product. Many researchers, study the former behavior of the customers. The demand of a product is met by the other mutually substitutable product. Two products which could serve the same purpose as the other product does is called as substitutable products. Moreover, they are identical in means of similarity and utility in the eyes of the customer. For example, two different soft drinks like Pepsi and coke, tea and coffee, butter and margarine, printed book and kindle etc. In an inventory system, when one product becomes out of stock, the demand of it is fulfilled from the inventory of other product. Substitution can be done in two ways, complete substitution and partial substitution. Most of the researchers have developed an inventory system consisting of two or more substitutable product and identified that only partial substitution of demand of the products bring forth enormous profit compared to complete substitution. Whenever, substitution takes place, an additional cost is incurred called as substitution cost. Complementary product is a product, which is composed of two or more components, packed together to be utilized for customers. For example, pencils and notebooks, burger and burger buns, DVD and DVD player, mobile phones and sim cards etc. The utility of the products will not be completed without both the components. For instance, the DVD is useless without the DVD player, a customer must buy a DVD player in order to make use of the DVD. Many examples are available for substitutable product composed of complementary components. Different brands of tennis balls and different brands of tennis rackets, different brand of mobile phones with different sim cards, different brands of toothpaste and different brands of tooth brush etc. Quantity discount is offered to the retailer in order to encourage them to buy products in bulk. The retailer also gets benefited as the unit price of the quantity is decreased when ordered in large quantities. This phenomenon of quantity discount is applied in the proposed model.

Many researchers have contributed in area of inventory model with product substitution, Benkherouf et al [1] developed an inventory model with product substitution and found an optimal order schedule minimizing the total cost. Whereas, Chen, Y. et al [2] studied an inventory model by considering partial substitution with shortages allowed, results indicated that partial substitution decreases the total expected cost. Most of the articles considered the EOQ framework and extended the idea of substitution of products, which are mutually substitutable. Drezner et al [3] considered an EOQ model with two substitutable products and showed that partial substitution or no substitution may be optimal because of the non-linearity of the total cost equation. The demand of the product is assumed to be quadratic in the articles by Durga, B. K., & Chandrasekaran, E. [4] in which they developed a model with two substitutable product under quadratic demand. Also, Durga, B. K., & Chandrasekaran, E. [5] showed the impact of substitution when obtaining the optimal ordering policy for an inventory system consisting of two substitutable products. Very few researchers have considered articleas on both substitutable and complementary products in EOQ framework. In particular, Durga, B. K., & Chandrasekaran, E. [6] developed a model with two products which are mutually substitutable and complementary products under quadratic demand. Edalatpour et al [7] presented a model based on multi product consisting of complementary and substitutable items. In this paper demand is assumed as a function of price sensitive. The inventory model is also considered in game theory approach by some of the researchers such as Giri et al [8], in which they considered two substitutable product and one complementary product and solved using game theoretical approach. Most articles study about the stock –out based substitution, like Goyal, S. K. [9] proposed a model with substitution between products and Gurnani, H., & Drezner, Z. [10] extended his work and considered a model with multiple products and determined the optimal ordering policy and substitutable quantities. Krommyda et al [11] studied about stock out based demand and product substitution to maximize the profit function. Maddah et al [12] formulated a joint replenishment model in EOQ framework for multiple products under product substitution. Mcgillivray, R., & Silver, E. [13] investigated a generalized inventory model with effect of product substitution by considering unit variable cost and shortage cost. The feature of quantity discount is considered byMishra, V. K. et al [14] in his article where he studied the aspect of substitution of product, quantity discount and substitution cost. Yet another important work on substitution and complementary product is done by Mokhtari, H. [15] in which he extended the classical approach of EOQ by considering the situation of some product being complement with others. Some of the other contribution in this area is as follows, Pan et al [16] shows that the visibility of the products stock level brings positive impact and increases the demand for a product. Salameh et al [17] combines the works on substitution and joint replenishment model under EOQ framework. Taleizadeh et al [18] discussed the effect of deterioration in the combination of complementary and substitutable products. Tang et al [19] in their model adopted fixed and variable pricing strategy to obtain the optimal ordering quantity. Transchel, S [20] examined a stochastic inventory model to determine the demand function in the stock-out based substitution. Wei et al [21] two complementary products is studied is under game theoretical approach. Yue, X. et al [22] complementary goods as a mixed bundle is studied.

In this article, we consider an inventory system with two products. Product 1 consists of two complementary components and product 2 also comprises of two complementary components. The inventory is depleted by its own demand. The demand of the products are assumed to be quadratic. When a product becomes out of stock, the demand of it is partially met from the inventory of product 2 with an additional cost of substitution. The unmet demand is assumed to be lost. The phenomenon of Quantity discount is utilized. The main objective of this model is to find an optimal ordering policy for the proposed model.

The rest of the article is organized as follows, Section 2 points to the notation and assumptions used throughout the model. Section 3 describes the formulation of the proposed model. Section 4 a solution procedure is suggested for the proposed model. In section 5, sensitivity analysis is done extensively to show the behavior of the model and finally concluded with some useful remarks in section 6.

2. NOTATIONS AND ASSUMPTIONS

The proposed model uses the following notations and assumptions

2.1 Notations

k_i	Ordering cost of i^{th} product, where $i=1,2$
D_i	Demand rate of i^{th} product, where $i=1,2$
h_i	Holding cost of i^{th} product, where $i=1,2$
a_i	Usage rates of complementary components of product 1, where $i=1,2$
a_j	Usage rates of complementary components of product 2, where $j=3,4$
c_{s_i}	Unit shortage cost of i^{th} product, where $i=1,2$
δ_{ij}	Unit substitution cost of i^{th} product when it is substituted by j^{th} product
T_i	Cycle time of i^{th} product, where $i=1,2$
t	Substitution period
q_i	Ordering quantities of complementary components of i^{th} product, where $i=1,2$
q_j	Ordering quantities of complementary components of j^{th} product, where $j=3,4$
ν_i	Substitution rate of i^{th} product by j^{th} product, where $i=1,2$
TC_i	Total cost in scenario 1 and 2 (with substitution)
TC_{wos}	Total cost in scenario 3 (without substitution)
TUC_i	Total cost per unit time in scenario 1 and 2 (with substitution)
TUC_{wos}	Total cost per unit time in scenario 3 (without substitution)

2.2 Assumptions:

- The inventory system consists of two products. Each product consists of two complementary components, which are mutually substitutable.

