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AN EFFICIENT HIGHER ORDER COMPACT FINITE DIFFERENCE SCHEMES FOR LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

E. DHANANJAYA[1,*], R. BHUVANA VIJAYA[2]

[1]Jawaharlal Nehru Technological University Anantapur, Ananthapuramu, Andhra Pradesh, India

[2]Department of Mathematics, Jawaharlal Nehru Technological University Anantapur, Ananthapuramu, Andhra Pradesh, India

Abstract: To approximate the partial derivative of Linear Hyperbolic Partial Differential Equations (LPHDE), a system of compact schemes used at non-boundary nodes. For the spatial discretization of the advection term, instead of using the upwind schemes, central difference based compact 4^{th} order and conventional 2^{nd} order schemes are experimented. The main aim of the numerical experiments carried out is, to assess the ability of the compact 4^{th} order scheme in capturing the convection process, in comparison with the conventional 2^{nd} order scheme.

Keywords: central difference based compact schemes; linear hyperbolic partial differential equations; incompressible fluid flow; explicit scheme; implicit scheme; spatial discretization.

2010 AMS Subject Classification: 65M20.

1. INTRODUCTION

The general Linear Hyperbolic Partial Differential Equations is the advection equation. Its mathematical form in one-dimension is given by,

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*Corresponding author
E-mail address: dhanuelisetty@gmail.com
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$$\frac{\partial f}{\partial t}+a\frac{\partial f}{\partial x_1}=0 \tag{1.1}$$

It is also called as wave equation, when this equation is used to model the propagation of sufficiently small perturbations with a phase speed “a”. Equation (1.1) should be satisfied for the distance on the real line.

It is important to define a initial condition to satisfy the well-posedness of the problem and to find a unique solution. One of the distinguishing feature of this equation is that, the solution is constant along any characteristic curve $x_1 - at = 0$. In the next two subsections, the time and the spatial discretization using the 4^{th} order compact schemes of the one dimensional wave equation and the numerical algorithm to solve the fully discretized equation is explained. Herein, the fourth order compact schemes relevant discretize the first order spatial derivative on a periodic and non-periodic computational domains are presented.

The last part of this section, the developed numerical schemes for the equation (1.1) is tested on several test problems. Each of the test problems selected represent to a particular physical situation, which exhibit an unique dispersive behavior of the quantity being transported or propagated. Since, central difference based compact schemes for the first derivative do not induce artificial numerical dissipation, itis essential their performance is evaluated more rigorously with such linear problems, before employing them for the non-linear PDE. To this end, the aim of the numerical experiments is to demonstrate the superiority of the central difference based compact 4^{th} order schemes to model a convection process, in comparison with the central difference based 2^{nd} order schemes.

2. TIME DISCRETIZATION OF ONE-DIMENSIONAL ADVECTION EQUATION

The explicit Adams-Bashforth (AB) method for the skew-symmetric form of the spatial first derivative, the time discretized form of the equation (1.1) is given by,

$$f_i^{n+1}=f_i^n+\frac{\Delta t}{2}\left[\frac{-3}{2}\left[a\frac{\partial f_i^n}{\partial x_1}+\frac{\partial\left(af_i^n\right)}{\partial x_1}\right]+\frac{1}{2}\left[a\frac{\partial f_i^{n-1}}{\partial x_1}+\frac{\partial\left(af_i^{n-1}\right)}{\partial x_1}\right]\right] \tag{1.2}$$

The skew-symmetric form of the convection term

$$\frac{a\dfrac{\partial f_i^n}{\partial x_1}+\dfrac{\partial\left(af_i^n\right)}{\partial x_1}}{2}$$

Appearing in the RHS of the equation (1.2) is nothing but the average of its advection and divergence forms. The reason for using skew symmetric form is, to assess the accuracy of the compact schemes, developed for this term, because of its importance in the discrete conservation of kinetic energy, in the simulation of the incompressible fluid flows.

3. Compact Schemes for Spatial Discretization of One Dimensional Advection Equation on Periodic Computational Domain

At boundary nodes ψ_0 and ψ_{x_1} are known, the partial derivative, $\dfrac{\partial f}{\partial x_1}$ approximated on uniform grid at ψ_i, where $i=1........\left(N_{x_1}-1\right)$. The coefficients in the matrix representation is given by,

$$[A_x]F_x = [B_x]f_R \tag{1.3}$$

Where A_x and B_x represent the matrices of the coefficients and F_x, f_R represent the column vectors of non-boundary values in LHS and RHS respectively.

