NEW COMPOUND PROBABILITY DISTRIBUTION USING BIWEIGHT KERNEL FUNCTION AND EXPONENTIAL DISTRIBUTION

AHMAD M. H. AL-KHAZALEH[*], LOAI ALZOUBI

Department of Mathematics, Al al-Bayt University, Mafraq (25113), Jordan

Abstract: In this paper, a new continuous probability distribution is proposed for fitting real data using Biweight kernel function and the exponential distribution. The suggested distribution is named the Biweight-exponential distribution (BiEd). Some statistical properties of this distribution are derived and illustrated mathematically. The probability density function and the cumulative distribution function are derived. Some reliability analysis functions are defined. The moments and moment generating function are derived. Re'nyi entropy is derived. The maximum likelihood method of estimation is used to derive the parameter estimates. The Bonferroni and Lorenz curves and Gini index equations are derived. The distribution of the order statistic and the quantile function are derived as well. The mean and median absolute deviations of the new distribution are derived. A numerical study was conducted to the quantile equation. An application to real data set is conducted to investigate the usefulness of the suggested distribution. In the real data application, the values of Cramer-von misses (W), Anderson Darling statistic (A), Kolmogorov-Smirnov (D) statistic, the p-value, the maximum likelihood estimates (MLE), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan Quinn information

[*]Corresponding author

E-mail address: ahmed_2005kh@yahoo.com

criterion (HQIC) and minimum value of the log-likelihood function are obtained. The distribution is compared with the exponential (base) distribution based on these criteria. The results showed that the BiED fits better than the exponential distribution. This means that the suggested distribution can replace the exponential distribution for analysis of some real data sets.

Keywords: biweight kernel function; moments; entropy; order statistics; quantile function; MLE.

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1. INTRODUCTION

The most significant one parameter of life distributions is the exponential distribution. There are many important problems where the real data does not follow any of the classical known probability distributions [1]. There are many generalizations of new continuous distribution based on exponential distribution such that: generalized exponential [2], beta exponential [3], beta generalized exponential [4], Kumaraswamy exponential [5], gamma exponentiated exponential [6], Transmuted exponentiated exponential distribution [7]. Numerous researchers proposed new distributions using different families. For examples; [8] suggested the transmuted Janardan distribution. [9] generated the transmuted Burr type XII distribution: a generalization of the Burr type XII distribution. [10] suggested a generalization of the new Weibull-Pareto distribution. The transmuted two-parameter Lindley distribution is proposed by [11]. [12] proposed the transmuted Shanker distribution. [13] used the same map to develop Mukherjee-Islam distribution. [14] worked out the transmuted Ishita distribution using the quadratic transmutation map. In this article, we present a new generalization of the exponential distribution via Biweight kernel function [15] and call it the Biweight exponential distribution (BED).

The rest of this article is organized as follows: in Section 2, we defined the materials and methods of this article. Section 3 defines the new distribution and its pdf and cdf. The reliability analysis is defined in Section 4. The moments and moment generating function are derived in Section 5. We have used the maximum likelihood method of estimation to estimate the parameters in Section 6.

The quantile function and the densities of order statistics are derived in Sections 7 and 8. In Section 9 we have defined the Re'nyi entropy. While in Section 10 we have derived the equations of Bonferroni and Lorenz curves and the Gini index equation. Real data applications are run in Section 12. Finally in Section 13, we have sum up our conclusions.

2. MATERIALS AND METHODS

The kernel is any function having the following properties $\int K(u)du=1$, $\int uK(u)du=0$ and $\int u^2 K(u)du=k_2<\infty$. Many kinds of kernel function can be initiated in the related literature [16] – [18].

One of the symmetric kernels is the Biweight kernel function [24]. It is defined as

$$(2.1) \qquad K(u) = \begin{cases} \dfrac{15}{16}\left(1-u^2\right)^2 & ; \qquad |u|<1 \\ 0 & ; \qquad |u|\geq 1 \end{cases}$$

Definition: A random variable X is said to have a Biweight probability Distribution with its cumulative distribution function (CDF) $G(x)$ and probability density function (PDF) $g(x)$ are obtained respectively as follows

$$(2.2) \qquad G(x) = 2\int_{0}^{F(x)} k(u)du = 2\int_{0}^{F(x)} \frac{15}{16}\left(1-u^2\right)^2 du = \left[\frac{15}{8}F(x)-\frac{5}{4}\left(F(x)\right)^3+\frac{3}{8}\left(F(x)\right)^5\right],$$

where $k(u)$ is the Biweight kernel function and $F(x)$ is the base distribution. The PDF can be obtained by differentiating the CDF. Thus

$$(2.3) \qquad g(x) = \frac{15}{8}f(x)\left[1-\left[F(x)\right]^2\right]^2,$$

where $F(x)$ and $f(x)$ are CDF and PDF of the base distribution; respectively.