- The inventory of the products depletes by its own demand, which is assumed to be quadratic.
- During the stock out, partial substitution between products takes place with a rate of substitution and a substitution cost associated with it.
- Lead time is zero and replenishment is instantaneous.
- The distributor offer the following discount based on quantity to the retailer is given below.

Product 1				Product 2			
Component α_1	Unit price of α_1 (c_{1i})	Component α_2	Unit price of α_2 (c_{2i})	Component β_1	Unit price of β_1 (d_{1j})	Component β_2	Unit price of β_2 (d_{2j})
$0 \le q_1 \le q_1^1$	c_{11}	$0 \le q_2 \le q_2^1$	c_{21}	$0 \le q_3 \le q_3^1$	d_{11}	$0 \le q_4 \le q_4^1$	d_{21}
$q_1^1 \le q_1 \le q_1^2$	c_{12}	$q_2^1 \le q_2 \le q_2^2$	c_{22}	$q_3^1 \le q_3 \le q_3^2$	d_{12}	$q_4^1 \le q_4 \le q_4^2$	d_{22}
$q_1^2 \le q_1 \le q_1^3$	c_{13}	$q_2^2 \le q_2 \le q_2^3$	c_{23}	$q_3^2 \le q_3 \le q_3^3$	d_{13}	$q_4^2 \le q_4 \le q_4^3$	d_{23}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$q_1^{n-1} \le q_1 \le q_1^n$	c_{1n}	$q_2^{m-1} \le q_2 \le q_2^m$	c_{2m}	$q_3^{r-1} \le q_3 \le q_3^r$	d_{1r}	$q_4^{s-1} \le q_4 \le q_4^s$	d_{2s}

3. FORMULATION OF MODEL

Consider an inventory system consisting of two products which are mutually substitutable. Product 1 comprises of two complementary components α_1 and α_2 .Specifically, product 1 include a_1 units of component α_1 and a_2 units of component α_2, where a_1 and a_2 are the utilization rates of the complementary components α_1 and α_2. Product 2 consists of two complementary components β_1 and β_2 with utilization rates a_3 and a_4 respectively. The demand rates of the products are assumed to be quadratic. When product 1 depletes, the further demand of it is met from the inventory of product 2 partially, the unmet demand is assumed to be lost and vice-versa. Whenever substitution takes place, there incurs an additional cost called substitution cost. In this model we have considered partial substitution. The rate in which substitution takes places is knowns as rate of substitution. ν_1 is the fraction of demand met from the inventory of product 2 to product 1. Similarly ν_2 is the fraction of demand met from the inventory of product 1 to product 2. Three scenarios has been discussed based on the time interval of the products. i.e. when $T_1 \le T_2$, $T_1 \ge T_2$ and $T_1 = T_2$, where T_1 and T_2 are the cycle time of product 1 and product 2 respectively.

Scenario 1

Consider a situation where the product 1 inventory is completed first. The future demand is partially substituted from the inventory of product 2 and the unmet demand is assumed to be lost. In this case the cycle time of product 1 is less than that of product 2 i.e. $T_1 \le T_2$.

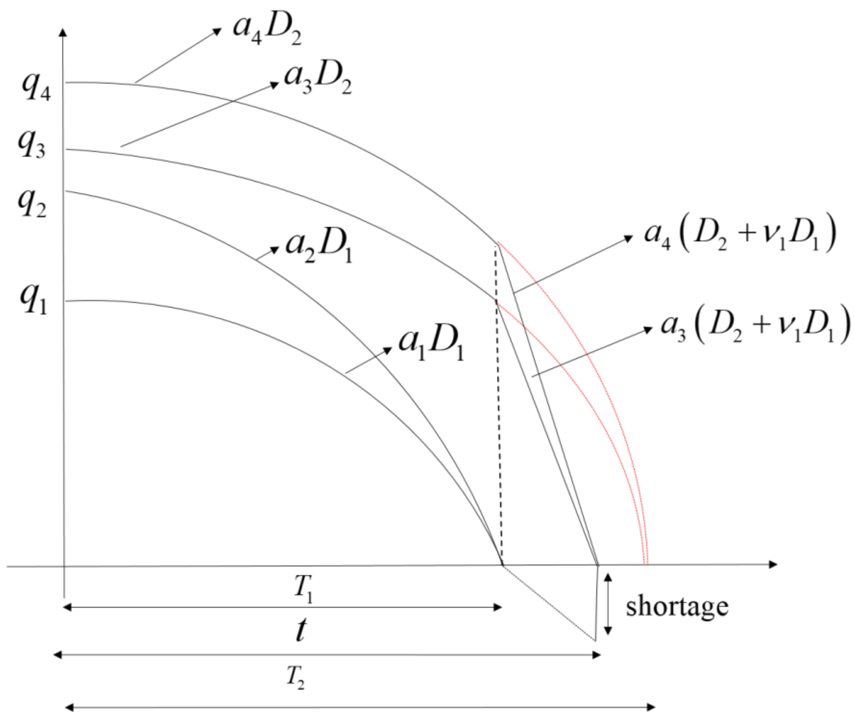

Fig 1. Inventory level for scenario 1 ($T_1 \le T_2$)

The total cost equation in this situation comprises of the following cost that is listed below.

Ordering cost:

$$OC_1 = k_1 + k_2$$

Holding cost:

$$HC_1 = \frac{2}{3}\left[h_1\left(\frac{c_{1i}q_1^2}{a_1D_1} + \frac{c_{2i}q_2^2}{a_2D_1}\right) + h_2\left(\frac{d_{1j}q_3^2}{a_3D_2} + \frac{d_{2j}q_4^2}{a_4D_2}\right)\right]$$

Holding cost in $[T,t]$:

$$HC_{sub_1} = \frac{h_2 d_{1j} d_{2j} D_1^2 (a_1+a_2)^2 (1-\nu_1)^2 (q_1+q_2+q_3+q_4)}{2\left[(a_3+a_4)D_2 + (a_1+a_2)D_1\right]^2 \left[(a_3+a_4)D_2 + \nu_1(a_1+a_2)D_1\right]}$$

Substitution cost:

$$SC_{sub_1} = \frac{\delta_{12}(a_1+a_2)D_1\nu_1\left((a_1+a_2)D_1(q_1+q_2+q_3+q_4)(1-\nu_1)\right)}{\left((a_3+a_4)D_2 + \nu_1(a_1+a_2)D_1\right)\left((a_1+a_2)D_1 + (a_3+a_4)D_2\right)}$$

Shortage cost:

$$SC_1 = \frac{D_1^2(a_1+a_2)^2 c_{s_1}(q_1+q_2+q_3+q_4)(1-\nu_1)^2}{\left((a_3+a_4)D_2 + \nu_1 D_1(a_1+a_2)\right)\left((a_3+a_4)D_2 + D_1(a_1+a_2)\right)}$$

The total cost is given by

$$TC_1 = OC_1 + HC_1 + HC_{sub_1} + SC_{sub_1} + SC_1$$

The total cost per unit time is given as $TCU_1 = \frac{TC_1}{t}$, where $t = \frac{(q_1+q_2)+(q_3+q_4)}{(a_3+a_4)D_2 + \nu_1(a_1+a_2)D_1}$.

