Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 6193-6204
https://doi.org/10.28919/jmcs/6050
ISSN: 1927-5307

NEW RESULTS ON NONLOCAL FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

AMOL D. KHANDAGALE[1], AHMED A. HAMOUD[2,*], KIRTIWANT P. GHADLE[1]

[1]Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India

[2]Department of Mathematics, Taiz University, Taiz, Yemen

Abstract. This paper considers nonlinear fractional mixed Volterra-Fredholm integro-differential equation with a nonlocal initial condition. We propose a fixed-point approach to investigate the existence and uniqueness of solutions. Results of this paper are based on nonstandard assumptions and hypotheses and provide a supplementary result concerning the regularity of solutions.

Keywords: Caputo fractional derivative; Volterra-Fredholm integro-differential equation; Krasnoselskii's fixed point theorem; Banach fixed point theorem; Banach space.

2010 AMS Subject Classification: 45J05, 34A12, 47H10.

1. Introduction

The theory of fractional differential and partial differential have become an important area of investigation in the past two decades because of their applications to various problems arising in communications, control technology, impact mechanics, electrical engineering, medicine, and biology. Fractional systems has the advantages to describe the processes associate with previous history. Due to its tremendous applications many researchers are working in this field. Initially it was developed by several authors in their monograph [21, 22, 26, 27, 28, 29].

*Corresponding author

E-mail address: maths.aah@bamu.ac.in

Received May 16, 2021

In the fractional calculus the various integral inequalities plays an important role in the study of qualitative and quantitative properties of solution of differential and integral equations.

The work on nonlocal Cauchy problem was first initiated by Byszewski [4] in Banach spaces. He proved the existence and uniqueness of mild, strong and classical solutions of nonlocal Cauchy problem using the method of semigroups and the Banach fixed point theorem. The nonlocal condition can be applied to physics with better effect than the classical condition, since the nonlocal condition is usually more precise for physical measurements than the classical one [1, 5, 6, 7, 8, 21, 23].

Volterra-Fredholm integro-differential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aero elastic phenomena, visco elasticity, visco elastic panel in super sonic gas flow, fluid dynamics, electro dynamics of complex medium, many models of population growth [22].

In recent years, many authors focus on the development of techniques for discussing the solutions of fractional differential and integro-differential equations. For instance, we can remember the following works:

Ibrahim and Momani [19] studied the existence and uniqueness of solutions of a class of fractional order differential equations, Karthikeyan and Trujillo [20] proved existence and uniqueness of solutions for fractional integro-differential equations with boundary value conditions, Bahuguna and Dabas [2] applied the method of lines to establish the existence and uniqueness of a strong solution for the partial integro-differential equations, Matar [23] deliberated the existence of solutions for nonlocal fractional semilinear integro-differential equations in Banach spaces via Banach fixed point theorem. Momani et al. [25] proved the Local and global uniqueness result by using Bihari's inequality for the fractional integro-differential equation,

$$ {}^{c}D^{\alpha}u(t) = f(t,u(t)) + \int_{t_0}^{t} Z(t,s,u(s))ds, \; 0 < \alpha \leq 1 $$

with the initial condition

$$ u(0) = u_0, $$

Ahmad and Sivasundaram [1] studied some existence and uniqueness results in a Banach space for the fractional integro-differential equation,

$$ {}^{c}D^{\alpha}u(t) = f(t,u(t)) + \int_{t_0}^{t} Z(t,s,u(s))ds,\ 0 < \alpha < 1, $$

with the nonlocal condition

$$ u(0) = u_0 - g(u). $$

Recently, many authors [3, 11, 25, 32, 33] obtained the result on existence and uniqueness of solutions for fractional integro-differential equations with nonlocal conditions using the fixed point theorem of Banach space couple with contraction mapping principle.