3. BIWEIGHT-EXPONENTIAL DISTRIBUTIONS

A random variable X is said to have an exponential distribution with parameter $\lambda > 0$ if its pdf and cdf are given by respectively

(3.1)
$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0,$$

(3.2)
$$F(x) = 1 - e^{-\lambda x}$$

Now using (2.2) and (3.2), the cdf of **BiED** is defined as

(3.3)
$$G(x) = \left[1 - \frac{5}{2} e^{-3\lambda x} + \frac{15}{8} e^{-4\lambda x} - \frac{3}{8} e^{-5\lambda x} \right]$$

Hence, the pdf of **BiED** is given by

(3.4)
$$g(x) = \frac{15}{2} \lambda e^{-3\lambda x} - \frac{15}{2} \lambda e^{-4\lambda x} + \frac{15}{8} \lambda e^{-5\lambda x}$$

Note that the **BiED** is an extended model to analyze more complex data and it generalizes some of the generally used distributions.

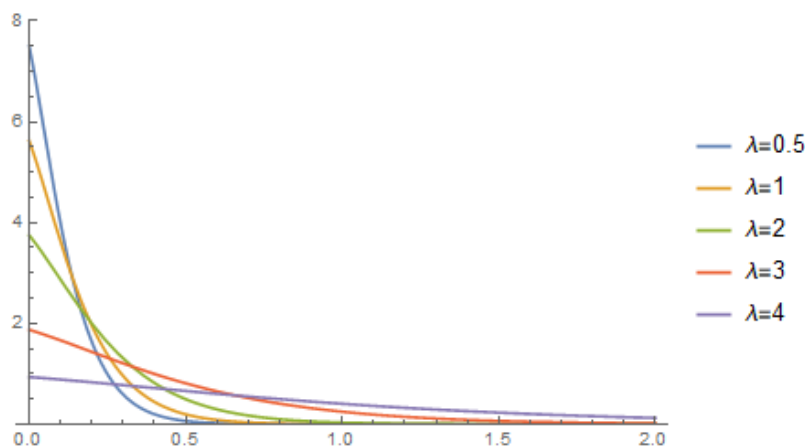Figure 1 shows the plot of the pdf of a BiED for selected values of the parameter $\lambda = 0.5, 1, 2, 3,$ and 4.

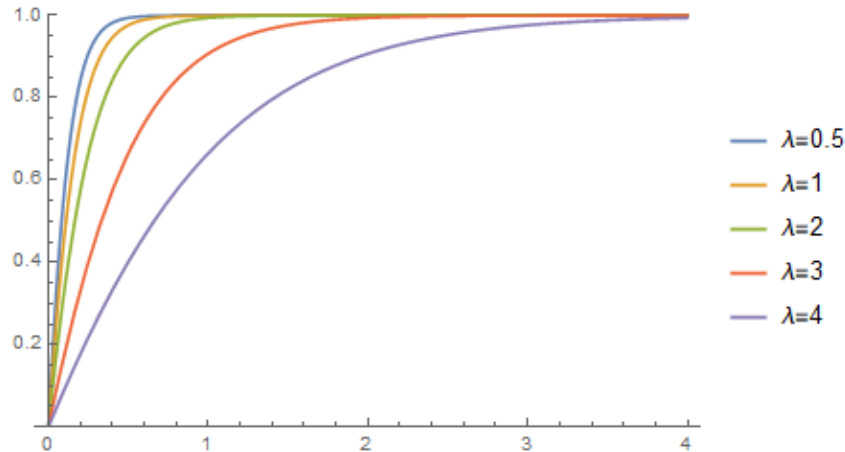Figure 1 pdf of **BiED** for $\lambda = 0.5, 1, 2, 3,$ and 4

Figure 2 cdf of **BiED** for $\lambda = 0.5, 1, 2, 3, \text{ and } 4$

The asymptotic behaviour of the cdf of a BiED as follows

$$\lim_{x \to \infty} G(x) = \lim_{x \to \infty} \left[1 - \frac{5}{2} e^{-3\lambda x} + \frac{15}{8} e^{-4\lambda x} - \frac{3}{8} e^{-5\lambda x} \right] = 1$$

$$\lim_{x \to 0} G(x) = \lim_{x \to 0} \left[1 - \frac{5}{2} e^{-3\lambda x} + \frac{15}{8} e^{-4\lambda x} - \frac{3}{8} e^{-5\lambda x} \right] = 0$$

4. RELIABILITY ANALYSIS

The reliability is concerned with the calculation and prediction of the probability of the limit state at any phase during the structures life. The life time distribution and the reliability function (survival distribution) are complementary functions.

The reliability function or the survival function is defined as $R(x) = P(X > x); \quad x > 0$

The reliability of the BiED is defined as:

(4.1)
$$R(x) = 1 - G(x) = \frac{5}{2} e^{-3\lambda x} - \frac{15}{8} e^{-4\lambda x} + \frac{3}{8} e^{-5\lambda x}$$

The hazard rate function of the BiED is defined as

(4.2)
$$H(t) = \frac{g(t)}{1 - G(t)} = 15\lambda \left[\frac{4 - 4e^{-\lambda x} + e^{-2\lambda x}}{20 - 15e^{-\lambda x} + 3e^{-2\lambda x}} \right]$$

5. MOMENTS

The rth moment of the BiED is given by the following theorem:

Theorem 5.1: Let X be a random variable that follows a BiED, then the rth moment

is

(5.1)
$$E\left(X^r\right) = \left[\frac{5}{2}\left(\frac{1}{3\lambda}\right)^r - \frac{15}{8}\left(\frac{1}{4\lambda}\right)^r + \frac{3}{8}\left(\frac{1}{5\lambda}\right)^r\right]\Gamma(r+1)$$