Using equation (1.3), to compute the values of $\dfrac{\partial f}{\partial x_1}$ at the $(n\ -\ 1)^{th}$ and n^{th} time levels are deduced respectively as,

$$F_x^{n-1} = [A_x]^{-1}\left[B_x\right] f_R^{n-1} \tag{1.4}$$

and

$$F_x^{n} = [A_x]^{-1}\left[B_x\right] f_R^{n} \tag{1.5}$$

Compact schemes to interpolate the values of the function, f on the non-boundary nodes, at φ_i where, $i=1........\left(N_{x_1}-1\right)$ located on one-dimensional structured Cartesian uniform grid

(Figure 1.1) is given by,

$$\alpha f_{\varphi_{N_{x_1}}} + f_{\varphi_1} + \beta f_{\varphi_2} = r\frac{\left(f_{\psi_1} + f_{\psi_{N_{x_1}}}\right)}{\Delta x_1} + o\left(\Delta x_1^{\,4}\right) \tag{1.6a}$$

$$\alpha f_{\varphi_{i-1}} + f_{\varphi_i} + \beta f_{\varphi_{i+1}} = r\frac{\left(f_{\psi_i} + f_{\psi_{i-1}}\right)}{\Delta x_1} + o\left(\Delta x_1^{\,4}\right), i = 1........\left(N_{x_1} - 1\right) \tag{1.6b}$$

$$\alpha f_{\varphi_{N_{x_1}-1}} + f_{\varphi_{N_{x_1}}} + \beta f_{\varphi_1} = r\frac{\left(f_{\psi_{N_{x_1}}} + f_{\psi_{N_{x_1}-1}}\right)}{\Delta x_1} + o\left(\Delta x_1^{\,4}\right) \tag{1.6c}$$

The above equations solved simultaneously, with the known values of the function, f on the left boundary node, at ψ_0 and some guessed values on the non-boundary nodes, at ψ_i where, $i = 1........\left(N_{x_1} - 1\right)$. Using Taylor series expansion the values of the coefficients involved in the equation (1.6) are given in the Table 1.1. Equation (1.6) is in matrix form of given by,

$$[A_x]f_x = [B_x]f_R \tag{1.7}$$

where, A_x and B_x represent the coefficient matrices and f_x , f_R represent the column vectors of equations (1.6a) -(1.6c)

Similarly, Using equation (1.7), to find the values of the function, f at the $\left(n - 1\right)^{th}$ and n^{th} time levels as,

$$f_x^{\,n\text{-}1} = [A_x]^{-1}[B_x]f_R^{\,n-1} \tag{1.8}$$

and

$$f_x^{\,n} = [A_x]^{-1}[B_x]f_R^{\,n} \tag{1.9}$$

The calculations in the equations (1.8) or (1.9) involve tri-diagonal matrix and a matrix vector multiplication.

A system of compact schemes to approximate the partial derivative, $\dfrac{\partial\left(af\right)}{\partial x_1}$ on the right boundary node and on the non-boundary nodes along the one-dimensional Cartesian uniform grid (Figure 1.2) are given by,

$$\alpha\left(af'\right)_{\psi_{N_{x_1}}}+\left(af'\right)_{\psi_1}+\beta\left(af'\right)_{\psi_2}=r\frac{\left(\left(af\right)_{\varphi_2}-\left(af\right)_{\varphi_1}\right)}{\Delta x_1}+o\left(\Delta x_1^{\;4}\right) \qquad (1.10a)$$

$$\alpha\left(af'\right)_{\psi_{i-1}}+\left(af'\right)_{\psi_i}+\beta\left(af'\right)_{\psi_{i+1}}=r\frac{\left(\left(af\right)_{\varphi_{i+1}}-\left(af\right)_{\varphi_i}\right)}{\Delta x_1}+o\left(\Delta x_1^{\;4}\right), \qquad (1.10b)$$
$$i=1........\left(N_{x_1}-1\right)$$

$$\alpha\left(af'\right)_{\psi_{N_{x_1}-1}}+\left(af'\right)_{\psi_{N_{x_1}}}+\beta\left(af'\right)_{\psi_1}=r\frac{\left(\left(af\right)_{\varphi_1}-\left(af\right)_{\varphi_{N_{x_1}}}\right)}{\Delta x_1}+o\left(\Delta x_1^{\;4}\right) \qquad (1.10c)$$

The above (1.10a) - (1.10c) are simultaneously solved on the non-boundary nodes using the known values of the function f. Using Taylor series expansion, the coefficient values involved in the equation (1.10) are given in the Table 1.2

The equation (1.10) represented in matrix form is given by,

$$\left[A_x\right]F_x=\left[B_x\right]f_R \qquad (1.11)$$

Where, A_x and B_x represent the matrices of the coefficients of equations (1.10a) - (1.10c) and F_x, f_R represent the column vectors of values.

Using equation (1.11), to compute the partial derivative $\frac{\partial\left(af\right)}{\partial x_1}$ values of at the $\left(n\ -\ 1\right)^{th}$ and n^{th} time levels as,

$$F_x^{\;n-1}=\left[A_x\right]^{-1}\left[B_x\right]f_R^{\;n-1} \qquad (1.12)$$

and

$$F_x^{\;n}=\left[A_x\right]^{-1}\left[B_x\right]f_R^{\;n} \qquad (1.13)$$

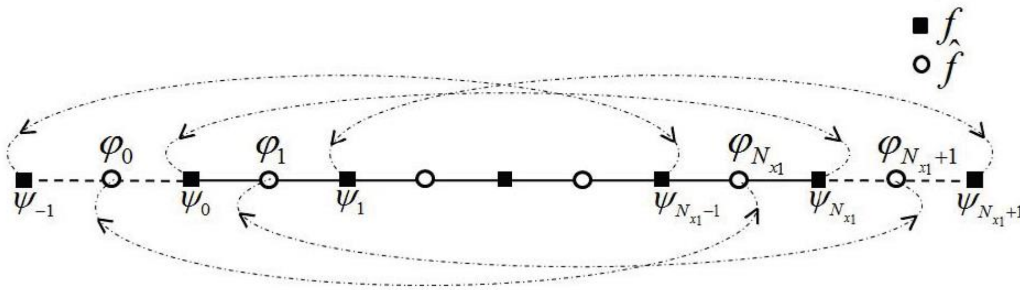

Figure 1.1: One-dimensional structured Cartesian uniform grid used to approximate the function value, $\hat{f}$ at the φ_i locations, using the known function value, f at the ψ_i locations on a periodic domain

