Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 6936-6948
https://doi.org/10.28919/jmcs/6068
ISSN: 1927-5307

ON THE CORDIAL LABELING OF CERTAIN TRIGRAPHS

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Abstract. Let G be a graph that has n vertices and m edges. Let $f\colon V(G) \to \{1,2,\ldots,k\}$ be a function that assigns to each vertex $v \in G$ a positive integer $f(v) \in \{1,2,\ldots,k\}$. We assign to each edge $uv \in E(G)$ a label which is the $\gcd(f(u), f(v))$. The function f is called k-prime cordial labeling of G if $|v_f(i) - v_f(j)| \leqslant 1$ for all $i, j \in \{1,2,\ldots,k\}$ and $|e_f(0) - e_f(1)| \leqslant 1$, where $v_f(i)$ denotes the number of vertices labeled with i, $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1, respectively. In this paper, we introduce the concept of trigraph of a graph G, $T_3(G)$, and we show that the trigraph of a path P_n, $T_3(P_n)$, and the trigraph of a cycle C_n, $T_3(C_n)$ are 4-prime cordial graphs.

Keywords: cordial labeling; 4-prime cordial graphs; path; cycle.

2010 AMS Subject Classification: 05C78.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a finite simple graph, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. Let n be the number of vertices in $V(G)$ and m the number of edges in $E(G)$. The number of edges incident to a vertex v is called the degree of v, and is denoted by $d_G(v)$. A graph G is connected if every pair of vertices are joined by a path. We say two vertices u, $v \in V(G)$ are adjacent or neighbors if uv is an edge of G. We say that a graph that has n vertices

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Received May 19, 2021

and m edges an (n,m)-graph. The concept of cordial labeling of graphs was introduced in [1] by Cahit. In [3], M. Sandaran, R. Ponraj and S. Somasundararam, introduced the concept of prime cordial labeling and discussed the prime labeling of certain graphs. In [4] and [5], the authors introduced k-prime cordial labeling of certain graphs. In this paper we will show that the trigraphs, $T_3(P_n)$ and $T_3(C_n)$, are 4-prime cordial graphs. But first we need to introduce the following definitions.

Definition 1.1. Let G be an (n,m)-graph and let $f\colon V(G) \to \{1,2,\dots,k\}$ be a function. For each edge uv, we assign the label $\gcd(f(u),f(v))$. We say that f is a k-prime cordial labeling of G if $|v_f(i)-v_f(j)| \leqslant 1$ for all $i,j \in \{1,2,\dots,k\}$ and $|e_f(0)-e_f(1)| \leqslant 1$ where $v_f(x)$ denotes the number of vertices labeled with x, $e_f(1)$ and $e_f(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1.

Definition 1.2. A graph G is called a k-prime cordial graph if it admits a k-prime cordial labeling [3].

Definition 1.3. A shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, G' and G'' and joining each vertex $v' \in G'$ to the neighbors of the corresponding vertex $v'' \in G''$ and vice-versa [3].

Definition 1.4. A trigraph of G, $T_3(G)$, of a connected graph G is constructed by taking two copies of G, G' and G'' and joining each vertex $v \in G$ to the neighbors of the corresponding vertex $v' \in G'$, $v'' \in G''$ and joining v' to the neighbors of the corresponding vertex $v'' \in G''$ [4].

Example 1.1. The graphs below are $D_2(P_4)$ and $T_3(P_4)$ where P_4 is a path on 4 vertices.

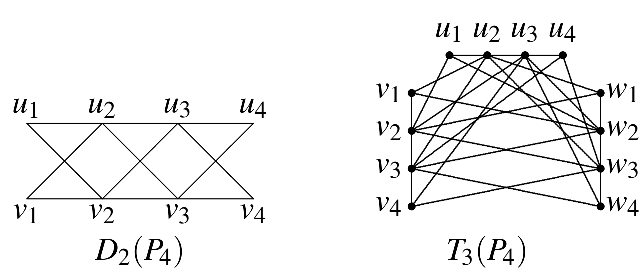

Figure 1

2. 4-Prime Cordial Labeling of a Path P_n

In this section, we will discus the structure of the trigraph $T_3(P_n)$ and show that $T_3(P_n)$ is a 4-prime cordial graph.