Proof: the rth moment of the BiED is defined as

$$E(x^r) = \int_0^\infty x^r g(x)dx = \int_0^\infty x^r\left[\frac{15}{2}\lambda e^{-3\lambda x} - \frac{15}{2}\lambda e^{-4\lambda x} + \frac{15}{8}\lambda e^{-5\lambda x}\right]dx$$

(5.2)
$$= \frac{15}{2}\lambda\int_0^\infty x^r e^{-3\lambda x}dx - \frac{15}{2}\lambda\int_0^\infty x^r e^{-4\lambda x}dx + \frac{15}{8}\lambda\int_0^\infty x^r e^{-5\lambda x}dx$$

The integrals can be solved by the u-substitution as

$$u = 3\lambda x \quad \frac{du}{3\lambda} = dx \quad x = \frac{u}{3\lambda};$$

$$u = 4\lambda x \quad \frac{du}{4\lambda} = dx \quad x = \frac{u}{4\lambda}$$

$$u = 5\lambda x \quad \frac{du}{5\lambda} = dx \quad x = \frac{u}{5\lambda}$$

Then, we substitute these assumptions in equation (5.2), we have

$$= \frac{15}{2}\lambda\int_0^\infty\left(\frac{u}{3\lambda}\right)^r e^{-u}\frac{du}{3\lambda} - \frac{15}{2}\lambda\int_0^\infty\left(\frac{u}{4\lambda}\right)^r e^{-u}\frac{du}{4\lambda} + \frac{15}{8}\lambda\int_0^\infty\left(\frac{u}{5\lambda}\right)^r e^{-u}\frac{du}{5\lambda}.$$

Simplify and rearrange, we have

$$= \left[\frac{5}{2}\left(\frac{1}{3\lambda}\right)^r - \frac{15}{8}\left(\frac{1}{4\lambda}\right)^r + \frac{3}{8}\left(\frac{1}{5\lambda}\right)^r\right]\int_0^\infty (u)^r e^{-u}du$$

□

We can find the first moment (mean)

$$\mu = E(X) = \left[\frac{5}{2}\left(\frac{1}{3\lambda}\right) - \frac{15}{8}\left(\frac{1}{4\lambda}\right) + \frac{3}{8}\left(\frac{1}{5\lambda}\right)\right] = \frac{0.439}{\lambda},$$

and the second moment to find the variance as

$$E(X^2) = \left[\frac{5}{9} - \frac{15}{64} + \frac{3}{100}\right]\left(\frac{1}{\lambda}\right)^2 = \frac{0.3512}{\lambda^2}$$

$$\sigma^2 = \frac{0.3512}{\lambda^2} - \left(\frac{0.439}{\lambda}\right)^2 = \frac{0.1579}{\lambda^2}$$

The following theorem defines the moment generating function of BiED.

Theorem 5.2: Let X be a random variable that follows a BiED, therefore the moment generating function (MGF) is defined as:

(5.3)
$$E\left(e^{tX}\right) = \frac{15\lambda}{2}\left[\frac{1}{(3\lambda - t)} - \frac{1}{(4\lambda - t)} + \frac{1}{4(5\lambda - t)}\right]$$

Proof: the MGF of the BiED is defined as

$$E\left(e^{tX}\right) = \int_0^\infty e^{tx} g(x)dx = \int_0^\infty e^{tx}\left[\frac{15}{2}\lambda e^{-3\lambda x} - \frac{15}{2}\lambda e^{-4\lambda x} + \frac{15}{8}\lambda e^{-5\lambda x}\right]dx$$

(5.4)
$$= \frac{15}{2}\lambda \int_0^\infty e^{-x(3\lambda - t)}dx - \frac{15}{2}\lambda \int_0^\infty e^{-x(4\lambda - t)}dx + \frac{15}{8}\lambda \int_0^\infty e^{-x(5\lambda - t)}dx$$

By using the u-substitution as follows

$$u = x(3\lambda - t) \quad dx = \frac{du}{3\lambda - t}$$

$$u = x(4\lambda - t) \quad dx = \frac{du}{4\lambda - t}$$

$$u = x(5\lambda - t) \quad dx = \frac{du}{5\lambda - t},$$

and substitute in equation (5.4), we have

$$= \frac{15\lambda}{2}\left[\frac{1}{(3\lambda - t)} - \frac{1}{(4\lambda - t)} + \frac{1}{4(5\lambda - t)}\right]\int_0^\infty e^{-u}du$$

$\square$

6. MAXIMUM LIKELIHOOD ESTIMATES

Maximum likelihood method of estimation is a well-known method of estimating the distribution parameters. The likelihood function used the parameter as a variable conditional to the observations. For the BiED, the maximum likelihood estimator of the distribution parameter is

given as the following. The joint pdf of $X_1, ..., X_n$ is given by

$$L(\lambda) = g(x_1, ..., x_n \setminus \lambda) = \prod_{i=1}^{n} \frac{15\lambda}{2} \left[e^{-3\lambda x_i} \left(1 - e^{-\lambda x_i} + \frac{1}{4}\lambda e^{-2\lambda x_i} \right) \right]$$

$$= \left(\frac{15\lambda}{2} \right)^n e^{-3\lambda \sum\limits_{i=1}^{n} x_i} \left(1 - e^{-\lambda x_i} + \frac{1}{4}\lambda e^{-2\lambda x_i} \right)^n$$

