INDEPENDENCE POLYNOMIAL OF THE SIERPINSKI GASKET GRAPH AND THE TOWER OF HANOI GRAPH

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Abstract. In dynamical systems, the most well-known fractals are the Sierpinski gasket graph, also known as the Sierpinski fractal, and the Tower of Hanoi graph. In this paper, we investigate the nth generated independence polynomial of these graphs as well as its properties. In order to partition its spanning subgraphs, we use iterative patterns of the Sierpinski graph and the Hanoi graph. Furthermore, we consider the relationship between the Tower of Hanoi graph and the Sierpinski fractal in terms of their independence polynomial.

Keywords: Sierpinski fractal; independence polynomial; graph of Tower of Hanoi; dynamical systems; fractals.

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1. INTRODUCTION

The study of system which is moving or changing in time is called dynamics .Fractals are geometric objects that are static images which is self similar and whose dimension exceeds its

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topological dimension.The Sierpinski fractal, also known as the Sierpinski gasket graph, is a well-known fractal and is denoted by *SGg*.The Tower of Hanoi is a classic puzzle created in 1883 by Edward Lucas , and the Tower of Hanoi graph is denoted by *THg*.The graphs such as Tower of Hanoi and Sierpinski are came from these games like sources. As a result, it is interesting to investigate graphs whose drawings can be approximated to Sierpinski or Hanoi graphs.In the past few years, we have used these structures extensively in topology, algorithm analysis, discrete mathematical structures, and various types of networks [1].Both graphs have the same vertex set but can be labeled differently.This distinction allows for different approaches to both graphs.The relationship between these two graphs is useful in studying the classic TH task[2].

Determining the independence number and polynomial for a general graph is NP-hard[3], and its exact computation is extremely difficult.As a result, we choose these two graphs that are exactly tractable, and this is a common research path for NP-complete problems.The problem of determining independence polynomial has found widespread application in various of fields, most particularly theoretical computer science, coding theory, and wireless networks [4].

Definition 1.1. *[5] An independent set in a graph G is a vertex subset $S \subseteq V(G)$ that contains no edge of G.A graph's independence number is the maximum size of an independent set of vertices ie, $I(G)$.The independence polynomial was introduced by I.Gutman and F.Harary.*

Definition 1.2. *[5] Let s_k denote the number of independent sets of size k,which are induced sub graphs of G,then $I(G,z) = \sum_{k=0}^{I(G)} s_k z^k$ where $I(G)$ is the independence number of G.*

The independence polynomials are almost everywhere, but it is an NP complete problem to determine the independence polynomial of a graph.

2. SIERPINSKI TRIANGLE

The Sierpinski Gasket is a fascinating mathematical term that visualizes a variety of mathematical concepts. It was named after the Polish mathematician who suggested the concept first. The definition is a fractal, which is an equilateral triangle that can be subdivided into four different equilateral triangles, each $\frac{1}{4}$ the size of the original. These subdivided triangles can

then be subdivided again, and the fractal continues in this manner. The mathematical concept of self-similarity is one of the most important ideas derived from the Sierpinski Gasket.

The SGg can be constructed recursively.Let S_n represent the nth generation graph where $n \geq$ 1.If $n = 1$,S_1 is an equilateral triangle with three vertices and three edges.If $n = 2$, remove from the middle a triangle whose dimensions are exactly half that of the original triangle.This produces three smaller equilateral triangles, each with one-half the dimensions of the original triangle, generating S_2.For $n > 2$,S_n is obtained from S_{n-1} by performing the aforementioned operations on each triangle in S_{n-1}.The graphs obtained through n iterations of this method that leads to the Sierpinski gasket graphs and denoted by S_n.The Sierpinski fractal is used in digital image steganography, dynamic systems, probability, and psychology [6].