Inspired by the discussion, we consider Caputo fractional Volterra-Fredholm integro-differential equation:

$$ {}^{c}D^{\alpha}u(t) = f(t,u(t)) + \int_{0}^{t} Z_1(t,s,u(s))ds + \int_{0}^{T} Z_2(t,s,u(s))ds, \tag{1} $$

with the nonlocal condition

$$ u(0) = u_0 - g(u), \tag{2} $$

where ${}^{c}D^{\alpha}$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$, $f : [0,T] \times X \longrightarrow X$, $Z_1, Z_2 : [0,T] \times [0,T] \times X \longrightarrow X$ and $g : C([0,T],X) \longrightarrow X$ are appropriate functions satisfying some conditions which will be stated later.

The main objective of the present paper is to study the new existence and uniqueness results by means of Krasnoselskii's fixed point and the Banach fixed point theorems for Caputo fractional Volterra-Fredholm integro-differential equations.

The rest of the paper is organized as follows: In Section 2, some essential notations, definitions and Lemmas related to fractional calculus are recalled. In Section 3, the new existence and uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential equation have been proved. Finally, we will give a report on our paper and a brief conclusion is given in Section 4.

2. PRELIMINARIES

In this section, we recall the necessary theory that is used throughout the work in order to obtain new results. For more details, see [9, 10, 12, 13, 14, 15, 16, 17, 22, 26, 31].

Let $C(J,X)$, $C^n(J,X)$ are the Banach space of all continuous bounded functions and continuously differentiable functions up to order $(n-1)$ on $J=[0,T]$, respectively. For any function $h \in C(J,X)$, $\|h\|_{C(J,X)} = \sup\{|h(t)| : t \in J\}$. $L^1(J)$ denotes the space of all real functions defined on J which are Lebesgue integrable.

Definition 2.1. [21] ***(Riemann-Liouville fractional integral).*** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function h is defined as*

$$\begin{aligned} J^{\alpha}h(t) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}h(s)ds, \qquad \alpha \in \mathbb{R}^+, \\ J^0 h(t) &= h(t), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. [21] ***(Caputo fractional derivative).*** *The fractional derivative of $h(t)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD^{\alpha}h(t) &= J^{m-\alpha}D^m h(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (x-s)^{m-\alpha-1}\frac{\partial^m h(s)}{\partial s^m}ds, & m-1<\alpha<m, \\ \\ \frac{\partial^m h(t)}{\partial t^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{3}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

(1) $J^{\alpha}J^{v}h = J^{\alpha+v}h, \quad \alpha, v > 0.$
(2) $J^{\alpha}h^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}h^{\beta+\alpha},$
(3) $D^{\alpha}h^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}h^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1.$
(4) $J^{\alpha}D^{\alpha}h(t) = h(t) - h(a),\ 0 < \alpha < 1.$
(5) $J^{\alpha}D^{\alpha}h(t) = h(t) - \sum_{k=0}^{m-1} h^{(k)}(0^+)\frac{(t-a)^k}{k!}, \quad t > 0.$

Definition 2.3. [21] ***(Riemann-Liouville fractional derivative).*** *The Riemann Liouville fractional derivative of order* $\alpha > 0$ *is normally defined as*

$$D^{\alpha}h(t) = D^{m}J^{m-\alpha}h(t), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{4}$$

Definition 2.4. [26] *Let* $T : X \longrightarrow X$ *be a mapping on a normed space* $(X, \|.\|)$*. A point* $x \in X$ *for which* $Tx = x$ *is called a fixed point of* T*.*

Definition 2.5. [26] *The mapping* T *on a normed space* $(X, \|.\|)$ *is called contraction if there is a non-negative real number* $c \in (0,1)$*, such that* $\|Tx - Ty\| \leq c\|x - y\|$ *for all* $x, y \in X$*.*

Theorem 2.1. [34] ***(Banach fixed point theorem)*** *Let* $(X, \|.\|)$ *be a complete normed space, and let the mapping* $T : X \longrightarrow X$ *be a contraction mapping. Then* T *has exactly one fixed point.*