Considering product 1, since the cycle time of both the components are same, we have the relation $q_2 = \frac{a_2}{a_1}q_1$. Similarly, considering product 2, we have $q_4 = \frac{a_4}{a_3}q_3$. Substituting this relation in the total cost equation gives,

$$TC_1 = k_1 + k_2 + \frac{2h_1q_1^2\left(a_1c_{1i} + a_2c_{2i}\right)}{3D_1a_1^2} + \frac{2h_2q_3^2\left(a_3d_{1j} + a_4d_{2j}\right)}{3D_2a_3^2}$$
$$+\frac{D_1^2d_{1j}d_{2j}\left(a_1+a_2\right)^2\left(1-\nu_1\right)^2\left(q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)\right)}{2\left(D_2\left(a_3+a_4\right)+D_1\nu_1\left(a_1+a_2\right)\right)\left(D_1\left(a_1+a_2\right)+D_2\left(a_3+a_4\right)\right)^2}$$
$$+\frac{D_1^2c_s\left(a_1+a_2\right)^2\left(1-\nu_1\right)^2\left(q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)\right)}{\left(D_2\left(a_3+a_4\right)+D_1\nu_1\left(a_1+a_2\right)\right)\left(D_1\left(a_1+a_2\right)+D_2\left(a_3+a_4\right)\right)}$$
$$+\frac{D_1^2\delta_{12}\nu_1\left(a_1+a_2\right)^2\left(1-\nu_1\right)\left(q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)\right)}{\left(D_2\left(a_3+a_4\right)+D_1\nu_1\left(a_1+a_2\right)\right)\left(D_1\left(a_1+a_2\right)+D_2\left(a_3+a_4\right)\right)}$$

and the total cost per unit time is $TCU_1 = \frac{TC_1}{t}$ where $t = \frac{q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)}{\left(a_3+a_4\right)D_2+\nu_1\left(a_1+a_2\right)D_1}$.

Scenario 2

Consider the case where $T_1 \geq T_2$, product 2 depletes faster than product 1. Further demand of product 2 is met from the inventory of the product 1, the unmet demand is assumed to be lost.

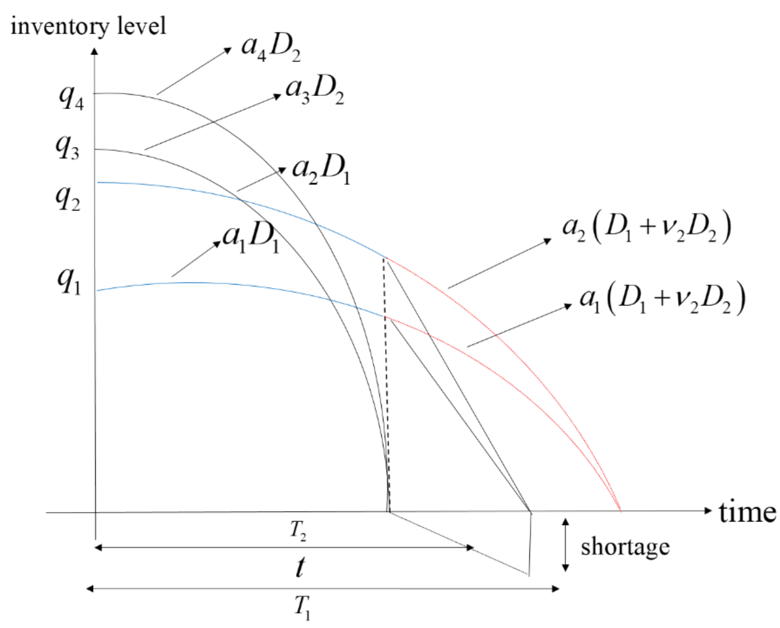

Fig 2. Inventory level for Scenario 2 ($T_1 \geq T_2$)

The total cost comprises of the following cost.

Ordering cost:

$$OC_2 = k_1 + k_2$$

Holding cost:

$$HC_2 = \frac{2h_1q_1^2\left(a_1c_{1i} + a_2c_{2i}\right)}{3D_1a_1^2} + \frac{2h_2q_3^2\left(a_3d_{1j} + a_4d_{2j}\right)}{3D_2a_3^2}$$

Holding cost in $[T, t]$:

$$HC_{sub_2} = \frac{D_2^2c_{1i}c_{2i}\left(a_3 + a_4\right)^2\left(1 - \nu_2\right)^2\left(q_1\left(1 + \frac{a_2}{a_1}\right) + q_3\left(1 + \frac{a_4}{a_3}\right)\right)}{2\left(D_1\left(a_1 + a_2\right) + D_2\nu_2\left(a_3 + a_4\right)\right)\left(D_1\left(a_1 + a_2\right) + D_2\left(a_3 + a_4\right)\right)^2}$$

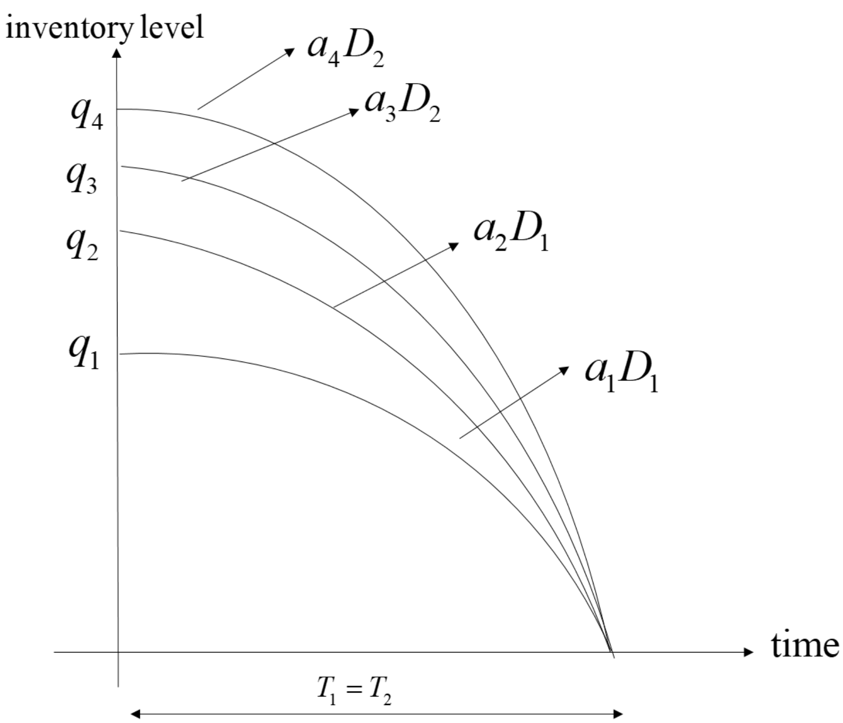

Fig 3. Inventory level for scenario 3 ($T_1 = T_2$)