By taking the log of $L(\lambda)$ as

$$\ln L(\lambda) = \ln \left[\left(\frac{15\lambda}{2} \right)^n e^{-3\lambda \sum\limits_{i=1}^{n} x_i} \left(1 - e^{-\lambda x_i} + \frac{1}{4}\lambda e^{-2\lambda x_i} \right)^n \right]$$

(6.1) $\quad = n \left[\ln\left(\frac{15}{2} \right) + \ln(\lambda) \right] - 3\lambda \sum\limits_{i=1}^{n} x_i + n \sum\limits_{i=1}^{n} \ln\left(1 - e^{-\lambda x_i} + \frac{1}{4}\lambda e^{-2\lambda x_i} \right)$

By taking the partial derivative with respect to λ, we have

$$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - 3\sum\limits_{i=1}^{n} x_i + n\sum\limits_{i=1}^{n} \frac{x_i e^{-\lambda x_i}\left[\frac{1}{2}\lambda x_i e^{-2\lambda x_i} - \frac{1}{4}e^{-2\lambda x_i} \right]}{\left(1 - e^{-\lambda x_i} + \frac{1}{4}\lambda e^{-2\lambda x_i} \right)}$$

This solution of equation $\frac{\partial l(\lambda)}{\partial \lambda} = 0$ gives the maximum likelihood estimates of parameter λ. The

solution can be solved numerically with use the appropriate software like R when data set are

available.

7. QUANTILE FUNCTION

The quantile q of the random variable, say X that follows BiED is the solution of the equation

$G\left(x_q \right) = q$. Hence,

$$G(x) = q = \left[1 - \frac{5}{2}e^{-3\lambda x} + \frac{15}{8}e^{-4\lambda x} - \frac{3}{8}e^{-5\lambda x} \right], \quad \text{so} \quad 1 - q = \frac{5}{2}e^{-3\lambda x} - \frac{15}{8}e^{-4\lambda x} + \frac{3}{8}e^{-5\lambda x}$$

Simplify and rearrange we have

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$$(7.1) \qquad 8 - 8q = 20\left(e^{-\lambda x}\right)^3 - 15\left(e^{-\lambda x}\right)^4 + 3\left(e^{-\lambda x}\right)^5$$

This equation does not have an exact but it can be solved numerically. The solutions of the quantile

function are illustrated in Table 1.

Table 1: Numerical solutions of Equation 7.1 of BiED for $\lambda = 0.5,\ 1,\ 2,\ 3,\ 4$ and the values of

$q = \{0.01, 0.2, ..., 0.99\}$

q	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	q	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
0	0	0	0	0	0	0.5	0.660	0.330	0.165	0.110	0.083
0.01	0.011	0.005	0.003	0.002	0.001	0.51	0.678	0.339	0.169	0.113	0.085
0.02	0.022	0.011	0.005	0.004	0.003	0.52	0.696	0.348	0.174	0.116	0.087
0.03	0.032	0.016	0.008	0.005	0.004	0.53	0.714	0.357	0.178	0.119	0.089
0.04	0.043	0.022	0.011	0.007	0.005	0.54	0.733	0.366	0.183	0.122	0.092
0.05	0.054	0.027	0.014	0.009	0.007	0.55	0.751	0.376	0.188	0.125	0.094
0.06	0.065	0.033	0.016	0.011	0.008	0.56	0.771	0.385	0.193	0.128	0.096
0.07	0.076	0.038	0.019	0.013	0.010	0.57	0.791	0.395	0.198	0.132	0.099
0.08	0.087	0.044	0.022	0.015	0.011	0.58	0.811	0.405	0.203	0.135	0.101
0.09	0.099	0.049	0.025	0.016	0.012	0.59	0.831	0.415	0.208	0.138	0.104
0.1	0.110	0.055	0.027	0.018	0.014	0.6	0.852	0.426	0.213	0.142	0.106
0.11	0.121	0.061	0.030	0.020	0.015	0.61	0.873	0.436	0.218	0.145	0.109
0.12	0.133	0.066	0.033	0.022	0.017	0.62	0.895	0.447	0.224	0.149	0.112
0.13	0.144	0.072	0.036	0.024	0.018	0.63	0.917	0.459	0.229	0.153	0.115
0.14	0.156	0.078	0.039	0.026	0.019	0.64	0.940	0.470	0.235	0.157	0.118
0.15	0.167	0.084	0.042	0.028	0.021	0.65	0.964	0.482	0.241	0.161	0.120
0.16	0.179	0.090	0.045	0.030	0.022	0.66	0.987	0.494	0.247	0.165	0.123
0.17	0.191	0.096	0.048	0.032	0.024	0.67	1.012	0.506	0.253	0.169	0.127
0.18	0.203	0.102	0.051	0.034	0.025	0.68	1.037	0.519	0.259	0.173	0.130
0.19	0.215	0.108	0.054	0.036	0.027	0.69	1.063	0.532	0.266	0.177	0.133
0.2	0.227	0.114	0.057	0.038	0.028	0.7	1.090	0.545	0.273	0.182	0.136
0.21	0.240	0.120	0.060	0.040	0.030	0.71	1.118	0.559	0.279	0.186	0.140
0.22	0.252	0.126	0.063	0.042	0.032	0.72	1.146	0.573	0.287	0.191	0.143
0.23	0.265	0.132	0.066	0.044	0.033	0.73	1.175	0.588	0.294	0.196	0.147
0.24	0.277	0.139	0.069	0.046	0.035	0.74	1.206	0.603	0.301	0.201	0.151
0.25	0.290	0.145	0.073	0.048	0.036	0.75	1.237	0.618	0.309	0.206	0.155
0.26	0.303	0.151	0.076	0.050	0.038	0.76	1.269	0.635	0.317	0.212	0.159
0.27	0.316	0.158	0.079	0.053	0.039	0.77	1.303	0.652	0.326	0.217	0.163
0.28	0.329	0.164	0.082	0.055	0.041	0.78	1.338	0.669	0.335	0.223	0.167
0.29	0.342	0.171	0.086	0.057	0.043	0.79	1.375	0.688	0.344	0.229	0.172