Theorem 2.2. [21] ***(Krasnoselskii)*** *Let* M *be a closed convex and nonempty subset of a Banach space* X*. Let* A, B *be two operators such that:*

1. $Ax + By \in M$ *whenever* $x, y \in M$*;*

2. A *is compact and continuous;*

3. B *is a contraction mapping.*

Then there exists $z \in M$ *such that* $z = Az + Bz$*.*

3. Main Results

In this section, we shall give an existence and uniqueness results of Eq.(1), with the condition (2). Before starting and proving the main results, we introduce the following hypotheses:

(A1) $g : C(J, X) \longrightarrow X$ is continuous, bounded and there exists $0 < b < \frac{1}{2}$ such that

$$|g(u) - g(v)| \leq b\,|u - v|\,,\ u, v \in X.$$

(A2) $f : J \times X \longrightarrow X$ is continuous and there exist $\mu \in L^{1}([0,T], \mathbb{R}_{+})$ such that

$$|f(t,u)| \leq \mu(t), t \in J := [0,T].$$

(A3) $Z_1, Z_2 : J \times J \times X \longrightarrow X$ is continuous on D and there exist $\sigma_1(t), \sigma_2(t) \in L^1([0,T], \mathbb{R}_+)$ such that

$$\|Z_1(t,s,u(s))\| \leq \sigma_1(t),$$

$$\|Z_2(t,s,u(s))\| \leq \sigma_2(t),\ (t,s) \in D,$$

where $D = \{(t,s) : 0 \leq s \leq t \leq T\}$.

First, we will state the following axiom lemma.

Lemma 3.1. *Let $0 < \alpha \leq 1$. Assume that f, Z_1 and Z_2 are continuous functions. If $u \in C(J,X)$ then u satisfies the problem (1)-(2) if and only if u satisfies the integral equation*

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[f(s,u(s)) + \int_s^t Z_1(\sigma,s,u(s))d\sigma + \int_s^T Z_2(\sigma,s,u(s))d\sigma \Big] ds.$$

Now, we will prove the uniqueness result.

Theorem 3.1. *Assume that*

(B1) $\|g(u) - g(v)\| \leq L_g \|u - v\|$, *for* $u, v \in X$.

(B2) $\|f(t,u) - f(t,v)\| \leq L_f \|u - v\|,\ t \in J,\ u,v \in X$.

(B3) $\|Z_1(t,s,u) - Z_1(t,s,v)\| \leq L_z \|u - v\|,\ (t,s) \in D,\ u,v \in X$.

$\|Z_2(t,s,u) - Z_2(t,s,v)\| \leq L_{z^*} \|u - v\|$.

and if

$$\Theta := L_g + \frac{L_f T^\alpha}{\Gamma(\alpha+1)} + \frac{L_z \alpha T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{L_{z^*} \alpha T^{\alpha+1}}{\Gamma(\alpha+2)} < 1.$$

Then the fractional integro-differential equation (1)-(2) *has a unique solution continuous on J.*

Proof. Let the operator $F : C(J,X) \longrightarrow C(J,X)$ be defined as define the operator $F : C(J,X) \longrightarrow C(J,X)$ by

$$\begin{aligned} Fu(t) &= u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[f(s,u(s)) + \int_s^t Z_1(\sigma,s,u(s))d\sigma \\ &\quad + \int_s^T Z_2(\sigma,s,u(s))d\sigma \Big] ds. \end{aligned}$$

Let us set $M_f = \sup_{t \in [0,T]} \|f(t,0)\|$, $M_z = \sup_{t,s \in [0,T]} \|Z_1(t,s,0)\|, M_{z^*} = \sup_{t,s \in [0,T]} \|Z_2(t,s,0)\|$ and $M_g = \sup_{t \in [0,T]} \|g(t)\|$.