Table 1.1: Values of the coefficients in the equations (1.6a) - (1.6c)

Values of the coefficients in the LHS	Values of the coefficients in the RHS
$\alpha = \frac{1.0}{6.0}$	----
	$\gamma = \frac{2.0}{3.0}$
$\beta = \frac{1.0}{6.0}$	----

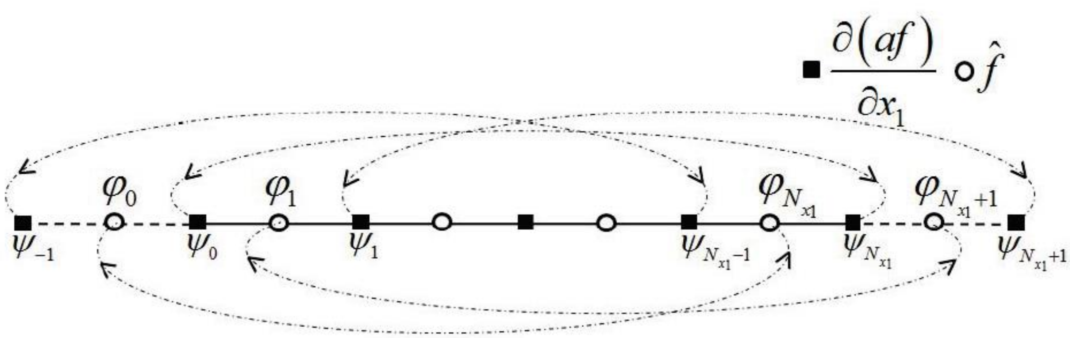

Figure 1.2: One-dimensional structured Cartesian uniform grid used to approximate the divergence form of the convection term $\frac{\partial(af)}{\partial x_1}$ on a periodic domain

Table 1.2: Values of the coefficients in the equations (1.10a) - (1.10c)

Values of the coefficients in the LHS	Values of the coefficients in the RHS
$\alpha = \frac{1.0}{22.0}$	----
----	$\gamma = \frac{12.0}{11.0}$
$\beta = \frac{1.0}{22.0}$	----

4. ALGORITHM TO SOLVE THE FULLY DESCRETIZED ONE-DIMENSIONAL ADVECTION EQUATION ON PERIODIC COMPUTATIONAL DOMAIN

The summation of the terms in the RHS of the equation (1.2) corresponding to the right boundary node, at N_{x_1} and the non-boundary nodes, at ψ_i where, $i = 1........(N_{x_1} - 1)$ are carried out, using the known values of the partial derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial (af)}{\partial x_1}$ and the function f. After storing the RHS value and function value in the equation (1.2) corresponding to each node of interest in the column vector C_R and f_R respectively, the function value at the $(n+1)^{th}$ time level is computed, in a straight forward manner and is given as,

$$f_R^{n+1} = C_R^n \tag{1.14}$$

It is worth mentioning that, during the first iteration of the time evolution of the solution, the calculations in the RHS of the equation (1.2) are carried out, by setting as equal, the values of the function at the $(n\text{-}1)^{th}$ and n^{th} time levels. In the subsequent iterations of the time marching, the value of the function at the n^{th} time level is assigned as the function value at $(n\text{-}1)^{th}$ time level and the value of the function computed during current time step (i.e. at the $(n+1)^{th}$ time level) is assigned as the function value at n^{th} time level. The solution is marched in time, by solving the equation (1.2) recursively, for the succeeding time steps, following the steps as discussed above.

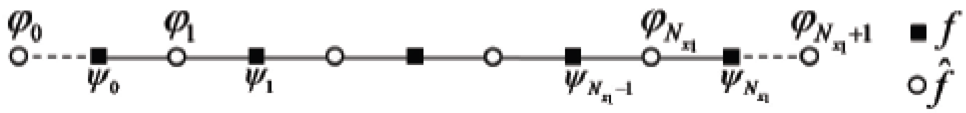

Figure 1.3: Uniform grid to calculate the function value, f at φ_i locations, using the known value of the function at the ψ_i locations on a non-periodic domain

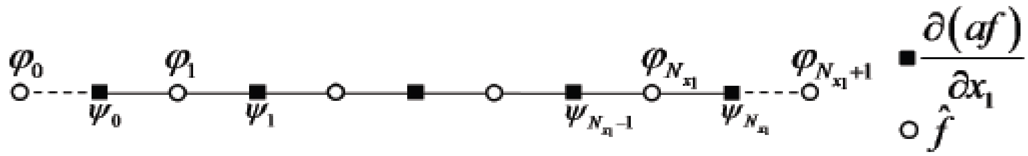

Figure 1.4: Uniform grid used to approximate the convection term $\frac{\partial(af)}{\partial x_1}$ on a non-periodic Computational Domain

5. Algorithm to Solve the Fully Discretized One-Dimensional Advection Equation on Non-Periodic Computational Domain

The terms in the RHS of the equation (1.2) to the non-boundary nodes, at ψ_i where, $i = 1........(N_{x_1} - 1)$ are carried out, using the known values of the partial derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial(af)}{\partial x_1}$ and the function f.