Theorem 2.1. *The trigraph $T_3(P_n)$ is a 4-prime cordial graph for $n \geqslant 7$.*

Proof. Let $V(T_3(P_n)) = \{u_i, v_i, w_i \mid 1 \leqslant i \leqslant n\}$ and

$$\begin{aligned} E(T_3(P_n)) = &\{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} \mid 1 \leqslant i \leqslant n-1\} \\ &\cup \{u_iv_{i+1}, v_iu_{i+1} \mid 1 \leqslant i \leqslant n-1\} \\ &\cup \{u_iw_{i+1}, w_iu_{i+1} \mid 1 \leqslant i \leqslant n-1\} \\ &\cup \{v_iw_{i+1}, w_iv_{i+1} \mid 1 \leqslant i \leqslant n-1\}. \end{aligned}$$

In a trigraph of a path on n vertices, $T_3(P_n)$, we have $3n$ vertices and $9n-9$ edges. That is $|V(T_3(P_n))| = 3n$ and $|E(T_3(P_n))| = 9n-9$. Note that $\deg(u_1) = \deg(u_n) = \deg(v_1) = \deg(v_n) = \deg(w_1) = \deg(w_n) = 3$ and $\deg(v_i) = \deg(w_i) = \deg(w_i) = 6$ for all $i \neq 1, n$.

Let us define a function $f\colon V(G) \to \{1,2,3,4\}$ as in the following. But first we have to divide the discussion into four cases, depending on the value of n.

Case 1: Suppose that $n \equiv 0 \pmod 4$. That is, $n = 4t$. Define f as

$$\begin{aligned} f(u_i) &= 2 \qquad 1 \leqslant i \leqslant \frac{n}{2}, \\ f(v_i) &= 4 \qquad 1 \leqslant i \leqslant \frac{n}{2}, \\ f(w_i) &= 2 \qquad i = 1,3,\ldots,\frac{n}{2}-1, \\ f(w_i) &= 4 \qquad i = 2,4,\ldots,\frac{n}{2}. \end{aligned}$$

This implies that 2 is assigned to $\frac{n}{2}$, v_i vertices and $\frac{n}{4}$, w_i vertices. That is the number of vertices that 2 is assigned to is $\frac{n}{2} + \frac{n}{4} = \frac{3n}{4}$ vertices. By the same argument 4 is assigned to $\frac{n}{2} + \frac{n}{4} = \frac{3n}{4}$ vertices. That is $v_f(2) = \frac{3n}{4}$ and $v_f(4) = \frac{3n}{4}$. In this part of the graph all edges are either labeled with 2, since $2 = \gcd(2,2) = \gcd(2,4)$ and $4 = \gcd(4,4)$. That is the number of edges that are labeled with 2 or 4 equals $\frac{6(3n-6)+3(6)}{2} = \frac{9n-9}{2}$. That is $e_f(0) = \frac{9n-9}{n}$. We continue to define the function f for the remaining vertices as

$$f(v_i) = f(u_i) = f(w_i) = 1 \qquad \text{for } i = \frac{n}{2}+1, \frac{n}{2}+3, \ldots$$
$$f(v_i) = f(u_i) = f(w_i) = 3 \qquad \text{for } i = \frac{n}{2}+2, \frac{n}{2}+4, \ldots.$$

This implies $v_f(1) = \frac{n}{2} + \frac{n}{4} = \frac{3n}{2}$ and $v_f(3) = \frac{n}{2} + \frac{n}{4} = \frac{3n}{2}$. Also, all the edges in the second part of the graph are labeled with $1 = \gcd(1,3) = \gcd(1,4) = \gcd(1,2)$.It follows that $e_f(1) = \frac{6(3n-3)+3(3)}{2} = \frac{9n-9}{2}$. We summarize the previous information in the following table:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{9n-9}{2}$	$\frac{9n-9}{2}$

Table 1

From Table 1, f satisfies the definition of 4-prime cordial labeling for $T_3(P_n)$, since $|v_f(i) - v_f(j)| = 0 \leqslant 1$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$ for all $i, j = 1, 2, 3, 4$.