Proposition 2.1. *[7] S_n has $\frac{3}{2}[3^{n-1}+1]$ vertices and 3^n edges.*

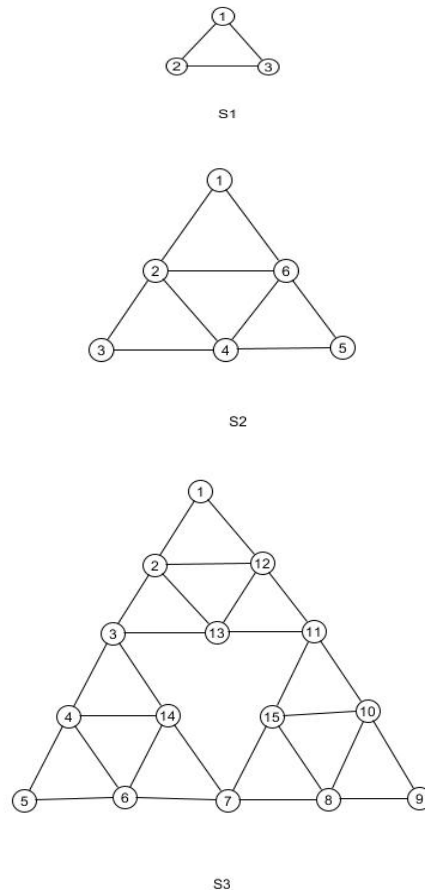Fig.1 illustrates the first three iterations of the SGg of order n, S_n for $n = 1, 2, 3$.

Fig.1

3. THE TOWER OF HANOI

The Tower of Hanoi and its variations, which were first published in 1883 under the title N. Claus and sold as a puzzle game, have been the subject of hundreds of scientific papers in fields ranging from mathematics to computer science to psychology[8].In fact, when we look at the set of all possible puzzle configurations as defined in Analyzing the Tower of Hanoi, as well as the relationships between those configurations, we find a surprising connection to the Sierpinski gasket.

THg can be constructed by the recursive modular method [9].For $n = 0$,H_0 is the complete graph K_3.For $n \geq 1$,H_n is obtained from three copies of H_{n-1} joined by three new edges, each one connecting a pair of vertices from two different replicas of H_{n-1} and these three new edges are called special edges.
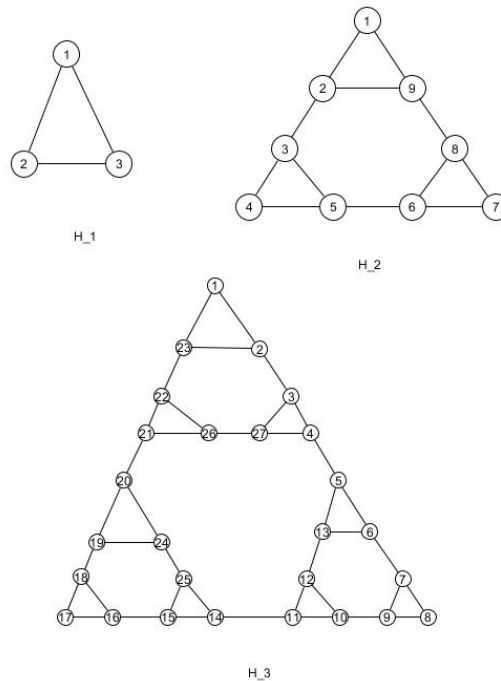

Fig.2 : THg of order 1,2,and 3: H_1, H_2 and H_3 .

Geometrically the Tower of Hanoi graph can be interpreted as the intersection graph of the remaining triangles after the nth iteration of S_n.

4. INDEPENDENCE POLYNOMIAL OF *SGg* OF ORDER 1,2 AND 3

We can calculate independent sets of different orders of *SGg* by labeling the vertices differently.

For $n = 1$, $I(S_1, z)$ equals $1 + 3z$.

For $n = 2$, S_2 contains the following independent sets of different orders.There is only one independent set of order three, and there are six independent sets of order two.Because all of the vertices of S_1 are order 1 independent sets, there are a total of six elements.The empty set is regarded as an independent set of order zero.Therefore independence polynomial of S_2 is given by $z^3 + 6z^2 + 6z + 1$.