Using the equation (1.2) corresponding to each node of interest in the column vector c_R and f_R respectively, the function value at the $(n+1)^{th}$ time level is computed, in a straight forward manner and is given as,

$$f_R^{n+1} = c_R^n \tag{1.15}$$

6. TEST PROBLEMS

The exact solutions of these problems satisfy the PDE, subjected to an initial and a Dirichlet boundary condition. In the case of numerical scheme developed for the LHPDE, the time discretization is carried out using a fully explicit Euler method.

PROBLEM 1:

$$f\left(x_1,t\right)=0.5\exp\left(-\frac{\left(x_1-at\right)^2}{3}\ln\left(2\right)\right), -20\le x_1\le 420, \Delta x=1, a=1 \tag{1.16a}$$

with the initial condition given by,

$$f\left(x_1,0\right)=0.5\exp\left(-\frac{\left(x_1\right)^2}{3}\ln\left(2\right)\right) \tag{1.16b}$$

and the boundary condition given by,

$$f\left(-20,t\right)=0.5\exp\left(-\frac{\left(-20-at\right)^2}{3}\ln\left(2\right)\right) \tag{1.16c}$$

Where, a is the wave propagation speed. Herein, the interest is to examine the nature of the wave profile at the time $t=400$, after it is convected to a distance of $x_1=400$. In the numerical experiments carried out on this problem, the values of the grid spacing and the wave propagation speed, a are fixed, as one unit. This in turn means, the CFL number is equal to the value of the time step, used in the calculation.

There are two numerical experiments carried out on this problem. In the first problem, the convection of the wave profile in a non-periodic computational domain with $CFL=0.0005$, subjected to the initial and the boundary condition, given by the equations (1.16b) and (1.16c), respectively is simulated. In the second problem, the convection of the wave profile in a periodic domain, subjected to the boundary condition of $f\left(-20,t\right)=f\left(450,t\right)$ and the initial condition, given by the equation (1.16b) is simulated.

RESULTS

In the Figure 1.5, the exact solution and the numerical solutions are compared based on the compact 4^{th} order and the conventional 2^{nd} order schemes is presented.

Similarly, in the Figure 1.6, the exact solution and the numerical solutions are compared based on the compact 4^{th} order and the conventional 2^{nd} order schemes with the exact solution is presented. From these figures, it is found that, the solution obtained by imposing periodic boundary condition is identical, with the same, produced by imposing non-periodic boundary conditions. This finding, confirms the correct implementation of the periodic tridiagonal algorithm, in the solver.

From this numerical experiment, it is found that, the usage of compact 4^{th} order central scheme, to approximate the spatial first derivatives in the wave equation, confirms, its good dispersion preserving character. Generally, in such propagation problems, if the central difference based schemes are used to approximate the first derivatives, it will cause spurious oscillations in the solutions, due to the inherent dispersion error. This behavior is proved correct, with the solutions arrived, using the 2^{nd} order central scheme. Rather, the solution arrived using the compact schemes shows only mild oscillations, along with only a small dampening in the amplitude. This numerical behavior proves its ability to maintain better trade-off between the dispersion and dissipation error than the 2^{nd} order schemes.

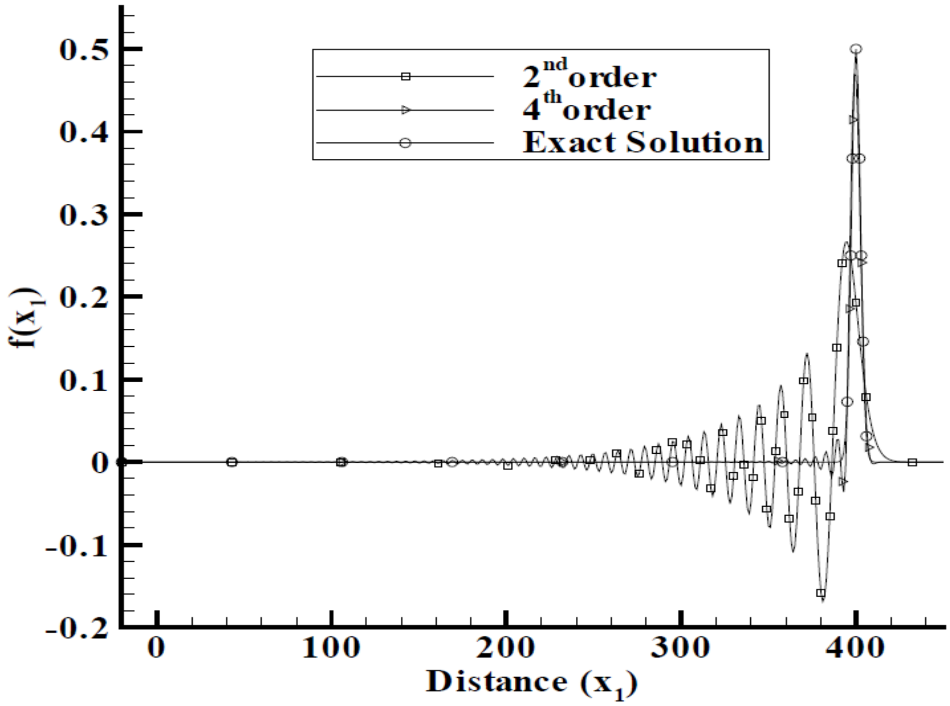

Figure 1.5: (a) Nature of wave profile at $t = 400$ obtained by imposing non-periodic boundary condition.