Case 2: Suppose that $n \equiv 1 \pmod 4$. Then $n = 1 + 4t$ for some positive integer t, so that $t = \frac{n-1}{4}$. In this case we define f follows

$$f(u_i) = 2 \qquad \text{for } 1 \leqslant i \leqslant 2t+1,$$
$$f(v_i) = 4 \qquad \text{for } 1 \leqslant i \leqslant 2t,$$
$$f(w_i) = 2 \qquad \text{for } i = 1, 3, \ldots, 2t-1,$$
$$f(w_i) = 4 \qquad \text{for } i = 2, 4, \ldots, 2t,$$
$$f(w_{2t-1}) = 1, \quad f(v_{2t+1}) = 3, \quad f(u_{2t+1}) = 4,$$
$$f(v_i) = f(u_i) = f(w_i) = 1 \qquad \text{for } i = 2t+2, 2t+4, \ldots,$$
$$f(v_i) = f(u_i) = f(w_i) = 3 \qquad \text{for } i = 2t+3, 2t+5, \ldots.$$

This implies that:

$$
\begin{aligned}
v_f(1) &= \left(\frac{2t}{2}\right)3+1 = 3t+1 = 3\left(\frac{n-1}{4}\right)+1 = \frac{3n+1}{4},\\
v_f(2) &= 2t+t+1 = 3t+1 = \frac{3n+1}{4},\\
v_f(3) &= 3t = 3\left(\frac{n-1}{4}\right) = \frac{3n-3}{4},\\
v_f(4) &= 2t+t+1 = 3\left(\frac{n-1}{4}\right)+1 = \frac{3n+1}{4}.
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
e_f(1) &= \frac{6(3(2t-1)+2)+12}{2} = \frac{36t-18+18}{2} = \frac{36t}{2} = \frac{9(4t)}{2} = \frac{9(n-1)}{2} = \frac{9n-9}{2},\\
e_f(0) &= (9n-9) - \left(\frac{9n-9}{2}\right) = \frac{9n-9}{2}.
\end{aligned}
$$

We summarize the previous results:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{9n-9}{2}$	$\frac{9n-9}{2}$

Table 2

It follows that f satisfies the conditions of the 4-prime cordial labeling.

Case 3: Suppose that $n \equiv 2 \pmod 4$. That is $n = 2+4t$, so that $t = \frac{n-2}{4}$. We can label the first $2t$ and the last $2t$ vertices as we did in the previous two cases. Now we label the following vertices as:

$$
\begin{aligned}
f(w_{2t+1}) &= f(w_{2t+2}) = 1,\\
f(u_{2t+1}) &= f(v_{2t+1}) = 2,\\
f(v_{2t+2}) &= 3,\\
f(u_{2t+2}) &= 4.
\end{aligned}
$$

This implies that:

$$\begin{aligned}
v_f(1) &= 3\left(\frac{2t}{2}\right)+2 = 3t+2 = 3\left(\frac{n-2}{4}\right)+2 = \frac{3n+2}{4},\\
v_f(2) &= 2t+t+2 = 3t+2 = \frac{3n+2}{4},\\
v_f(3) &= 3\left(\frac{2t}{2}\right)+1 = 3t+1 = \frac{3n-2}{4},\\
v_f(4) &= 2t+t+1 = 3t+1 = \frac{3n-2}{4}.
\end{aligned}$$

It follows that

$$\begin{aligned}
e_f(1) &= \frac{3(6(2t-2))+9+9+6+6+4+2+6+2}{2} = \frac{9n-10}{2},\\
e_f(0) &= 9n-9-\left(\frac{9n-10}{2}\right) = \frac{9n-8}{2}.
\end{aligned}$$

We summarize the above results as:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n+2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{9n-8}{2}$	$\frac{9n-10}{2}$

Table 3

From the table above, we have $|v_f(i)-v_f(j)| \leqslant 1$ and $|e_f(0)-e_f(1)| = \frac{2}{2} = 1 \leqslant 1$, so that the graph is 4-prime cordial graph.