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0.3	0.356	0.178	0.089	0.059	0.044	0.8	1.413	0.707	0.353	0.236	0.177
0.31	0.369	0.185	0.092	0.062	0.046	0.81	1.453	0.727	0.363	0.242	0.182
0.32	0.383	0.191	0.096	0.064	0.048	0.82	1.496	0.748	0.374	0.249	0.187
0.33	0.396	0.198	0.099	0.066	0.050	0.83	1.540	0.770	0.385	0.257	0.192
0.34	0.410	0.205	0.103	0.068	0.051	0.84	1.586	0.793	0.397	0.264	0.198
0.35	0.425	0.212	0.106	0.071	0.053	0.85	1.636	0.818	0.409	0.273	0.205
0.36	0.439	0.219	0.110	0.073	0.055	0.86	1.689	0.844	0.422	0.281	0.211
0.37	0.453	0.227	0.113	0.076	0.057	0.87	1.746	0.873	0.436	0.291	0.218
0.38	0.468	0.234	0.117	0.078	0.058	0.88	1.806	0.903	0.451	0.301	0.226
0.39	0.483	0.241	0.121	0.080	0.060	0.89	1.871	0.936	0.468	0.312	0.234
0.4	0.498	0.249	0.124	0.083	0.062	0.9	1.943	0.972	0.486	0.324	0.243
0.41	0.513	0.257	0.128	0.086	0.064	0.91	2.022	1.011	0.505	0.337	0.253
0.42	0.528	0.264	0.132	0.088	0.066	0.92	2.109	1.054	0.527	0.351	0.264
0.43	0.544	0.272	0.136	0.091	0.068	0.93	2.208	1.104	0.552	0.368	0.276
0.44	0.560	0.280	0.140	0.093	0.070	0.94	2.321	1.160	0.580	0.387	0.290
0.45	0.576	0.288	0.144	0.096	0.072	0.95	2.453	1.227	0.613	0.409	0.307
0.46	0.592	0.296	0.148	0.099	0.074	0.96	2.615	1.307	0.654	0.436	0.327
0.47	0.609	0.304	0.152	0.101	0.076	0.97	2.821	1.411	0.705	0.470	0.353
0.48	0.626	0.313	0.156	0.104	0.078	0.98	3.109	1.554	0.777	0.518	0.389
0.49	0.643	0.321	0.161	0.107	0.080	0.99	3.595	1.798	0.899	0.599	0.449

8. ORDER STATISTICS

If $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \cdots, X_n$ from a continuous population with cdf $G_X(x)$ and pdf $g_X(x)$ then the pdf of $X_{(j)}$ is defined as

$$g_j(x) = \frac{n!}{(j-1)!(n-j)!} g(x) [G(x)]^{j-1} [1-G(x)]^{n-j} \quad ; \ j = 1, 2, ..., n$$

Therefore, the pdf of BiED of $X_{(j)}$ is given by

(8.1)
$$= \frac{1}{1-\rho} \ln \frac{15\lambda}{2} \int_0^\infty \left[e^{-3\rho\lambda x} \left(1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right)^\rho \right] dx$$

By using the binomial theorem

$$g_j(x) = \binom{n-1}{j-1} \frac{15\lambda n}{2} y^{3(n-j+1)} \left[1 - y + \frac{1}{4} y^2 \right] \left[1 - \frac{5}{2} y^3 + \frac{15}{8} y^4 - \frac{3}{8} y^5 \right]^{j-1} \left[\frac{5}{2} - \frac{15}{8} y + \frac{3}{8} y^2 \right]^{n-j},$$

where $y = e^{-\lambda x}$

Moreover, the pdf of the largest order statistic $X_{(n)}$ of the BiED is given by

$$g_n(x) = \frac{15\lambda n}{2} e^{-3\lambda x} \left[1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right] \left[1 - \frac{5}{2} e^{-3\lambda x} + \frac{15}{8} e^{-4\lambda x} - \frac{3}{8} e^{-5\lambda x} \right]^{n-1}$$

And the pdf of the smallest order statistic $X_{(1)}$ of the BiED is given by

$$g_1(x) = \frac{15\lambda n}{2} e^{-3\lambda x n} \left[1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right] \left[\frac{5}{2} - \frac{15}{8} e^{-\lambda x} + \frac{3}{8} e^{-2\lambda x} \right]^{n-1}$$

9. RÉNYI ENTROPY

The Rényi entropy for the BiED is defined by the following theorem:

Theorem 9.1: The Rényi entropy for the random variable X that follows a BiED with pdf $g(x)$

defined by:

$$E_R = \frac{1}{1-\rho} \ln \frac{15}{2} \sum_{i=0}^{\rho} \sum_{m=0}^{i} (-1)^m \binom{\rho}{i} \binom{i}{m} \left(\frac{1}{4} \right)^{(\rho-i)} \frac{1}{(5\rho+m-2i)}$$

Proof:

$$(9.1) \qquad E_R = \frac{1}{1-\rho} \ln \left[\int_0^{\infty} (g(x))^\rho \, dx \right] = \frac{1}{1-\rho} \ln \frac{15\lambda}{2} \int_0^{\infty} \left[e^{-3\lambda x} \left(1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right)^\rho \right] dx$$

$$= \frac{1}{1-\rho} \ln \frac{15\lambda}{2} \int_0^{\infty} \left[e^{-3\rho\lambda x} \left(1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right)^\rho \right] dx$$

$$\left(1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right)^\rho = \sum_{i=0}^{\rho} \binom{\rho}{i} \left(1 - e^{-\lambda x} \right)^i \left(\frac{1}{4} e^{-2\lambda x} \right)^{\rho-i} \text{, and}$$

$$\left(1 - e^{-\lambda x} \right)^i = \sum_{m=0}^{i} \binom{i}{m} (-1)^m \left(e^{-\lambda x} \right)^m \text{ then the term } \left(1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right)^\rho \text{ becomes}$$

as

$$\left(1 - e^{-\lambda x} + \frac{1}{4} e^{-2\lambda x} \right)^\rho = \sum_{i=0}^{\rho} \sum_{m=0}^{i} \binom{\rho}{i} \binom{i}{m} (-1)^m \left(e^{-\lambda x} \right)^m \left(\frac{1}{4} e^{-2\lambda x} \right)^{\rho-i}$$

Thus equation (9.1) becomes

$$= \frac{1}{1-\rho} \ln \frac{15\lambda}{2} \sum_{i=0}^{\rho} \sum_{m=0}^{i} \binom{\rho}{i}\binom{i}{m}(-1)^m \int_0^{\infty}\left[e^{-3\rho\lambda x}\left(e^{-\lambda x}\right)^m\left(\frac{1}{4}e^{-2\lambda x}\right)^{\rho-i}\right]dx$$

Rearrange and simplify, we have

$$E_R = \frac{1}{1-\rho} \ln \frac{15\lambda}{2} \sum_{i=0}^{\rho} \sum_{m=0}^{i} (-1)^m \binom{\rho}{i}\binom{i}{m}\left(\frac{1}{4}\right)^{(\rho-i)} \int_0^{\infty}\left[e^{-3\rho\lambda x}\left(e^{-m\lambda x}\right)\left(e^{-2(\rho-i)\lambda x}\right)\right]dx$$

$$= \frac{1}{1-\rho} \ln \frac{15\lambda}{2} \sum_{i=0}^{\rho} \sum_{m=0}^{i} (-1)^m \binom{\rho}{i}\binom{i}{m}\left(\frac{1}{4}\right)^{(\rho-i)} \int_0^{\infty}\left[e^{-\lambda x(5\rho+m-2i)}\right]dx$$

$\square$

10. BONFERRONI AND LORENZ CURVES AND GINI INDEX OF THE BIED

Assume that the random variable X is non-negative with continuous and twice differentiable cumulative distribution function G(x). The Bonferroni Curve of the random variable X follows BiED is defined as

(10.1)
$$B(p) = \frac{1}{p\mu} \int_0^{q} xg(x)dx = \frac{1}{p\mu}\left(\mu - \int_q^{\infty} xg(x)dx\right)$$

Where $q = G^{-1}(p)$ and $p \in (0,1]$. The integral in equation (10.1) can be computed for the BiED as follows

$$\int_q^{\infty} xg(x)dx = \int_q^{\infty} x\left[\frac{15}{2}\lambda e^{-3\lambda x} - \frac{15}{2}\lambda e^{-4\lambda x} + \frac{15}{8}\lambda e^{-5\lambda x}\right]dx$$

$$= \frac{15}{2}\left(\frac{e^{-3\lambda q}(1+3\lambda q)}{9\lambda}\right) - \frac{15}{2}\left(\frac{e^{-4\lambda q}(1+4\lambda q)}{16\lambda}\right) + \frac{15}{8}\left(\frac{e^{-5\lambda q}(1+5\lambda q)}{25\lambda}\right)$$

$$B(p) = \left(\frac{1}{p} - \left[\frac{15}{2p\mu\lambda}\left[\left(\frac{e^{-3\lambda q}(1+3\lambda q)}{9}\right) - \left(\frac{e^{-4\lambda q}(1+4\lambda q)}{16}\right) + \frac{1}{4}\left(\frac{e^{-5\lambda q}(1+5\lambda q)}{25}\right)\right]\right]\right)$$

Lorenz Curves of BiED is defined as

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$$(10.2) \qquad L(p) = \frac{1}{\mu} \int_0^q xg(x)dx = \frac{1}{\mu}\left(\mu - \int_q^\infty xg(x)dx\right)$$