Substitution cost:

$$SC_{sub_2}=\frac{D_2^2\delta_{21}\nu_2\left(a_3+a_4\right)^2\left(1-\nu_2\right)\left(q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)\right)}{\left(D_1\left(a_1+a_2\right)+D_2\nu_2\left(a_3+a_4\right)\right)\left(D_1\left(a_1+a_2\right)+D_2\left(a_3+a_4\right)\right)}$$

Shortage cost:

$$SC_2=\frac{D_2^2c_{s_2}\left(a_3+a_4\right)^2\left(1-\nu_2\right)^2\left(q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)\right)}{\left(D_1\left(a_1+a_2\right)+D_2\nu_2\left(a_3+a_4\right)\right)\left(D_1\left(a_1+a_2\right)+D_2\left(a_3+a_4\right)\right)}$$

The total cost is given by

$$TC_2=OC_2+HC_2+HC_{sub_2}+SC_{sub_2}+SC_2$$

The total cost per unit time is given as $TCU_2=\frac{TC_2}{t}$, where $t=\frac{q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)}{\left(a_1+a_2\right)D_1+\nu_2\left(a_3+a_4\right)D_2}$

Scenario 3

In this case we consider that there is no substitution taking place. Then the total cost equation includes the ordering cost and the holding cost of the products alone.

Ordering cost:

$$OC_{wos}=k_1+k_2$$

Holding cost:

$$HC_{wos}=\frac{2}{3}\frac{h_1q_1^2\left(a_1c_{1i}+a_2c_{2i}\right)}{D_1a_1^2}+\frac{2}{3}\frac{h_2q_3^2\left(a_3d_{1j}+a_4d_{2j}\right)}{D_2a_3^2}$$

The total cost is given by

$$TC_{wos}=OC_{wos}+HC_{wos}$$

The total cost per unit time is

$$TCU_{wos}=\frac{TC_{wos}}{t}\quad\text{where}\quad t=\frac{q_1\left(1+\frac{a_2}{a_1}\right)+q_3\left(1+\frac{a_4}{a_3}\right)}{D_1\left(a_1+a_2\right)+D_2\left(a_3+a_4\right)}$$

4. SOLUTION PROCEDURE

In this section a method to obtain an optimal ordering quantity for Scenario 1 and 2 is recommended. By proving TCU_1 and TCU_2 as pseudo- convex function, will guarantee a unique optimal policy.

The pseudo convexity of the total cost per unit time has been established below.

Theorem 1

TCU_1 is pseudo convex

Proof: Appendix A

Theorem 2

TCU_2 is pseudo convex

Proof: Appendix B

The following algorithm helps to find a unique optimal ordering quantities as the total cost per unit time is pseudo convex.

Algorithm to obtain optimal order quantities under quantity discount

Step 1 Initialize all the parameters

Step 2 Solve the optimization problem with the given constraint

(i) $\min_{q_1, q_3} TCU_1$ subject to $T_1 \le T_2$

(ii) $\min_{q_1, q_3} TCU_2$ subject to $T_1 \ge T_2$

Step 3 Obtain the optimal ordering quantities $\left(q_1^*, q_3^*\right)$ and hence the optimal total cost.

Step 4 If Step 3 is not feasible, take different combinations of order quantity and then go to Step 2.

Step 5 Repeat Step 4 until appropriate order quantities are found.

Step 6 Stop the algorithm

5. SENSITIVITY ANALYSIS

In this section, numerical example with sensitivity analysis has been done extensively to illustrate the behavior of the proposed model. We initialize the parametric values as follows: $k_1 = 150$; $k_2 = 150$; $h_1 = 10$; $h_2 = 10$; $a_1 = 1$; $a_2 = 2$; $a_3 = 2$; $a_4 = 2$; $D_1 = 200$; $D_2 = 100$; $\nu_1 = 0.2$; $\nu_2 = 0.8$; $c_{s_1} = 1$; $c_{s_2} = 30$; $c_{1i} = 5$; $c_{2i} = 4$; $d_{1j} = 3$; $d_{2j} = 2$; $\delta_{12} = 2$; $\delta_{21} = 2$; First, we solve the constrained optimization problem in Step 2 (i). MATLAB software is used for solving the problem, the optimal ordering quantities obtained are $q_1 = 17.92$ units, $q_3 = 31.064$ units and $TCU_1 = Rs.3044.69$. Similarly, solving Step 2 (ii) the optimal ordering quantities are obtained as $q_1 = 22.36$ units, $q_3 = 22.36$ units and $TCU_2 = Rs.5181.08$. On comparing, we choose the optimal ordering policy as $q_1^* = 17.92$ units, $q_3^* = 31.064$ units and $TCU_1^* = Rs.3044.69$. For the same initial parameter, total cost per unit time for the case of without substitution is calculated and also solved using the algorithm. The optimal ordering quantities obtained are $q_{1_{\mathrm{WOS}}}^* = 22.36$ units, $q_{2_{\mathrm{WOS}}}^* = 22.36$ units and $TCU_{\mathrm{WOS}} = Rs.5366.56$. The percentage improvement of total cost with substitution over without substitution is 43.27%.

Sensitivity analysis has been done extensively by increasing the parametric values and it is presented in Table 1,2,3 & 4. It is seen that total cost per unit times increases when the ordering cost, holding cost increases where as it decreases when shortage cost and substitution rate increases. Further, we investigate the percentage improvement of various parameter in the model in case of substitution over without substitution and the results of it is presented in Figure 4,5,6 & 7.