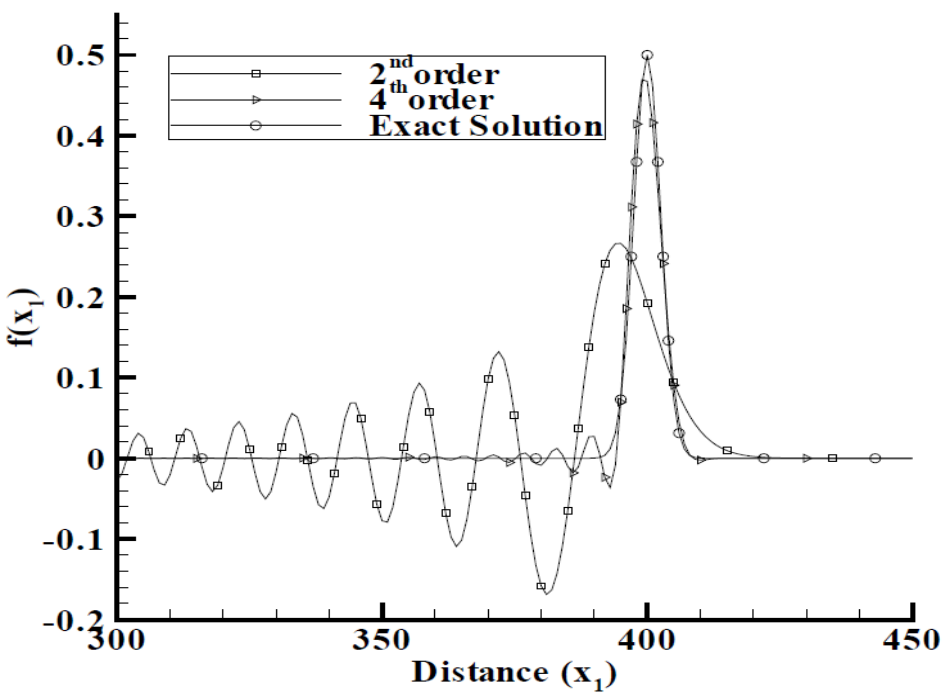

Figure 1.5: (b) Truncated view of the wave profile between $x_1 = 300$ and $x_1 = 450$

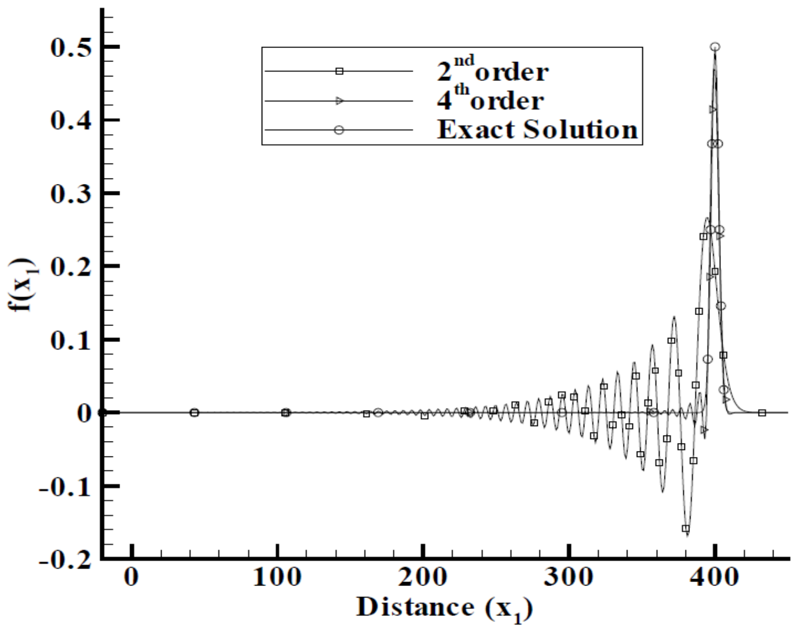

Figure 1.6: (a) Nature of wave profile at $t=400$ obtained by imposing non-periodic boundary condition.