Case 4: Suppose that $n \equiv 3 \pmod 4$. That is $n = 3+4t$, so that $t = \frac{n-3}{4}$. We can label the first $2t$ and the last $2t$ vertices as in the previous cases. Now we label the following vertices

$$u_{2t+1}, w_{2t+1}, v_{2t+1}, u_{2t+2}, w_{2t+2}, v_{2t+2}, u_{2t+3}, w_{2t+3}, v_{2t+3}$$

as

$$\begin{aligned}
&f(u_{2t+1}) = 2, \qquad f(w_{2t+1}) = 4, \quad f(v_{2t+1}) = 3,\\
&f(u_{2t+2}) = 1, \qquad f(w_{2t+2}) = f(v_{2t+2}) = 4,\\
&f(u_{2t+3}) = 1, \qquad f(w_{2t+3}) = 2, \qquad f(w_{2t+3}) = 3.
\end{aligned}$$

This implies that:

$$\begin{aligned} v_f(1) &= 3\left(\frac{2t}{2}\right)+2 = 3t+2 = 3\left(\frac{n-3}{4}\right)+2 = \frac{3n-1}{4}, \\ v_f(2) &= 3t+2 = 3\left(\frac{n-3}{4}\right)+2 = \frac{3n-1}{4}, \\ v_f(3) &= 3t+2 = 3\left(\frac{n-3}{4}\right)+2 = \frac{3n-1}{4}, \\ v_f(4) &= 3t+3 = 3\left(\frac{n-3}{4}\right)+3 = \frac{3n+3}{4}. \end{aligned}$$

It follows that:

$$e_f(1) = \frac{3(6(2t-1))+9+10+10+6}{2} = \frac{36t+17}{2} = \frac{9(4t)+17}{2} = \frac{9(n-3)+17}{2} = \frac{9n-10}{2}.$$

$$e_f(0) = (9n-9) - \left(\frac{9n-10}{2}\right) = \frac{9n-8}{2}.$$

We summarize the above results in the following table:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n+3}{4}$	$\frac{9n-10}{2}$	$\frac{9n-8}{2}$

Table 4

From the table above, we have $|v_f(i)-v_f(j)| \leqslant 1$ for all $i,j = 1,2,3,4$ and $|e_f(0)-e_f(1)| = 1 \leqslant 1$, so that the graph is 4-prime cordial graph. $\square$

3. Prime-Cordial Labeling of a Cycle C_n on n Vertices

In this section we will prove that the trigraph $T_3(C_n)$ is a 4-prime cordial graph.

Theorem 3.1. *The trigraph $T_3(C_n)$ is a 4-prime cordial graph, for $n \geqslant 9$.*

Proof. Let $V(T_3(C_n)) = \{u_i, v_i, w_i \mid 1 \leqslant i \leqslant n\}$ and

$$\begin{aligned} E(T_3(C_n)) = &\{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} \mid 1 \leqslant i \leqslant n-1\} \\ &\cup \{u_iw_{i+1}, u_iv_{i+1}, v_iu_{i+1}, w_iu_{i+1}, v_iw_{i+1}w_iv_{i+1} \mid 1 \leqslant i \leqslant n-1\} \\ &\cup \{u_nv_1, u_nw_1, v_nu_1, v_nw_1, w_nv_1, w_nu_1\}. \end{aligned}$$

In a trigraph of a cycle, $T_3(C_n)$, we have $3n$ vertices and $9n$ edges since the degree of each vertex is 6 and we have $3n$ vertices, so that the number of edges equals $\frac{6(3n)}{2} = 9n$. That is $|V(T_3(C_N))| = 3n$ and $|E(T_3(C_n))| = 9n$. We divide the discussion into four cases, depending on n, the number of vertices in each cycle.