Table.1 Sensitivity analysis with respect to ordering cost

Parameter (k_1)	Unit price of components of product 1	Unit price of components of product 2	optimal ordering policy with substitution			optimal ordering policy without substitution			%improvement
			q_1	q_3	TCU_{WS}	q_1	q_3	TCU_{WOS}	
150	$c_{1i}=5,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	11.77	58.83	2386.36	11.77	58.83	3922.32	39.16
	$c_{1i}=5,c_{2i}=3$	$d_{1j}=1,d_{2j}=2$	15.38	44.44	2658.65	15.38	44.44	4443.74	40.17
	$c_{1i}=5,c_{2i}=3$	$d_{1j}=3,d_{2j}=3$	21.23	25.95	3054.90	21.23	25.95	5190.26	41.14
	c_{1i}=3,c2i=3	d_{1j}=1,d2j=1	18.26	54.77	2245.52	18.26	54.77	3651.48	38.50
250	c_{1i}=4,c2i=3	$d_{1j}=2,d_{2j}=3$	19.54	36.48	3514.00	19.54	36.48	6079.78	42.20
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=1,d_{2j}=2$	18.97	50.6	2978.92	18.97	50.6	5059.64	41.12
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=3,d_{2j}=3$	26.26	29.17	3389.96	26.26	29.17	5834.60	41.90
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	17.8	65.25	2608.89	17.8	65.25	4350.26	40.03
350	$c_{1i}=5,c_{2i}=3$	$d_{1j}=2,d_{2j}=2$	19.36	48.41	3706.79	19.36	48.41	6454.97	42.57
	$c_{1i}=5,c_{2i}=3$	$d_{1j}=2,d_{2j}=2$	21.73	47.07	3613.97	21.73	47.07	6276.46	42.42
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=3,d_{2j}=3$	22.93	35.68	4066.20	22.93	35.68	7135.06	43.01
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=3,d_{2j}=2$	23.14	40.1	3828.10	23.14	40.1	6683.83	42.73
450	$c_{1i}=5,c_{2i}=3$	$d_{1j}=3,d_{2j}=1$	21.21	53.03	4026.01	21.21	53.03	7071.07	43.06
	$c_{1i}=5,c_{2i}=3$	$d_{1j}=3,d_{2j}=3$	23.84	39.74	4488.51	23.84	39.74	7947.19	43.52
	$c_{1i}=5,c_{2i}=3$	$d_{1j}=2,d_{2j}=3$	28.76	42.19	4008.75	28.76	42.19	7031.23	42.99
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=3,d_{2j}=3$	28.18	37.57	4263.61	28.18	37.57	7514.69	43.26

Table.2 Sensitivity analysis with respect to holding cost

Parameter (h_1)	Unit price of components of product 1	Unit price of components of product 2	optimal ordering policy with substitution			optimal ordering policy without substitution			%improvement
			q_1	q_3	TCU_{WS}	q_1	q_3	TCU_{WOS}	
2	$c_{1i}=4, c_{2i}=5$	$d_{1j}=3, d_{2j}=3$	31.98	31.98	2307.18	31.98	31.98	3752.33	38.51
	$c_{1i}=4, c_{2i}=4$	$d_{1j}=1, d_{2j}=2$	40.82	40.82	1876.39	40.82	40.82	2939.39	36.16
	$c_{1i}=4, c_{2i}=3$	$d_{1j}=1, d_{2j}=1$	47.43	47.43	1662.26	47.43	47.43	2529.82	34.29
4	$c_{1i}=4, c_{2i}=5$	$d_{1j}=1, d_{2j}=1$	26.76	49.95	2078.37	26.76	49.95	3330.03	37.59
	$c_{1i}=4, c_{2i}=5$	$d_{1j}=3, d_{2j}=3$	27.85	27.85	2596.20	27.85	27.85	4308.13	39.74
	$c_{1i}=3, c_{2i}=3$	$d_{1j}=3, d_{2j}=2$	32.35	32.35	2281.42	32.35	32.35	3709.45	38.50
6	$c_{1i}=5, c_{2i}=5$	$d_{1j}=3, d_{2j}=3$	24.49	24.5	2903.44	24.49	24.5	4898.98	40.73
	$c_{1i}=4, c_{2i}=5$	$d_{1j}=1, d_{2j}=2$	21.85	40.78	2468.69	21.85	40.78	4078.44	39.47
	$c_{1i}=4, c_{2i}=4$	$d_{1j}=3, d_{2j}=3$	26.11	26.11	2745.71	26.11	26.11	4595.65	40.25
8	$c_{1i}=5, c_{2i}=5$	$d_{1j}=3, d_{2j}=1$	17.93	35.86	2835.13	17.93	35.86	4780.91	40.70
	$c_{1i}=5, c_{2i}=5$	$d_{1j}=3, d_{2j}=3$	19.93	26.57	3119.09	19.93	26.57	5313.69	41.30
	$c_{1i}=5, c_{2i}=3$	$d_{1j}=1, d_{2j}=3$	22.74	33.35	2661.47	22.74	33.35	4446.94	40.15
10	$c_{1i}=4, c_{2i}=5$	$d_{1j}=1, d_{2j}=3$	15.86	37.01	2914.92	15.86	37.01	4934.35	40.93
	$c_{1i}=4, c_{2i}=4$	$d_{1j}=3, d_{2j}=3$	19.93	26.57	3119.09	19.93	26.57	5313.69	41.30
	$c_{1i}=3, c_{2i}=4$	$d_{1j}=1, d_{2j}=3$	19.19	35.18	2788.07	19.19	35.18	4690.42	40.56