For $n = 3$, S_3 contains independent sets of 2 elements of order 6,52 elements of order 5,138 elements of order 4,166 elements of order 3,78 elements of order 2,15 elements of order 1 and empty set of order 0.Therefore independence polynomial of S_3 is given by $2z^6 + 52z^5 + 138z^4 + 166z^3 + 78z^2 + 15z + 1$.

The independence number of the SGg of order 2 is three, and the independence number of order 3 is six. Also, the maximum number of independent sets for S_2 and S_3 is one and two, respectively.

As a result, the independence polynomial of order 2 and 3 of *SGg* satisfies the following standard results.This demonstrates that the independence polynomial of our Sierpinski graph is unique.

Theorem 4.1. *[10] The independence number of the SGg of order n, $S_n, n \geq 2$ is $\alpha_n = \frac{3^{(n-1)}+3}{2}$.*

Theorem 4.2. *[10] For $n \geq 2$,the number of maximum independent sets of the SGg of order n,S_n is $2^{\frac{3^{(n-2)}-1}{2}}$.*

5. THE INDEPENDENCE POLYNOMIAL OF THE *SGg* OF ORDER n

The class of self-similar groups contains many interesting properties. The Sierpinski graph and the Tower of Hanoi graph use this self-similar representation properties.We must analyze the relationship between spanning subgraphs for the independence polynomial of order n. The following observations are true.

The restrictions of special vertices determine whether or not the spanning subgraphs belong to the recursive formula for SGg. We will use the following partitions to find the number of independent sets on S_n, which corresponds to the result obtained in [11].The graph's three outermost vertices of degree two play a crucial role in this partition.Furthermore, we should consider cases where the outermost vertices belong to the same connected components separately. We have the spanning subgraphs of S_n using [11] as

(1) $G_{3,n}$ denotes the collection of all subgraphs with the same number of vertices of S_n with the three outermost vertices composed of same component which are connected.

(2) $G_{1,n}^p$ denotes the collection of all subgraphs with the same number of vertices of S_n with only left and right vertices belong to the same component which are connected.

(3) $G_{1,n}^q$ denotes the collection of all subgraphs with the same number of vertices of S_n with only right and topmost vertices belong to the same component which are connected.

(4) $G_{1,n}^r$ denotes the collection of all subgraphs with the same number of vertices of S_n with only left and top most vertices belong to the same component which are connected.

(5) $G_{0,n}$ denotes the collection of all subgraphs with the same number of vertices of S_n with the three outermost vertices are composed of three different components which are connected.

The S_n subgraphs can be represented graphically as shown in Fig.3. If we connect two outmost vertices in our diagram with a solid line, it means they are in the same connected component, according to our convention.

From Fig.3, the subgraphs of S_n can be written as $S_n = G_{3,n} \cup G_{1,n}^p \cup G_{1,n}^q \cup G_{1,n}^r \cup G_{0,n}$ for each $n \geq 1$.

Consider how these subgraphs contribute to the nth independence polynomial of the Sierpinski graph.

Theorem 5.1. *For each $n > 0$,the independence polynomial $I_n(z)$ is given by $I_n(z) = I_{3,n} + 3I_{1,n} + I_{0,n}$ with initial conditions $I_{3,0}(z) = 1, I_{1,0}(z) = z, I_{0,0}(z) = 0$.*

Proof. The initial conditions are easy to verify. By the definition of independence polynomials,$I(G,z) = \sum_I z^{[I]}$. Independence polynomial $I_n(z) = \sum_{I \in G_{3,n}} z^{[I]} + \sum_{I \in G_{1,n}^p} z^{[I]} +$