Also, we define $B_r = \{u \in C(J,X) : \|u\| \le r\}$, for some $r > 0$, choosing

$$2\Big[\|u_0\| + M_g + \frac{M_f T^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha(M_z + M_{z^*})T^\alpha}{\Gamma(\alpha+2)}\Big] \le r,$$

Step 1. We show that $FB_r \subset B_r$. By the hypotheses, then for any $u \in B_r$ and for each $t \in J$, we have

$$\begin{aligned}
\|Fu(t)\| &\le \|u_0\| + M_g + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\|f(s,u(s))\| + \int_s^t \|Z_1(\sigma,s,u(s))\| d\sigma \\
&\quad + \int_s^T \|Z_2(\sigma,s,u(s))\| d\sigma\Big] ds \\
&\le \|u_0\| + M_g + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[[\|f(s,u(s)) - f(s,0)\| + \|f(s,0)\|] \\
&\quad + \int_s^t [\|Z_1(\sigma,s,u(s)) - Z_1(\sigma,s,0)\| + \|Z_1(\sigma,s,0)\|] d\sigma \\
&\quad + \int_s^T \|Z_2(\sigma,s,u(s)) - Z_2(\sigma,s,0)\| + \|Z_2(\sigma,s,0)\| d\sigma\Big] ds \\
&\le \|u_0\| + M_g + \frac{L_f r + M_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds + \frac{L_z r + M_z}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha} ds \\
&\quad + \frac{L_{z^*} r + M_{z^*}}{\Gamma(\alpha)}\int_0^T (t-s)^{\alpha} ds \\
&\le \|u_0\| + M_g + \frac{(L_f r + M_f)T^\alpha}{\Gamma(\alpha+1)} + \frac{(L_z r + M_z)\alpha T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(L_{z^*} r + M_{z^*})\alpha T^{\alpha+1}}{\Gamma(\alpha+2)} \\
&\le r.
\end{aligned}$$

It follows that $\|Fu\| \le r$, which means that $F : B_r \longrightarrow B_r$.

Step 2. We shall show that $F : B_r \longrightarrow B_r$ is a contraction mapping. Indeed, through the assumptions, then for any $u, v \in B_r$, and for $t \in J$, we can write

$$\begin{aligned}
\|(Fu)(t) - (Fv)(t)\| &\le \|g(u) - g(v)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\|f(s,u(s)) - f(s,v(s))\| \\
&\quad + \int_s^t \|Z_1(\sigma,s,u(s)) - Z_1(\sigma,s,v(s))\| d\sigma \\
&\quad + \int_s^T \|Z_2(\sigma,s,u(s)) - Z_2(\sigma,s,v(s))\| d\sigma\Big] ds \\
&\le \Big[L_g + \frac{L_f T^\alpha}{\Gamma(\alpha+1)} + \frac{L_z \alpha T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{L_{z^*}\alpha T^{\alpha+1}}{\Gamma(\alpha+2)}\Big]\|u - v\| \\
&= \Theta\|u - v\|.
\end{aligned}$$

As $\Theta < 1$, so the conclusion of the theorem follows by the contraction mapping principle. This implies that F is contraction mapping. As consequence of Lemma 2.1, there exists a fixed point $u \in C(J,R)$ such that $Fu = u$ which is the unique solution of (1)-(2) on J. $\square$

Next, we will prove the existence of solution for the problem (1)-(2) in the space $C(J,X)$ by means of Krasnoselskii's fixed point theorem.

Theorem 3.2. *Assume that (A1)-(A3) hold, and if*

$$\frac{\|\mu\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha\|\sigma_1\|_{L^1}T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{\alpha\|\sigma_2\|_{L^1}T^{\alpha+1}}{\Gamma(\alpha+2)} < 1. \tag{5}$$

Then the fractional integro-differential equations (1)-(2) *has a solution in* $C(J,X)$ *on* J.