Table.3 Sensitivity analysis with respect to substitution rate

Parameter (v_1)	Unit price of components of product 1	Unit price of components of product 2	optimal ordering policy with substitution			optimal ordering policy without substitution			%improvement
			q_1	q_3	TCU_{WS}	q_1	q_3	TCU_{WOS}	
0.2	$c_{1i}=4,c_{2i}=4$	$d_{1j}=2,d_{2j}=3$	19.05	30.48	2994.11	19.05	30.48	5080.01	41.06
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=2,d_{2j}=2$	20.65	34.41	2736.13	20.65	34.41	4588.31	40.37
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	18.26	54.77	2245.52	18.26	54.77	3651.48	38.50
0.4	$c_{1i}=5,c_{2i}=5$	$d_{1j}=1,d_{2j}=1$	11.77	58.83	2813.33	11.77	58.83	3922.32	28.27
	$c_{1i}=5,c_{2i}=4$	$d_{1j}=2,d_{2j}=2$	16.83	36.46	3416.50	16.83	36.46	4861.72	29.73
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	18.26	54.77	2640.00	18.26	54.77	3651.48	27.70
0.6	$c_{1i}=5,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	15.41	56.51	3093.94	15.41	56.51	3767.43	17.88
	$c_{1i}=5,c_{2i}=3$	$d_{1j}=3,d_{2j}=3$	21.23	25.95	4177.59	21.23	25.95	5190.26	19.51
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=3$	22.36	33.54	3630.09	22.36	33.54	4472.14	18.83
0.8	$c_{1i}=5,c_{2i}=4$	$d_{1j}=3,d_{2j}=2$	17.92	31.06	4686.03	17.92	31.06	5177.27	9.49
	$c_{1i}=4,c_{2i}=5$	$d_{1j}=1,d_{2j}=2$	14.46	44.99	4088.86	14.46	44.99	4499.00	9.12
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	18.26	54.77	3342.98	18.26	54.77	3651.48	8.45
1	$c_{1i}=5,c_{2i}=5$	$d_{1j}=1,d_{2j}=1$	11.77	58.83	3922.32	11.77	58.83	3922.32	0.00
	$c_{1i}=4,c_{2i}=5$	$d_{1j}=1,d_{2j}=1$	12.5	58.36	3890.39	12.5	58.36	3890.39	0.00
	$c_{1i}=3,c_{2i}=5$	$d_{1j}=1,d_{2j}=1$	13.34	57.82	3854.50	13.34	57.82	3854.50	0.00

Table.4 Sensitivity analysis with respect to shortage cost

Parameter (c_s)	Unit price of components of product 1	Unit price of components of product 2	Optimal ordering policy with substitution			Optimal ordering policy without substitution			%improvement
			q_1	q_3	TCU_{WS}	q_1	q_3	TCU_{WOS}	
1	$c_{1i}=5,c_{2i}=4$	$d_{1j}=1,d_{2j}=3$	16.83	36.46	2877.15	16.83	36.46	4861.72	40.82
	$c_{1i}=4,c_{2i}=4$	$d_{1j}=3,d_{2j}=3$	19.93	26.57	3119.09	19.93	26.57	5313.69	41.30
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	18.26	54.77	2245.52	18.26	54.77	3651.48	38.50
3	$c_{1i}=5,c_{2i}=5$	$d_{1j}=1,d_{2j}=2$	13.65	45.49	3173.97	13.65	45.49	4548.59	30.22
	$c_{1i}=5,c_{2i}=4$	$d_{1j}=3,d_{2j}=2$	17.92	31.06	3505.49	17.92	31.06	5177.27	32.29
	$c_{1i}=4,c_{2i}=4$	$d_{1j}=1,d_{2j}=1$	14.3	57.21	2790.75	14.3	57.21	3813.85	26.83
5	$c_{1i}=5,c_{2i}=4$	$d_{1j}=2,d_{2j}=3$	17.92	31.06	3966.29	17.92	31.06	5177.27	23.39
	$c_{1i}=4,c_{2i}=5$	$d_{1j}=1,d_{2j}=2$	14.46	44.99	3608.98	14.46	44.99	4499.00	19.78
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	18.26	54.77	3167.12	18.26	54.77	3651.48	13.26
7	$c_{1i}=5,c_{2i}=4$	$d_{1j}=2,d_{2j}=3$	17.92	31.06	4427.09	17.92	31.06	5177.27	14.49
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=2,d_{2j}=2$	20.65	34.41	4118.53	20.65	34.41	4588.31	10.24
	$c_{1i}=3,c_{2i}=3$	$d_{1j}=1,d_{2j}=1$	18.26	54.77	3627.92	18.26	54.77	3651.48	0.65
9	$c_{1i}=4,c_{2i}=5$	$d_{1j}=2,d_{2j}=2$	15.86	37.01	4759.27	15.86	37.01	4934.35	3.55
	$c_{1i}=4,c_{2i}=4$	$d_{1j}=3,d_{2j}=3$	19.93	26.57	4962.29	19.93	26.57	5313.69	6.61
	$c_{1i}=4,c_{2i}=3$	$d_{1j}=2,d_{2j}=2$	20.65	34.41	4579.33	20.65	34.41	4588.31	0.20