G_3n   G_{1,n}^p   G_{1,n}^q
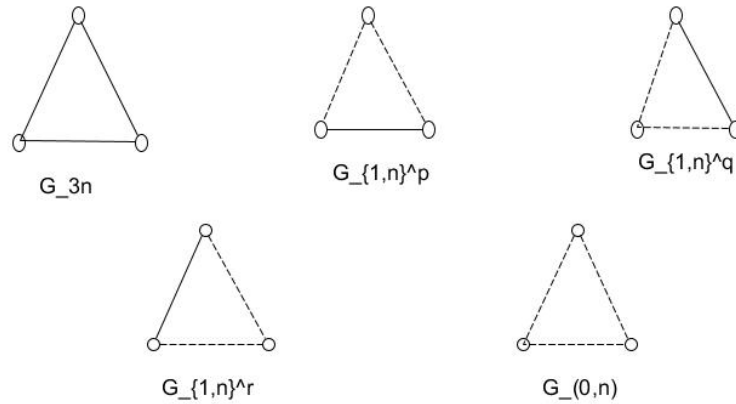
G_{1,n}^r   G_(0,n)

Fig.3

$\sum_{I \in G_{1,n}^q} z^{[I]} + \sum_{I \in G_{1,n}^r} z^{[I]} + \sum_{I \in G_{0,n}} z^{[I]}. = I_{3,n} + 3I_{1,n} + I_{0,n}$ since by rotational invariance of the graph S_n, we have $I_{1,n}^p(z) = I_{1,n}^q(z) = I_{1,n}^r(z)$. $\qquad \square$

Similar way we can consider the independence polynomial of the Tower of Hanoi graph and it has a surprising relationship with the Sierpinski gasket graph.

6. INDEPENDENCE POLYNOMIAL OF HANOI GRAPH OF ORDER 1 AND 2

For $n = 1$, the independence polynomial of the THg is $I(H_1, z) = 1 + 3z$.

For $n = 2$, it is easy to check that the number of independence sets of order 3 is 18, of order 2 is 24, of order 1 is 9 and of order 0 is 1. Therefore independence polynomial of THg of order two is $I(H_2, z) = 18z^3 + 24z^2 + 9z + 1$.

In comparison to the following corollary, the independence polynomial's uniqueness is established.

Corollary 6.1. *[12] The number of vertices of H_n is 3^{n+1} and the number of edges is $\frac{3^{n+2}-3}{2}$.*

7. INDEPENDENCE POLYNOMIAL OF THg OF ORDER n

In this case, we can easily ignore the labeling of the vertices of the Tower of Hanoi graph and consider it as an unlabeled graph.

We will use the following partitions to find the number of independent sets on H_n, which corresponds to the result obtained in [13]. We have the spanning subgraphs of H_n using [13] as

(1) $M_{3,n}$ denotes the collection of all subgraphs with the same number of vertices of H_n with the three outermost vertices composed of same component which are connected.

(2) $M_{1,n}^p$ denotes the collection of all subgraphs with the same number of vertices of H_n with only left and right vertices belong to the same component which are connected.

(3) $M_{1,n}^q$ denotes the collection of all subgraphs with the same number of vertices of H_n with only right and topmost vertices belong to the same component which are connected.

(4) $M_{1,n}^r$ denotes the collection of all subgraphs with the same number of vertices of H_n with only left and top most vertices belong to the same component which are connected.

(5) $M_{0,n}$ denotes the collection of all subgraphs with the same number of vertices of H_n with the three outermost vertices are composed of three different components which are connected.

So for each $n \geq 1$, $H_n = M_{3,n} \cup M_{1,n}^p \cup M_{1,n}^q \cup M_{1,n}^r \cup M_{0,n}$.

We provide a recursive formula for $I(H_n)(z)$ to analyze the relationship between spanning subgraphs of the Tower of Hanoi graph.