Case 1: Suppose that $n \equiv 0 \pmod 4$. In this case we define $f\colon V(T_3(C_n)) \to \{1,2,3,4\}$ as follows:

$$f(v_{2i}) = f(u_{2i-1}) = f(w_{2i-1}) = 2, \quad 1 \leqslant i \leqslant \frac{n}{4},$$
$$f(v_{2i+1}) = f(u_{2i}) = f(w_{2i}) = 4, \quad 1 \leqslant i \leqslant \frac{n}{4},$$
$$f(v_{\frac{n}{2}+2i+2}) = f(u_{\frac{n}{2}+2i+2}) = f(w_{\frac{n}{2}+2i+2}) = 1, \quad 1 \leqslant i \leqslant \frac{n-4}{2},$$
$$f(v_{\frac{n}{2}+2i+3}) = f(u_{\frac{n}{2}+2i+3}) = f(w_{\frac{n}{2}+2i+3}) = 3, \quad 1 \leqslant i \leqslant \frac{n-8}{4}.$$
$$f(v_1) = f(v_{\frac{n}{2}+3}) = f(w_{\frac{n}{2}+3}) = 1.$$
$$f(v_{\frac{n}{2}+1}) = f(u_{\frac{n}{2}+1}) = f(u_{\frac{n}{2}+2}) = f(u_{\frac{n}{2}+3}) = f(w_{\frac{n}{2}+1}) = f(w_{\frac{n}{2}+2}) = 3.$$

This implies that

$$v_f(1) = 3\left(\frac{n-4}{4}\right) + 3 = \frac{3n}{4},$$
$$v_f(2) = 3\left(\frac{n}{4}\right) = \frac{3n}{4},$$
$$v_f(3) = 3\left(\frac{n-8}{4}\right) + 6 = \frac{3n}{4},$$
$$v_f(4) = 3\left(\frac{n}{4}\right) = \frac{3n}{4}.$$

It follows that, the number of edges that are labeled with 2 or 4 $= \frac{3(6(\frac{n}{2}-3))+37}{2} = \frac{9n-17}{2}$, and the number of edges labeled with 3 is equal to 9. Thus

$$e_f(0) = \frac{9n-17}{2} + 9 = \frac{9n+1}{2},$$
$$e_f(1) = 9n - \frac{9n+1}{2} = \frac{9n-1}{2}.$$

We summarize the results in the following table:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{9n+1}{2}$	$\frac{9n-1}{2}$

Table 5

From Table 5, we have $|v_f(i)-v_f(j)| = 0 \leqslant 1$ and $|e_f(0)-e_f(1)| = 1 \leqslant 1$, so that the graph is 4-prime cordial graph.

Case 2: Suppose that $n \equiv 1 \pmod 4$. In this case we define $f\colon V(T_3(C_n)) \to \{1,2,3,4\}$ as follows:

$$f(u_{2i-1}) = 2, \quad 1 \leqslant i \leqslant \frac{n+3}{4},$$
$$f(u_{2i}) = 4, \quad 1 \leqslant i \leqslant \frac{n+3}{4},$$
$$f(v_{2i}) = f(w_{2i}) = 2, \quad 1 \leqslant i \leqslant \frac{n-1}{4}$$
$$f(v_{2i+1}) = f(w_{2i+1}) = 4, \quad 1 \leqslant i \leqslant \frac{n-1}{4},$$
$$f(v_{\frac{n+5}{2}+2i}) = f(u_{\frac{n+5}{2}+2i}) = f(w_{\frac{n+5}{2}+2i}) = 1, \quad 1 \leqslant i \leqslant \frac{n-5}{4},$$
$$f(v_{\frac{n+7}{2}+2i}) = f(u_{\frac{n+7}{2}+2i}) = f(w_{\frac{n+7}{2}+2i}) = 3, \quad 1 \leqslant i \leqslant \frac{n-9}{4},$$
$$f(v_1) = f(v_{\frac{n+7}{2}}) = f(v_{\frac{n+3}{2}}) = f(w_1) = 1,$$
$$f(v_{\frac{n+5}{2}}) = f(u_{\frac{n+5}{2}}) = f(u_{\frac{n+7}{2}}) = f(w_{\frac{n+3}{2}}) = f(w_{\frac{n+5}{2}}) = f(w_{\frac{n+7}{2}}) = 3.$$