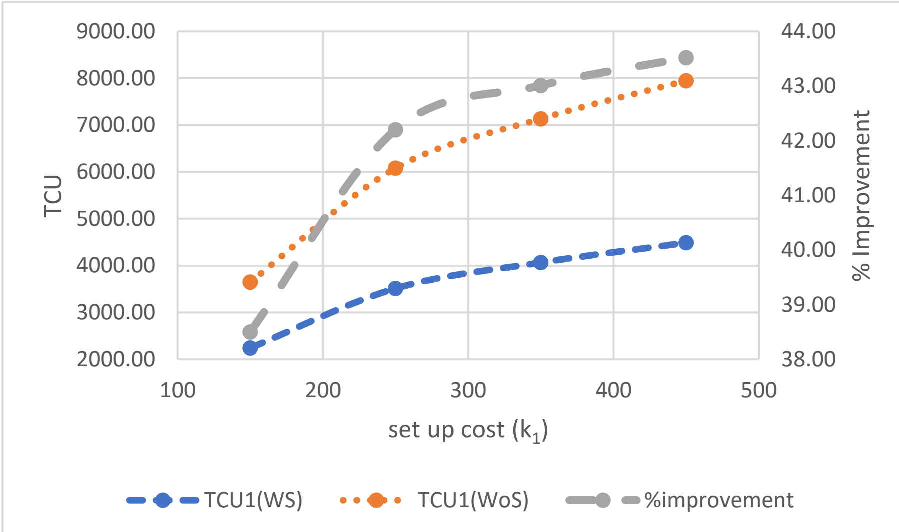

Fig.4 Sensitivity with respect to ordering cost

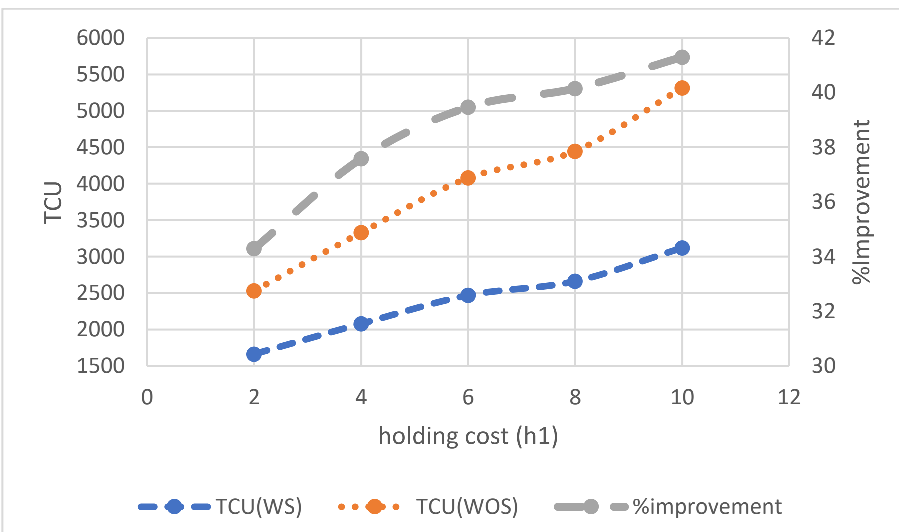

Fig.5 Sensitivity with respect to holding cost

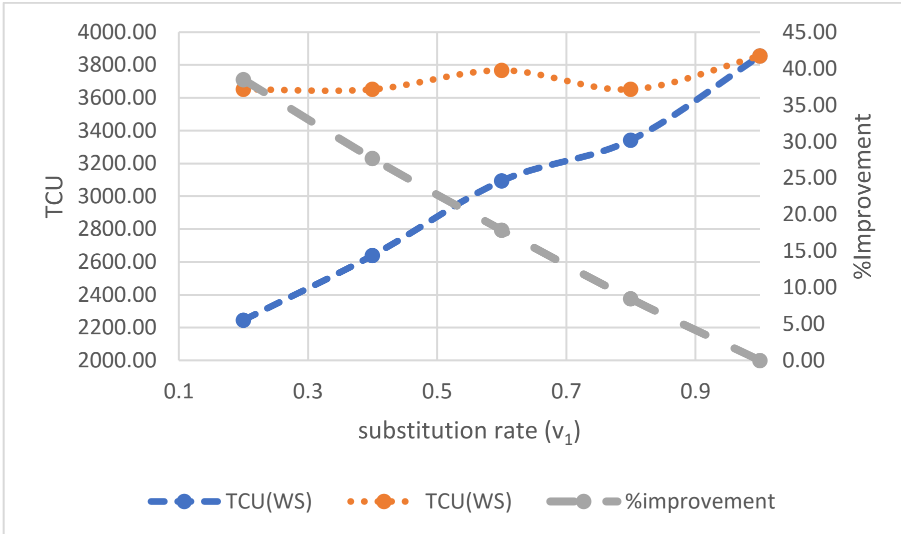

Fig.6 Sensitivity with respect to substitution rate

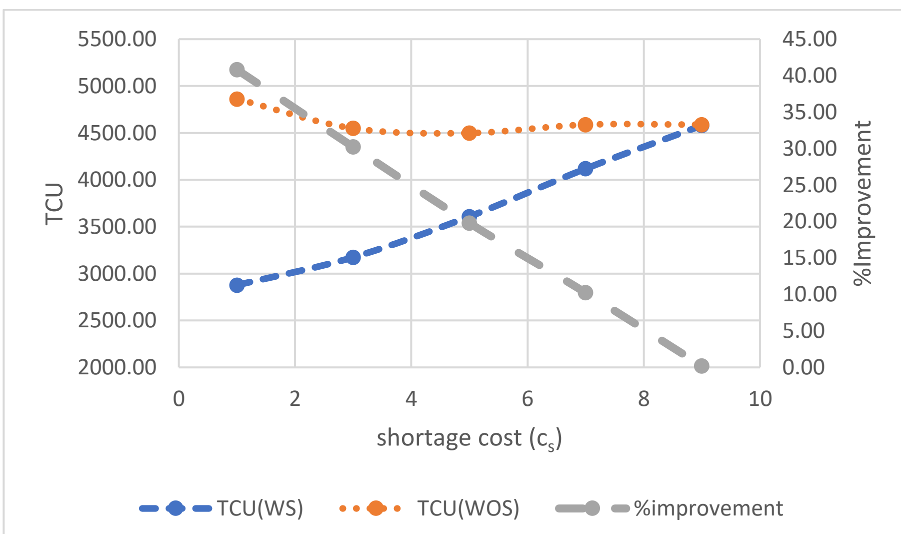

Fig.7 Sensitivity with respect to shortage cost

Table.5 Summary of sensitivity analysis

Parameter	Variation	TCU(WS)	TCU(WOS)	%Improvement
Ordering cost (k_1)	increases	increases	increases	increases
Holding cost (h_1)	increases	increases	increases	increases
Substitution rate (ν_1)	increases	increases	constant	decreases
Shortage cost (c_{s_1})	increases	increases	constant	decreases

5. CONCLUSION

An inventory system consisting of two mutually substitutable products where, each product consists of two complementary components is considered in this model. The inventory of both products is depleted by its own demand, which is assumed to be quadratic. When one product becomes out of stock, the demand of the other product is met from the inventory of the second product with a rate of substitution. In addition, substitution cost is incurred whenever substitution takes place. The unmet demand is assumed to be lost. The phenomenon of quantity discount is considered in this paper. The proposed model is applicable for products such as different brands of mobile and sim cards with different network, different brands of DVD players and different DVDs, etc. Three scenario has been discussed for the proposed model. A solution procedure is suggested in this paper to obtain the optimal total ordering quantities. Numerical example is provided to see the viability of the model. Sensitivity analysis has been extensively carried out to illustrate the percentage improvement in optimal total inventory cost with substitution over without substitution.

Further extension can be done by considering more than two product. The phenomenon of deterioration can be applied. In addition, it can be extended in the direction of probability by introducing, stochastic demand etc.