Proof. We transform the Cauchy problem (1)-(2) to be applicable to fixed point problem.

Let the operator $F : C(J,X) \longrightarrow C(J,X)$ be defined as in Theorem 3.1.

Before move ahead, we need to analyze the operator F into sum of two operators P and Q as follows

$$Pu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[f(s,u(s)) + \int_s^t Z_1(\sigma,s,u(s))d\sigma + \int_s^T Z_2(\sigma,s,u(s))d\sigma\Big]ds.$$
$$Qu(t) = u_0 - g(u).$$

For any function $u \in C(J,X)$, we define the norm $\|u\|_{L^1} = \sup\|u(t)\|_C : t \in J\}$. Note that the norms $\|u\|_{L^1}$ and $\|u\|_c$ are equivalent for $u \in C(J,X)$.

We prove that $Pu + Qv \in B_r \subset C(J,X)$, for every $u \in B_r$. Since f, Z_1 and Z_2 are respectively bounded on the compact sets $\Delta_1 = J \times X$ and $\Delta_2 = J \times J \times X$.

Let $\zeta_1 = \sup_{(t,u)\in\Delta_1}\|f(t,u)\|$, $\zeta_2 = \sup_{(t,s,u)\in\Delta_2}\|Z_1(t,s,u)\|$, $\zeta_3 = \sup_{(t,s,u)\in\Delta_2}\|Z_2(t,s,u)\|$, and r with

$$\|u_0\| + M_g + \frac{\|\mu\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\alpha\|\sigma_1\|_{L^1}T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{\alpha\|\sigma_2\|_{L^1}T^{\alpha+1}}{\Gamma(\alpha+2)} \le r,$$

and define $B_r = \{u \in C(J,X) : \|u\|_{L^1} \le r\}$.

Now, we show that operator P is uniformly bounded on B_r.

$$\begin{aligned}\|Pu(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\|f(s,u(s))\| + \int_s^t \|Z_1(\sigma,s,u(s))\|d\sigma \\ &\quad + \int_s^T \|Z_2(\sigma,s,u(s))\|d\sigma\Big]ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\|\mu\|_{L^1} + \int_s^t \|\sigma_1\|_{L^1}d\sigma + \int_s^T \|\sigma_1\|_{L^1}d\sigma\Big]ds \\ &\le \frac{\|\mu\|_{L^1}T^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha\|\sigma_1\|_{L^1}T^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{\alpha\|\sigma_2\|_{L^1}T^{\alpha+1}}{\Gamma(\alpha+2)}.\end{aligned}$$

Now, we prove the compactness of the operator P. For $t_1,t_2\in[0,T],\ u\in B_r$, we have