The integral in equation (10.2) can be computed for the BiED as follows

$$\int_q^\infty xg(x)dx = \frac{15}{2}\lambda\int_q^\infty xe^{-3\lambda x}dx - \frac{15}{2}\lambda\int_q^\infty xe^{-4\lambda x}dx + \frac{15}{8}\lambda\int_q^\infty xe^{-5\lambda x}dx$$

$$= \frac{15e^{-3\lambda q}}{2\lambda}\left[\left(\frac{1}{9}(1+3\lambda q)\right)-\left(\frac{e^{-\lambda q}}{16}(1+4\lambda q)\right)+\frac{1}{4}\left(\frac{e^{-2\lambda q}}{25}(1+5\lambda q)\right)\right].$$

Now, Lorenz Curves of BiED becomes as

$$L(p) = 1 - \frac{15e^{-3\lambda q}}{2\lambda\mu}\left[\frac{1}{9}(1+3\lambda q)-\frac{e^{-\lambda q}}{16}(1+4\lambda q)+\frac{1}{4}\frac{e^{-2\lambda q}}{25}(1+5\lambda q)\right]$$

The Gini index of the BiED is given by

$$(10.3) \qquad G = 1 - \frac{1}{\mu}\int_0^\infty (1-F(x))^2 dx = \frac{1}{\mu}\left(\int_0^\infty F(x)(1-F(x))dx\right)$$

By using the cdf of BiED and substitute in equation (10.3), we have

$$\int_0^\infty\left[1-\frac{5}{2}e^{-3\lambda x}+\frac{15}{8}e^{-4\lambda x}-\frac{3}{8}e^{-5\lambda x}\right]\left[1-\left(1-\frac{5}{2}e^{-3\lambda x}+\frac{15}{8}e^{-4\lambda x}-\frac{3}{8}e^{-5\lambda x}\right)\right]dx = \frac{1}{5\lambda}$$

So, the Gini index of the BiED is defined as

$$G = \frac{1}{5\lambda\mu}$$

11. MEAN AND MEDIAN DEVIATIONS OF THE BiED

The mean deviation about the mean $\xi_1(x)$ and the mean deviation about the median $\xi_2(x)$, to measure the scatter in the population are defined as follows

$$(11.1) \qquad \xi_1(x) = \int_0^\infty |x-\mu|g(x)dx = 2\mu G(\mu) - 2\int_0^\mu xg(x)dx$$

and

(11.2)
$$\xi_2(x) = \int_0^\infty |x-M| g(x) dx = \mu - 2 \int_0^M xg(x)dx$$

We substitute the pdf and cdf of the BiED in equations (11.1) and (11.2) to find the mean and

median deviations about the mean and median, respectively, are

(11.3)
$$\xi_1(x) = 2\mu G(\mu) - 2\int_0^\mu x\left[\frac{15}{2}\lambda e^{-3\lambda x} - \frac{15}{2}\lambda e^{-4\lambda x} + \frac{15}{8}\lambda e^{-5\lambda x}\right] dx$$

The integral in equation (11.3) can be computed as

$$\int_0^\mu xg(x)dx = \int_0^\mu x\left[\frac{15}{2}\lambda e^{-3\lambda x} - \frac{15}{2}\lambda e^{-4\lambda x} + \frac{15}{8}\lambda e^{-5\lambda x}\right] dx$$

$$= \frac{15}{\lambda}\left[\begin{array}{l} \frac{1}{18}\left(1-3\lambda\mu e^{-3\lambda\mu} - e^{-3\lambda\mu}\right) - \frac{1}{32}\left(1-4\lambda\mu e^{-4\lambda\mu} - e^{-4\lambda\mu}\right) \\ +\frac{1}{200}\left(1-5\lambda\mu e^{-5\lambda\mu} - e^{-5\lambda\mu}\right) \end{array}\right]$$

Now,

$$\xi_1(x) = 2\mu G(\mu) - \frac{15}{\lambda}\left[\begin{array}{l} \frac{1}{9}\left(1-3\lambda\mu e^{-3\lambda\mu} - e^{-3\lambda\mu}\right) - \frac{1}{16}\left(1-4\lambda\mu e^{-4\lambda\mu} - e^{-4\lambda\mu}\right) \\ +\frac{1}{100}\left(1-5\lambda\mu e^{-5\lambda\mu} - e^{-5\lambda\mu}\right) \end{array}\right]$$

Substituting $\mu = \left[\dfrac{5}{6\lambda} - \dfrac{15}{32\lambda} + \dfrac{3}{40\lambda}\right] = \dfrac{211}{480\lambda}$, we get

$$\xi_1(x) = \left[\frac{5}{3}e^{-\left[\frac{4}{3}\right]} - \frac{15}{16}e^{-\left[\frac{16}{9}\right]} + \frac{3}{20}e^{-\left[\frac{20}{9}\right]}\right] = 0.2971$$

Similarly, we can compute the integral in equation (11.2) as follows

$$2\int_0^M xg(x)dx = 2\frac{15}{\lambda}\left[\begin{array}{l} \frac{1}{18}\left(1-3\lambda M e^{-3\lambda M} - e^{-3\lambda M}\right) - \frac{1}{32}\left(1-4\lambda M e^{-4\lambda M} - e^{-4\lambda M}\right) \\ +\frac{1}{200}\left(1-5\lambda M e^{-5\lambda M} - e^{-5\lambda M}\right) \end{array}\right]$$

$$= \frac{1}{\lambda}\left(\frac{5}{3} - \frac{15}{16} + \frac{3}{10}\right) - 5\left(M + \frac{1}{3\lambda}\right)e^{-3\lambda M} + \frac{15}{4}\left(M + \frac{1}{4\lambda}\right)e^{-4\lambda M} - \frac{3}{2}\left(M + \frac{1}{5\lambda}\right)e^{-5\lambda M}$$