Appendix A

Proof of TCU_1 is pseudo convex

Consider TCU_1 equation. We first claim that TC_1 is convex function

$$TC_1 = k_1 + k_2 + \frac{2h_1 q_1^2 \left(a_1 c_{1i} + a_2 c_{2i}\right)}{3D_1 a_1^2} + \frac{2h_2 q_3^2 \left(a_3 d_{1j} + a_4 d_{2j}\right)}{3D_2 a_3^2}$$
$$+ \frac{D_1^2 d_{1j} d_{2j} \left(a_1 + a_2\right)^2 \left(1-\nu_1\right)^2 \left(q_1\left(1+\frac{a_2}{a_1}\right) + q_3\left(1+\frac{a_4}{a_3}\right)\right)}{2\left(D_2\left(a_3+a_4\right) + D_1\nu_1\left(a_1+a_2\right)\right)\left(D_1\left(a_1+a_2\right) + D_2\left(a_3+a_4\right)\right)^2}$$
$$+ \frac{D_1^2 c_s \left(a_1 + a_2\right)^2 \left(1-\nu_1\right)^2 \left(q_1\left(1+\frac{a_2}{a_1}\right) + q_3\left(1+\frac{a_4}{a_3}\right)\right)}{\left(D_2\left(a_3+a_4\right) + D_1\nu_1\left(a_1+a_2\right)\right)\left(D_1\left(a_1+a_2\right) + D_2\left(a_3+a_4\right)\right)}$$
$$+ \frac{D_1^2 \delta_{12} \nu_1 \left(a_1 + a_2\right)^2 \left(1-\nu_1\right) \left(q_1\left(1+\frac{a_2}{a_1}\right) + q_3\left(1+\frac{a_4}{a_3}\right)\right)}{\left(D_2\left(a_3+a_4\right) + D_1\nu_1\left(a_1+a_2\right)\right)\left(D_1\left(a_1+a_2\right) + D_2\left(a_3+a_4\right)\right)}$$

To show that TC_1 is convex, we claim that Hessian matrix is positive definite.

We have $H = \begin{bmatrix} \frac{\partial^2 TC_1}{\partial q_1^2} & \frac{\partial^2 TC_1}{\partial q_1 \partial q_3} \\ \frac{\partial^2 TC_1}{\partial q_1 \partial q_3} & \frac{\partial^2 TC_1}{\partial q_3^2} \end{bmatrix}$

We find that $\frac{\partial^2 TC_1}{\partial q_1^2} > 0$, $\frac{\partial^2 TC_1}{\partial q_3^2} > 0$ and $\frac{\partial^2 TC_1}{\partial q_1 \partial q_3} > 0$.

Therefore, $\left(\frac{\partial^2 TC_1}{\partial q_1^2}\right)\left(\frac{\partial^2 TC_1}{\partial q_3^2}\right) - \left(\frac{\partial^2 TC_1}{\partial q_1 \partial q_3}\right)^2 > 0$, which implies Hessian matrix is positive definite.

Hence TC_1 is convex.

Since $TCU_1 = \frac{TC_1}{t}$ where $t = \frac{q_1\left(1+\frac{a_2}{a_1}\right) + q_3\left(1+\frac{a_4}{a_3}\right)}{\left(a_3+a_4\right)D_2 + \nu_1\left(a_1+a_2\right)D_1}$

Now, we apply the result which states that "ratio of a positive convex function over a linear function is pseudo convex". Hence, TCU_1 is pseudo convex.

Appendix B

Proof of TCU_2 is pseudo convex.

Similar to proof of Theorem 1.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] L. Benkherouf, K. Skouri, I. Konstantaras, Inventory decisions for a finite horizon problem with product substitution options and time varying demand, Appl. Math. Model. 51 (2017), 669-685.

[2] Y. Chen, Yanru, et al. Joint replenishment decision considering shortages, partial demand substitution, and defective items, Computers Ind. Eng. 127 (2019), 420-435.

[3] Z. Drezner, H. Gurnani, B.A. Pasternack, An EOQ model with substitutions between products, J. Oper. Res. Soc. 46 (1995), 887-891.

[4] B.K. Durga, E. Chandrasekaran, Optimal ordering policy for substitutable products under quadratic demand, J. Adv. Res. Dyn. Control Syst. 11 (2019), 1553-1560.

[5] B.K. Durga, E. Chandrasekaran, Optimal ordering policy for substitutable products under quadratic demand with cost of substitution, J. Critic. Rev. 7 (2020), 200-205.

[6] B.K. Durga, E. Chandrasekaran, Optimal ordering policy for substitutable and complementary products under quadratic demand, J. Math. Comput. Sci. 11 (2021), 1849-1867.

[7] M.A. Edalatpour, S.M.J. Mirzapour Al-e-Hashem, Simultaneous pricing and inventory decisions for substitute and complementary items with nonlinear holding cost, Product. Eng. 13 (2019), 305-315.

[8] R.N. Giri, S.K. Mondal, M. Maiti, Analysis of pricing decision for substitutable and complementary products with a common retailer, Pac. Sci. Rev. A: Nat. Sci. Eng. 18 (2016), 190-202.

[9] S.K. Goyal, An EOQ model with substitutions between products, J. Oper. Res. Soc. 47 (1996), 476-477.

[10] H. Gurnani, Z. Drezner, Deterministic hierarchical substitution inventory models, J. Oper. Res. Soc. 51 (2000), 129-133.

[11] I.P. Krommyda, K. Skouri, I. Konstantaras, Optimal ordering quantities for substitutable products with stock-dependent demand, Appl. Math. Model. 39 (2015), 147-164.

[12] B. Maddah, et al. Joint replenishment model for multiple products with substitution, Appl. Math. Model. 40 (2016), 7678-7688.

[13] R. Mcgillivray, E. Silver, Some concepts for inventory control under substitutable demand, INFOR: Inform. Syst. Oper. Res. 16 (1978), 47-63.

[14] V.K. Mishra, K. Shanker, Optimal ordering quantity for substitutable products under quantity discount with cost of substitution, Proc. Nat. Acad. Sci. India Sect. A: Phys. Sci. 89 (2019), 497-508.

[15] H. Mokhtari, Economic order quantity for joint complementary and substitutable items, Math. Computers Simul. 154 (2018), 34-47.

[16] Q. Pan, et al. An inventory replenishment system with two inventory-based substitutable products, Int. J. Product. Econ. 204 (2018), 135-147.

[17] M.K. Salameh, et al. Joint replenishment model with substitution, Appl. Math. Model. 38 (2014), 3662-3671.

[18] A.A. Taleizadeh, et al. Pricing decision within an inventory model for complementary and substitutable products, Mathematics, 7 (2019), 568.

[19] C.S. Tang, R. Yin, Joint ordering and pricing strategies for managing substitutable products, Product. Oper. Manage. 16 (2007), 138-153.

[20] S. Transchel, Inventory management under price-based and stockout-based substitution, Eur. J. Oper. Res. 262 (2017), 996-1008.

[21] J. Wei, J. Zhao, Y. Li, Pricing decisions for complementary products with firms' different market powers, Eur. J. Oper. Res. 224 (2013), 507-519.

[22] X. Yue, S.K. Mukhopadhyay, X. Zhu, A Bertrand model of pricing of complementary goods under information asymmetry, J. Bus. Res. 59 (2006), 1182-1192.