$$\begin{aligned}\|(Pu)(t_1)-(Pu)(t_2)\| &= \Big\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Big[\|f(s,u(s))\| + \int_s^{t_1}\|Z_1(\sigma,s,u(s))\|d\sigma \\ &\quad + \int_s^T \|Z_2(\sigma,s,u(s))\|d\sigma\Big]ds \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\Big[\|f(s,u(s))\| + \int_s^{t_2}\|Z_1(\sigma,s,u(s))\|d\sigma \\ &\quad + \int_s^T \|Z_2(\sigma,s,u(s))\|d\sigma\Big]ds\Big\| \\ &= \frac{1}{\Gamma(\alpha)}\Big\|\int_{t_2}^{t_1}(t_1-s)^{\alpha-1}\Big[\|f(s,u(s))\| + \int_s^{t_1}\|Z_1(\sigma,s,u(s))\|d\sigma \\ &\quad + \int_s^T \|Z_2(\sigma,s,u(s))\|d\sigma\Big]ds - \int_0^{t_2}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}] \\ &\quad \times\Big[\|f(s,u(s))\| - \int_0^{t_2}[(t_2-s)^{\alpha-1}\int_{t_2}^s \|Z_1(\sigma,s,u(s))\|d\sigma \\ &\quad -(t_1-s)^{\alpha-1}\int_{t_1}^s \|Z_1(\sigma,s,u(s))\|d\sigma]ds \\ &\quad - \int_0^{t_2}[(t_2-s)^{\alpha-1}\int_{t_2}^T \|Z_2(\sigma,s,u(s))\|d\sigma \\ &\quad -(t_1-s)^{\alpha-1}\int_{t_1}^T \|Z_2(\sigma,s,u(s))\|d\sigma]\Big]ds\Big\| \\ &\le \frac{\zeta_1}{\Gamma(\alpha+1)}\Big|2(t_1-t_2)^\alpha + t_2^\alpha - t_1^\alpha\Big| + \frac{\alpha\zeta_2}{\Gamma(\alpha+2)}\Big|2(t_1-t_2)^{\alpha+1} \\ &\quad + t_2^{\alpha+1} - t_1^{\alpha+1}\Big| + \frac{\alpha\zeta_3}{\Gamma(\alpha+2)}\Big|2(t_1-t_2)^{\alpha+1} + t_2^{\alpha+1} - t_1^{\alpha+1}\Big| \\ &\le \frac{2\zeta_1}{\Gamma(\alpha+1)}\Big|(t_1-t_2)^\alpha\Big| + \frac{2\alpha(\zeta_2+\zeta_3)}{\Gamma(\alpha+2)}\Big|(t_1-t_2)^{\alpha+1}\Big|,\end{aligned}$$

which is independent of u. So P is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, P is compact on B_r. Thus all the assumptions of Theorem 3.1 are satisfied. Then the problem (1)-(2) has a solution in $C(J,X)$. □

4. CONCLUSION

The main purpose of this paper was to present new existence and uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as Krasnoselskii's fixed point, some properties of fractional calculus and Banach contraction mapping principle. Moreover, the results of references [1, 25] appear as a special case of our results.

ACKNOWLEDGEMENTS

The authors would like to thank the referees and the editor of this journal for their valuable suggestions and comments that improved this paper.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] B. Ahmad, S. Sivasundaram, Some existence results for fractional integro-differential equations with nonlinear conditions, Communications Appl. Anal. 12 (2008), 107-112.

[2] D. Bahuguna, J. Dabas, Existence and uniqueness of a solution to a partial integro-differential equation by the method of Lines, Electron. J. Qual. Theory Differ. Equ. 4 (2008), 1-12.

[3] K. Balachandran, J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integro-differential equations in Banach spaces, Nonlinear Anal. Theory Meth. Appl. 72 (2010), 4587-4593.

[4] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl. 162 (1991), 494-505.

[5] L. Dawood, A. Sharif, A. Hamoud, Solving higher-order integro-differential equations by VIM and MHPM, Int. J. Appl. Math. 33 (2020), 253-264.

[6] A. Hamoud, Uniqueness and stability results for Caputo fractional Volterra-Fredholm integro-differential equations, J. Sib. Fed. Univ. Math. Phys. 14 (2021), 313-325.

[7] A. Hamoud, Existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integro-differential equations, Adv. Theory Nonlinear Anal. Appl. 4 (2020), 321-331.

[8] A. Hamoud, N. Mohammed, K. Ghadle, Existence and uniqueness results for Volterra-Fredholm integro differential equations, Adv. Theory Nonlinear Anal. Appl. 4 (2020), 361-372.

[9] A. Hamoud, K. Ghadle, The approximate solutions of fractional Volterra-Fredholm integro-differential equations by using analytical techniques, Probl. Anal. Issues Anal. 7 (25) (2018), 41-58.

[10] A. Hamoud, K. Ghadle, Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations, J. Math. Model. 6(1) (2018), 91-104.

[11] A. Hamoud, K. Ghadle, Usage of the homotopy analysis method for solving fractional Volterra-Fredholm integro-differential equation of the second kind, Tamkang J. Math. 49(4) (2018), 301-315.