This implies that:

$$v_f(1) = 3\left(\frac{n-5}{4}\right) + 4 = \frac{3n+1}{4},$$
$$v_f(2) = 2\left(\frac{n-1}{4}\right) + \left(\frac{n+3}{4}\right) = \frac{3n+1}{4},$$
$$v_f(3) = 3\left(\frac{n-9}{4}\right) + 6 = \frac{3n-3}{4},$$
$$v_f(4) = 2\left(\frac{n-1}{4}\right) + \left(\frac{n+3}{4}\right) = \frac{3n+1}{4}.$$

It follows that the number of edges that are labeled with 2 or 4 $= \frac{6(\frac{n-1}{2})(3)-2-2-2-2-2-2-2+3+3}{2} = \frac{9n-15}{2}$, and the number of edges labeled 3 equals $\frac{16}{2}$. Thus,

$$e_f(0) = \frac{9n-15}{2} + \frac{16}{2} = \frac{9n+1}{2},$$
$$e_f(1) = 9n - \frac{9n+1}{2} = \frac{9n-1}{2}.$$

We summarize the above results in the following table:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{9n+1}{2}$	$\frac{9n-1}{2}$

Table 6

From Table 6, we have $|v_f(i) - v_f(j)| = 0$ or $1 \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$, so that the graph is 4-prime cordial graph.

Case 3: Suppose that $n \equiv 2 \pmod 4$. In this case we define $f\colon V(T_3(C_n)) \to \{1,2,3,4\}$ as follows:

$$f(u_{2i-1}) = 2, \quad 1 \leqslant i \leqslant \frac{n+2}{4},$$
$$f(u_{2i}) = 4, \quad 1 \leqslant i \leqslant \frac{n+2}{4},$$
$$f(v_{2i}) = f(w_{2i}) = 2, \quad 1 \leqslant i \leqslant \frac{n-2}{4},$$
$$f(v_{2i+1}) = f(w_{2i+1}) = 4, \quad 1 \leqslant i \leqslant \frac{n-2}{4},$$
$$f(v_{\frac{n+6}{2}+2i}) = f(u_{\frac{n+6}{2}+2i}) = f(w_{\frac{n+6}{2}+2i}) = 1, \quad 1 \leqslant i \leqslant \frac{n-6}{4},$$
$$f(v_{\frac{n+8}{2}+2i}) = f(u_{\frac{n+8}{2}+2i}) = f(w_{\frac{n+8}{2}+2i}) = 3, \quad 1 \leqslant i \leqslant \frac{n-10}{4},$$
$$f(w_1) = f(v_1) = f(v_{\frac{n+6}{2}}) = f(v_{\frac{n+4}{2}}) = f(u_{\frac{n+6}{2}}) = f(u_{\frac{n+8}{2}}) = f(w_{\frac{n+6}{2}}) = f(w_{\frac{n+8}{2}}) = 3,$$
$$f(v_{\frac{n+2}{2}}) = f(v_{\frac{n+4}{2}}) = f(v_{\frac{n+8}{2}}) = f(w_{\frac{n+2}{2}}) = f(w_{\frac{n+4}{2}}) = 1.$$

This implies that:

$$v_f(1)=3\left(\frac{n-6}{4}\right)+5=\frac{3n+2}{4},$$
$$v_f(2)=2\left(\frac{n-2}{4}\right)+\left(\frac{n+2}{4}\right)=\frac{3n-2}{4},$$
$$v_f(3)=3\left(\frac{n-10}{4}\right)+8=\frac{3n+2}{4},$$
$$v_f(4)=2\left(\frac{n-2}{4}\right)+\left(\frac{n+2}{4}\right)=\frac{3n-2}{4}.$$