So,

$$\xi_2(x) = \mu - \frac{1}{\lambda}\left(\frac{5}{3} - \frac{15}{16} + \frac{3}{10}\right) - 5\left(M + \frac{1}{3\lambda}\right)e^{-3\lambda M} + \frac{15}{4}\left(M + \frac{1}{4\lambda}\right)e^{-4\lambda M} - \frac{3}{2}\left(M + \frac{1}{5\lambda}\right)e^{-5\lambda M}$$

Again, we substitute $\mu = \left[\frac{5}{6\lambda} - \frac{15}{32\lambda} + \frac{3}{40\lambda}\right]$ then we have

$$\xi_2(x) = -\frac{3}{5\lambda} - 5\left(M + \frac{1}{3\lambda}\right)e^{-3\lambda M} + \frac{15}{4}\left(M + \frac{1}{4\lambda}\right)e^{-4\lambda M} - \frac{3}{2}\left(M + \frac{1}{5\lambda}\right)e^{-5\lambda M}$$

12. APPLICATIONS

In this section, we compare the performance of new probability distribution (BiED) with the performance of exponential distribution, The goodness of fit based on the following statistic; Cramer-von misses (W), statistic Anderson darling (A), Kolmogorov's D statistics, **p-value,** Maximum likelihood estimates (MLE), Akaike information criterion (AIC), Consistent Akaikes Information Criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Minimum value of *-2log-Lik*. For this purpose five datasets are used. The **first** dataset consists of data on the number of cycles of failure for 25 specimens of 100 cm specimens of yarn, tested at a particular strain level by [19] which presented as follows:

15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653.

The **second** dataset consist of thirty successive values of March precipitation (in inches) given by [20] and recorded as: 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05.

The **third** dataset represents the waiting times (in minutes) before service of 100 bank customers [21]; 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1,

7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6 ,21.3,21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

The **forth** real data set is given by [22] and it represents the number of million revolutions before failure for each of 23 ball bearings in a life test as follows; 17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.4, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

The **fifth** dataset refer the time between failures for repairable item [23]. The data as follows; 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17

The aim of generalizing any distribution is to make it more flexible. As presented in Tables 2-6 for all datasets, the BiED was recognized to be more flexible than the exponential distribution. Tables 2-6, the BiED has lowest values of W, A, D, MLE, AIC, CAIC, BIC, HQIC, Value and the larger value of p-value. Therefore, BiED is the best distribution for fitting all datasets.

Table 2: The goodness of fit statistics for BiED and exponential distribution to the first dataset

Dis	W	A	D	p-value	MLE	AIC	CAIC	BIC	HQIC	Value
BiED	0.05	0.337	0.185	0.356	0.0025	309.4	309.56	310.6	309.7	153.69
Exp.	0.06	0.379	0.199	0.275	0.0056	311.2	311.35	312.4	311.5	154.59

Table 3: The goodness of fit statistics for BiED and exponential distribution to the second dataset

Dis	W	A	D	p-value	MLE	AIC	CAIC	BIC	HQIC	Value
BiED	0.0143	0.111	0.214	0.1266	0.274	89.63	89.77	91.0	90.08	43.81
Exp	0.0136	0.103	0.235	0.0724	0.597	92.95	93.09	94.35	93.4	45.47

Table 4: The goodness of fit statistics for BiED and exponential distribution to the third dataset

Dis	W	A	D	p-value	MLE	AIC	CAIC	BIC	HQIC	Value
BiED	0.043	0.273	0.147	0.0259	0.046	652.8	652.87	655.4	653.9	325
Exp.	0.027	0.179	0.173	0.0050	0.101	660.0	660.08	662.7	661.1	329

Table 5: The goodness of fit statistics for BiED and exponential distribution to the forth dataset

Dis	W	A	D	p-value	MLE	AIC	CAIC	BIC	HQIC	Value
BiED	0.0429	0.2371	0.292	0.0398	0.0064	241.83	242	243	242	119.9
Exp	0.0384	0.2149	0.307	0.0264	0.0138	244.87	245	246.0	245	121.4

Table 6: The goodness of fit statistics for BiED and exponential distribution to the fifth dataset

Dis	W	A	D	p-value	MLE	AIC	CAIC	BIC	HQIC	Value
BiED	0.0222	0.1695	0.159	0.434	0.2926	85.8	85.95	87.2	86.256	41.9
Exp	0.0189	0.1439	0.185	0.259	0.6482	88.01	88.15	89.41	88.459	43

13. CONCLUSION

In this paper, we use the Biweight Kernel function (BKF) and exponential distributions to propose a new distribution called the Biweight exponential distribution. The new distribution is introduced without adding any new parameters to the original distributions. The statistical properties are considered including the estimation of model parameters and its application demonstrated using real datasets. Applications show that the new distribution fits better than the exponential distribution.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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