[12] A. Hamoud, K. Hussain, K. Ghadle, The reliable modified Laplace Adomian decomposition method to solve fractional Volterra-Fredholm integro differential equations, Dynamics of Continuous, Discr. Impuls. Syst. Ser. B: Appl. Algorithms, 26 (2019), 171-184.

[13] A. Hamoud, L. Dawood, K. Ghadle, S. Atshan, Usage of the modified variational iteration technique for solving Fredholm integro-differential equations, Int. J. Mech. Product. Eng. Res. Develop. 9(2) (2019), 895-902,

[14] A. Hamoud, K. Ghadle, Existence and uniqueness of the solution for Volterra-Fredholm integro-differential equations, J. Siberian Federal Univ. Math. Phys. 11(6) (2018), 692-701.

[15] A. Hamoud, K. Ghadle, Existence and uniqueness of solutions for fractional mixed Volterra-Fredholm integro-differential equations, Indian J. Math. 60(3) (2018), 375-395.

[16] A. Hamoud, K. Ghadle, M. Bani Issa, Giniswamy, Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations, Int. J. Appl. Math. 31(3) (2018), 333-348.

[17] A. Hamoud, K. Ghadle, S. Atshan, The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method, Khayyam J. Math. 5(1) (2019), 21-39.

[18] A. Hamoud, K. Ghadle, Some new existence, uniqueness and convergence results for fractional Volterra-Fredholm integro-differential equations, J. Appl. Comput. Mech. 5(1) (2019), 58-69.

[19] R. Ibrahim, S. Momani, On the existence and uniqueness of solutions of a class of fractional differential equations, J. Math. Anal. Appl. 334(1) (2007), 1-10.

[20] K. Karthikeyan, J. Trujillo, Existence and uniqueness results for fractional integro-differential equations with boundary value conditions, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012), 4037-4043.

[21] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Math. Stud., 204, Elsevier, Amsterdam, 2006.

[22] V. Lakshmikantham, M. Rao, Theory of Integro-Differential Equations, Gordon & Breach, London, 1995.

[23] M. Matar, Controllability of fractional semilinear mixed Volterra-Fredholm integro differential equations with nonlocal conditions, Int. J. Math. Anal. 4(23) (2010), 1105-1116.

[24] S. Momani, A. Jameel, S. Al-Azawi, Local and global uniqueness theorems on fractional integro-differential equations via Bihari's and Gronwall's inequalities, Soochow J. Math. 33(4) (2007), 619-627.

[25] S. Momani, A. Jameel, S. Al-Azawi, Local and global uniqueness theorems on fractional integro-differential equations via Bihari's and Gronwall's inequalities, Soochow J. Math. 33(4) (2007), 619-627.

[26] K. Miller, B. Ross, An introduction to the fractional calculus and differential equations, John Wiley, New York, 1993.

[27] Z. Odibat, Sunil Kumar, A robust computational algorithm of homotopy asymptotic method for solving systems of fractional differential equation, J. Comput. Nonlinear Dyn. 14(8) (2019), 1–10.

[28] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-verlag, New York, 1983.

[29] I. Podulbny, Fractional Differential Equations, vol. 198, Academic Press, San Diego, 1999.

[30] M. Rao, Ordinary Differential Equations, East-West Press PVT Limited, 1980.

[31] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Overdone, 1993.

[32] J. Wu, Y. Liu, Existence and uniqueness of solutions for the fractional integro-differential equations in Banach spaces, Electron. J. Differ. Equ. 2009 (2009), 1-8.

[33] J. Wu, Y. Liu, Existence and Uniqueness Results for Fractional Integro-Differential Equations with Nonlocal Conditions, 2nd IEEE International Conference on Information and Financial Engineering, 2010 (2010), 91-94.

[34] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific, Singapore, 2014.