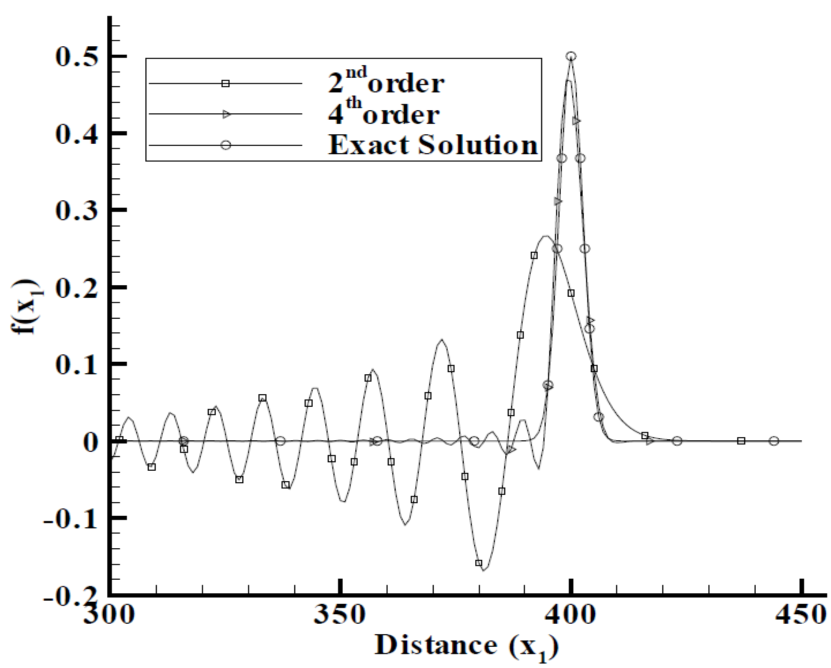

Figure 1.6: (b) Truncated view of the wave profile between $x_1=300$ and $x_1=450$

PROBLEM 2:

$$f(x_1,t) = 0.5\cos\left[10\pi(x_1 - at)\right], 0 \le x_1 \le 1, a = 0.05 \tag{1.17a}$$

with the boundary condition given by,

$$f(0,t) = 0 \tag{1.17b}$$

and the initial condition given by,

$$f(x_1,0) = 0.5\cos\left[10\pi(x_1)\right] \tag{1.17c}$$

Where, a is the wave propagation speed. The exact solution given by the equation (1.17a) represents a cosine profile of periodicity $\frac{1}{5}$ and amplitude 0.5. Ideally, in the case of wave equation, if the scalar quantity is defined on only one boundary, it means, it is entering the domain through the inflow boundary. In the meantime, if there is no boundary condition is imposed on the other boundary, it means, the scalar quantity leaves the domain through the outflow boundary, without any distortion or reflection. In this particular problem, a sudden discontinuity in the solution is introduced at the in let of the domain (i.e.) at $x_1 = 0$. As the cosine wave is propagating at a phase speed of 0.05, the discontinuity travels, along with the wave at this particular speed and leaves the domain, at $t = 20$. During the course of passing through the domain, the discontinuity introduces, spurious oscillations in the wave propagation. But it is physically expected, that these spurious oscillations are damped out, after the wave passes through the domain. That is, after several cycles, the wave must be seen along the length of the domain without any oscillations.

Therefore, the objective of this numerical simulation is to assess and compare the stability of the compact 4^{th} order and the conventional 2^{nd} order schemes. The numerical parameters for this problem are as follows: For the given wave speed, a very low CFL number of $5\text{x}10^{-4}$ is considered, by selecting the grid spacing value as 0.01. The time integration is performed for a long time of $t = 100$. The solutions obtained at different time intervals, viz., $t = 10, 25, 30, 60$ and 100 plotted over the length of the domain is shown, in the Figure 1.7.

RESULTS:

From these graphs, it is clear that, at $t = 10, 25$ and 30, spurious oscillations are more persistent in the domain, for the solution produced using 2^{nd} order scheme, when compared to that of the 4^{th} order scheme. It is because, in addition to the inherent oscillations in the solution, caused by the imposed discontinuity, the high dispersion error of the second order scheme, induces additional oscillatory behavior, in the wave propagation. The oscillation in the wave propagation is meager. However, the dissipation induced by this scheme in the solution is not sufficient to suppress the oscillations due to the discontinuity. At $t = 60$ and $t = 100$, the solutions based on both the schemes, oscillate very close to the $f = 0$ line. But one distinct feature about the compact scheme is that, after the time $t = 20$ at which, the wave leaves the domain, the oscillations observed at the intervals $t = 10, 25, 30, 60$ and 100 fluctuate very close to the $f = 0$ line. Whereas, at these intervals the solutions produced using the second order scheme show significant fluctuation about the $f = 0$ line. From this numerical simulation, the better stability and dispersion preserving capability of the compact scheme based numerical solution, in the long time integration of a wave propagation phenomenon is demonstrated.