Theorem 7.1. *For each $n > 0$, the independence polynomial $I(H_n)(z)$ is given by $I(H_n)(z) = I_{3,n} + 3I_{1,n} + I_{0,n}$ with initial conditions $I_{3,0}(z) = 1, I_{1,0}(z) = z$, $I_{0,0}(z) = 0$.*

Proof. The initial conditions are easy to verify. By the definition of independence polynomials, $I(G,z) = \sum_I z^{[I]}$. Independence polynomial $I_n(z) = \sum_{I \in M_{3,n}} z^{[I]} + \sum_{I \in M_{1,n}^p} z^{[I]} + \sum_{I \in M_{1,n}^q} z^{[I]} + \sum_{I \in M_{1,n}^r} z^{[I]} + \sum_{I \in M_{0,n}} z^{[I]}$. $= I_{3,n} + 3I_{1,n} + I_{0,n}$ since by rotational invariance of the graph H_n, we have $I_{1,n}^p(z) = I_{1,n}^q(z) = I_{1,n}^r(z)$. $\square$

8. NUMBER OF INDEPENDENT SETS ON H_n

The number of independent sets will be calculated using the above partitions of the Hanoi graph. Let r_n be the number of independent sets such that the vertex subset does not contain all three outmost vertices. Let s_n be the number of independent sets in which the vertex subset contains only one of the three outmost vertices. Let t_n be the number of independent sets in which the vertex subset contains the exact two specified vertices of the three outmost vertices. Let u_n be the number of independent sets such that the vertex subset contains all three outmost vertices.

Proposition 8.1. *The number of independent sets of H_n is given by $i_n = r_n + s_n + t_n + u_n$.*

Proof. It is simple to verify that the result holds for $n = 1, 2$ by doing the following:

For $n = 1$, independence polynomial of H_1 is $1 + 3z$. So $r_0 = 1, s_0 = 3$, $t_0 = u_0 = 0$.Therefore $i_0 = r_0 + s_0 + t_0 + u_0 = 4$. Thus the result hold for $n = 1$. For $n = 2$,independence polynomial of H_2 is $18z^3 + 24z^2 + 9z + 1$.It is easy to check that $r_1 = 18$, $s_1 = 24$, $t_1 = 9$ and $u_1 = 1$. Therefore total number of independent sets of H_2, $i_1 = r_1 + s_1 + t_1 + u_1 = 52$.Hence the result verified for $n = 2$.

For the general result,by utilizing the above outcome for each $n \geq 1$, $H_n = M_{3,n} \cup M_{1,n}^p \cup M_{1,n}^q \cup M_{1,n}^r \cup M_{0,n}$, we have the total number of independent sets is equal to the sum of the independent sets of all the above partitions. Hence the result.

$\square$

9. THE RELATIONSHIP BETWEEN THE INDEPENDENCE POLYNOMIAL OF SGg AND THE INDEPENDENCE POLYNOMIAL OF THg

SGg of order 3 is analogus to THg of order 3.The only difference between the independence polynomials of SGg and THg is that the special edges do not determine a spanning subgraph of S_n, and these edges do not belong to any of the copies of S_n. However, in the case of the Hanoi graph, we must specify how many special edges belong to its spanning subgraph.

Proposition 9.1. *[12] The number of independent sets of THg of order 3 is* 108144.

Remark 9.2. *The independence polynomial of SGg of order 3,S_3 is not same as the independence polynomial of THg of order 3. We have $I(S_3, z) = 2z^6 + 52z^5 + 138z^4 + 166z^3 + 78z^2 + 15z + 1$ and it contains 452 independent sets.But THg of order 3 contains 108144 independent sets and is a polynomial of degree 12.So both the graph has different independence polynomials.*

10. CONCLUSION

It is interesting to study the independence polynomial of a fractal network.In this paper,we investigate the independence polynomial of Sierpinski graph and Tower of Hanoi graph.First, we partition the independent sets of these graphs into five disjoint subsets, and then we inspect its various possible contributions.Finally, we derive the recursive formula for their independence

polynomial based on their self-similar structure. Using these formulas, we can define the exact number of independent sets and the independence number of these graphs.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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