It follows that number of edges labeled with 2 or 4 $=\frac{6(\frac{n-2}{2})(3)-2-2-1-2-2+3}{2}=\frac{9n-18}{2}$, and number of edges labeled 3 equals $\frac{3+3+3+3+3+3}{2}=\frac{18}{2}$. This implies that $e_f(0)=\frac{9n-18}{2}+\frac{18}{2}=\frac{9n}{2}$, so that $e_f(1)=9n-\frac{9n}{2}=\frac{9n}{2}$.

We summarize the above results as:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{9n}{2}$	$\frac{9n}{2}$

Table 7

From Table 7, we have $|v_f(i)-v_f(j)|=0$ or $1\leqslant 1$ and $|e_f(0)-e_f(1)|=0\leqslant 1$, so that the graph is 4-prime cordial graph.

Case 4: Suppose that $n\equiv 3 \pmod 4$. In this case we define $f\colon V(T_3(C_n))\to\{1,2,3,4\}$ as follows:

$$f(v_{2i})=f(u_{2i-1})=f(w_{2i})=2,\quad 1\leqslant i\leqslant\frac{n+1}{4}$$

$$f(v_{2i+1})=f(u_{2i})=f(w_{2i+1})=4,\quad 1\leqslant i\leqslant\frac{n+1}{4},$$

$$f(v_{\frac{n+3}{2}+2i})=f(u_{\frac{n+3}{2}+2i})=1,\quad 1\leqslant i\leqslant\frac{n-3}{4},$$

$$f(w_{\frac{n+3}{2}+2i})=1\quad 1\leqslant i\leqslant\frac{n-7}{4},$$

$$f(v_{\frac{n+5}{2}+2i})=f(u_{\frac{n+5}{2}+2i})=f(w_{\frac{n+5}{2}+2i})=3,\quad 1\leqslant i\leqslant\frac{n-7}{4},$$

$$f(u_{\frac{n+3}{2}})=f(u_{\frac{n+5}{2}})=f(v_{\frac{n+3}{2}})=f(v_{\frac{n+5}{2}})=f(w_{\frac{n+3}{2}})=f(w_{\frac{n+5}{2}})=3,$$

$$f(v_1)=f(w_1)=1,f(w_n)=4.$$

This implies that

$$v_f(1) = 3\left(\frac{n-3}{4}\right) + 2 = \frac{3n-1}{4},$$
$$v_f(2) = 3\left(\frac{n+1}{4}\right) = \frac{3n+3}{4},$$
$$v_f(3) = 3\left(\frac{n-7}{4}\right) + 6 - 1 = \frac{3n-1}{4},$$
$$v_f(4) = 2\left(\frac{n+1}{4}\right) + \left(\frac{n-3}{4}\right) = \frac{3n-1}{4}.$$

It follows that number of edges labeled with 2 or 4 $= \frac{6(\frac{n-1}{2})(3)-2-2-2+3-3-3-3+2}{2} = \frac{9n-19}{2}$, and number of edges labeled 3 equals $\frac{18}{2}$. Thus $e_f(0) = \frac{9n-19}{2} + \frac{18}{2} = \frac{9n-1}{2}$ so that $e_f(1) = 9n - \frac{9n-1}{2} = \frac{9n+1}{2}$.

We summarize the above results as:

$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$\frac{3n-1}{4}$	$\frac{3n+3}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{9n-1}{2}$	$\frac{9n+1}{2}$

Table 8

From Table 8, we have $|v_f(i) - v_f(j)| = 0$ or $1 \leqslant 1$ and $|e_f(0) - e_f(1)| = 0 \leqslant 1$, so that the graph is 4-prime cordial graph.

It follows that $T_3(C_n)$ is a 4-prime cordial graph for all $n \geqslant 9$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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