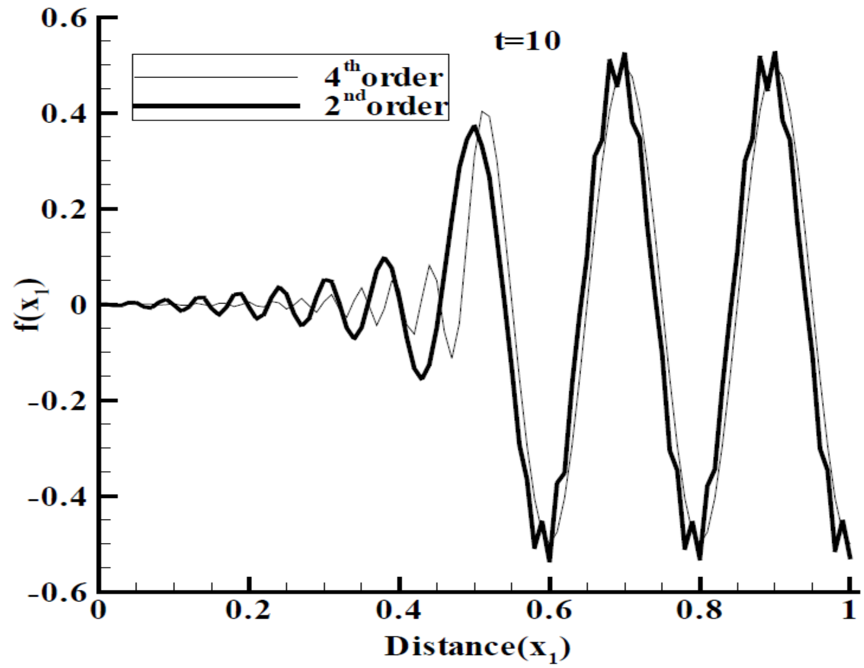

Figure 1.7: (a) Nature of wave profile at $t = 10$.

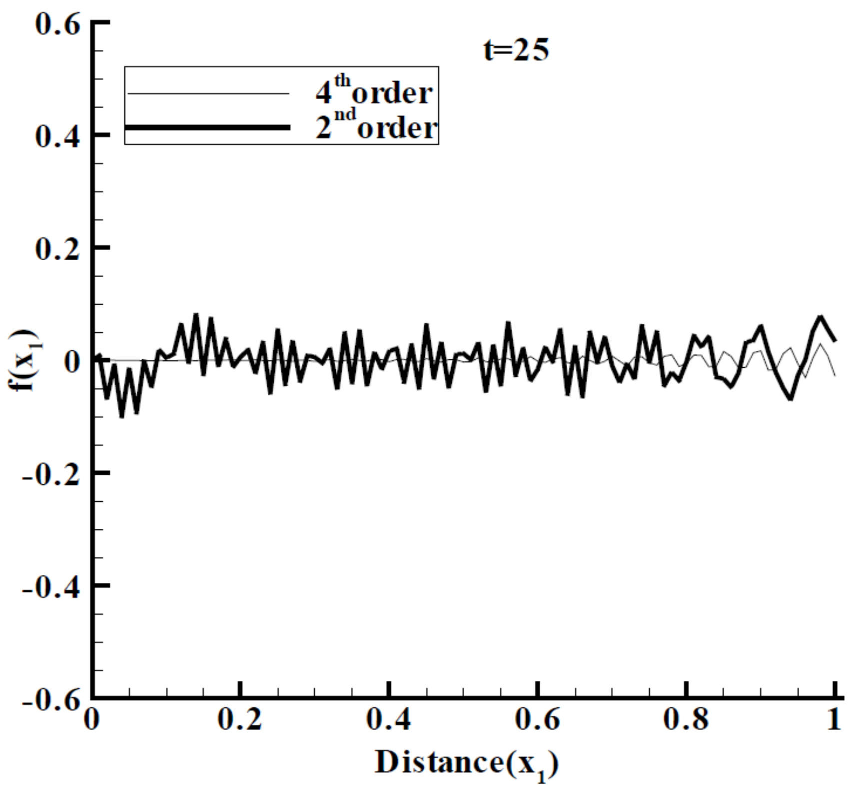

Figure 1.7: (b) Nature of wave profile at $t = 25$.

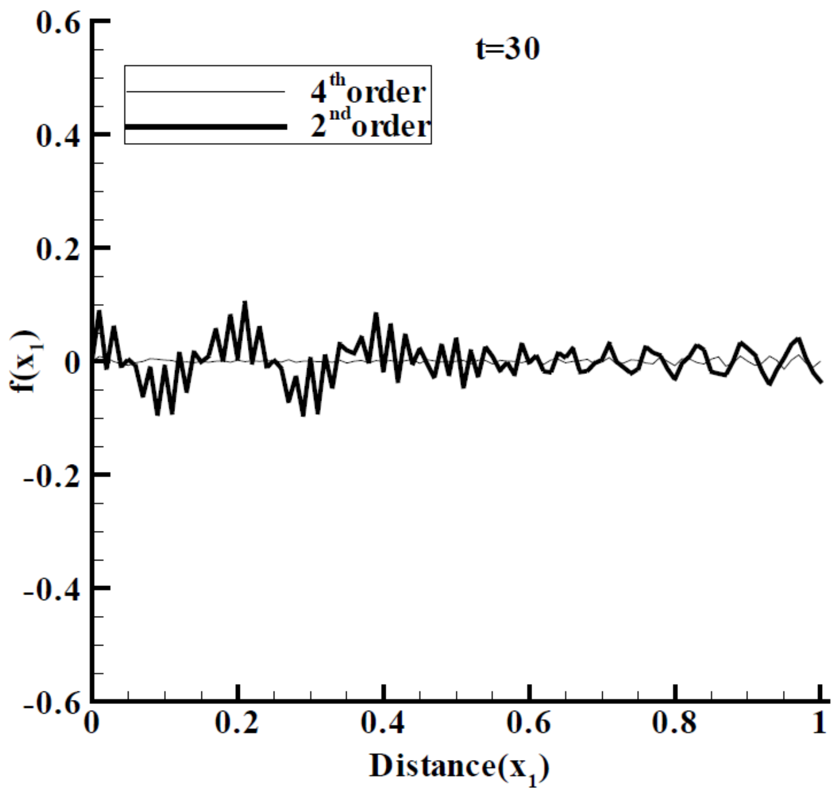

Figure 1.7: (c) Nature of wave profile at $t = 30$.

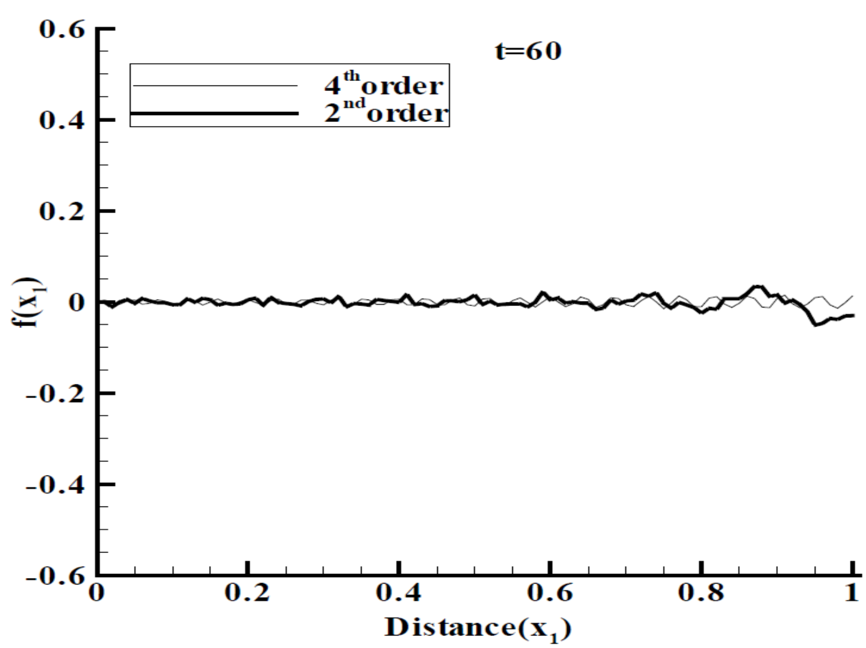

Figure 1.7: (d) Nature of wave profile at $t = 60$.

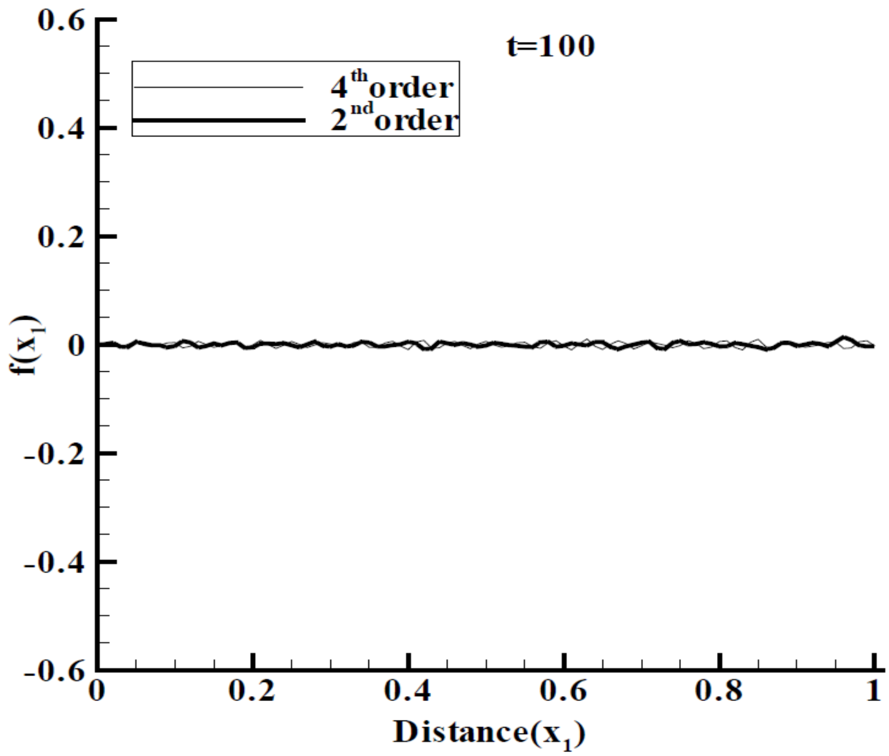

Figure 1.7:(e) Nature of wave profile at $t = 100$

7. CONCLUSION

As per the above two numerical experiments, In the first problem it is found that, the usage of compact 4^{th} order central scheme, to approximate the spatial first derivatives in the wave equation, confirms, its good dispersion preserving character. If the central difference based schemes are used to approximate the first derivatives, it will cause spurious oscillations in the solutions, due to the inherent dispersion error. This behavior is proved correct, with the solutions arrived, using the 2^{nd} order central scheme. Rather, the solution arrived using the compact schemes shows only mild oscillations, along with only a small dampening in the amplitude. In the second problem it is clear that, at $t = 10, 25$ and 30, spurious oscillations are more persistent in the domain, for the solution produced using 2^{nd} order scheme, when compared to that of the 4^{th} order scheme.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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