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SEMI-INCLINE LIGHTLIKE SUBMANIFOLDS OF INDEFINITE COSYMPLECTIC MANIFOLDS

RAJENDRA PRASAD, SHASHIKANT PANDEY*

Department of Mathematics and Astronomy, University of Lucknow, Lucknow, Uttar Pradesh, 226007, India

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Abstract. In this paper, we investigate the geometry of semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds and acquires portrayal the hypothesis for such manifolds. Next, we give the integrability states of appropriations on semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds. At last, we likewise acquire necessary and sufficient condition for foliation controlled by appropriations to be absolutely geodesic and give a few precedents.

Keywords: semi-incline lightlike manifolds; degenerate metric; indefinite cosymplectic manifolds; lightlike transversal vector bundle; radical distribution.

2010 AMS Subject Classification: 53C15, 53C40, 53C50.

1. INTRODUCTION

The hypothesis of submanifolds of semi-Riemannian manifolds is intriguing to think about the geometry of lightlike submanifolds due to the way that the crossing point of typical vector pack and the digression group is non-inconsequential. In this way, the examination turns out to be more fascinating and surprisingly unique in relation to the investigation of non-degenerate submanifolds. The geometry of lightlike submanifolds of inconclusive Kaehler manifolds was displayed in a book by Duggal furthermore, Bejancu [7]. B. Y. Chen has introduced the notion

*Corresponding author

E-mail address: shashi.royal.lko@gmail.com

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of slant immersions by generalizing the concept of holomorphic and totally real immersions [2] and [3]. Later, the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold was introduced by A. Lotta in [13]. Riemannian geometry of contact and symplectic manifolds was studied by D. E. Blair in [1]. The slant submanifolds of a Cosymplectic manifold was studied in [9] and lightlike submanifolds of indefinite Cosymplectic manifolds was studied in [12] and [10]. To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two (radical and screen) distributions. The radical distribution is totally lightlike and therefore it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Using these facts the notion of a slant lightlike submanifold of an indefinite Hermitian manifold was introduced by B. Sahin [15]. The geometry of semi-slant submanifolds of Kaehler manifolds was studied by N. Papaghuic in [14] and totally geodesic foliations was studied by Johnson and Whitt in [11]. The theory of slant, Cauchy-Riemann lightlike submanifolds of indefinite Kaehler manifolds has been studied in [7]. Lightlike submanifolds of indefinite Sasakian manifolds were studied in [6]. Lightlike submanifolds of indefinite Kaehler manifolds includes slant, Cauchy-Riemann lightlike submanifolds as its sub-cases. Lightlike geometry has its applications in general relativity, particularly in black hole theory, which gave impetus to study lightlike submanifolds of semi-Riemannian manifolds equipped with certain structures in [8]. The geometry of slant and semi-slant submanifolds of Sasakian manifolds was studied by J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez in [4] and [5]. On the other hand the theory of slant, contact Cauchy-Riemann lightlike submanifolds of indefinite Sasakian manifolds have been studied in [16]. Thus motivated sufficiently, we introduce the notion of semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds. Recently S.S. Shukla and A. Yadav introduce the notion of semi-slant lightlike submanifolds of indefinite Sasakian manifolds [18] and indefinite Kaehler manifolds in [17]. Motivated by above, we are interested to study semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds. We arranged the paper in the following manner:

In section second, we give a brief theory of lightlike geometry which involves definition and

some results also fundamentals which are needed for the paper. In section three, we investigate the geometry of semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds. Section four is devoted to the study of foliations determined by distributions on semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds. In section five, we provide some examples of semi-incline lightlike submanifolds of an indefinite Cosymplectic manifold.

2. PRELIMINARIES

An odd-dimensional semi-Riemannian manifold \bar{M} is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi_*, V, \eta, \bar{g})$, where ϕ_* is a (1,1) tensor field, V a vector field, η a 1-form and \bar{g} is the semi-Riemannian metric on \bar{M} satisfying

$$(2.1) \quad \phi_*^2 X = -X + \eta(X)V, \quad \eta \circ \phi_* = 0, \quad \eta(V) = \varepsilon, \quad \phi_* V = 0,$$

$$(2.2) \quad \bar{g}(\phi_* X, \phi_* Y) = \bar{g}(X, Y) - \varepsilon \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(T\bar{M})$, where $\varepsilon = 1$ or -1 . It follows that

$$(2.3) \quad \bar{g}(V, V) = \varepsilon,$$

$$(2.4) \quad \bar{g}(X, V) = \eta(X),$$

$$(2.5) \quad \bar{g}(X, \phi_* Y) = -\bar{g}(\phi_* X, Y),$$

for all $X, Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of vector fields on \bar{M} . An indefinite almost contact metric manifold \bar{M} is called an indefinite Cosymplectic structure iff [1],

$$(2.6) \quad (\tilde{\nabla}_X \phi_*)Y = 0,$$

A semi-Riemannian manifold endowed with an indefinite Cosymplectic structure is called an indefinite Cosymplectic manifold.

For any $X \in \Gamma(T\bar{M})$, we get

$$(2.7) \quad \tilde{\nabla}_X V = 0,$$

where $\tilde{\nabla}$ denote the Levi-Civita connection on \bar{M} .

Let $(\bar{M}, \bar{g}, \phi_*, V, \eta,)$ be an ε -almost contact metric manifold. If $\varepsilon = 1$, then \bar{M} is said to be a

spacelike ε -almost contact metric manifold and if $\varepsilon = -1$, then \bar{M} is called a timelike ε -almost contact metric manifold. In this paper, we consider indefinite Cosymplectic manifolds with spacelike characteristic vector field V .

(\bar{M}^{m+n}, \bar{g}) is called a lightlike submanifold if the metric g induced from \bar{g} is degenerate and the radical distribution $Rad(TM)$ is of rank r , where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM , i.e.,

$$(2.8) \quad TM = Rad(TM) \oplus_{orth} S(TM).$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in TM^\perp . Since, for any local basis ξ_i of $Rad(TM)$, there exists a local null frame N_i of sections with values in the orthogonal complement of $S(TM^\perp)$ in $S(TM^\perp)$ such that $\bar{g}(\xi_i, N_i) = \delta_{ij}$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by N_i . Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then,

$$(2.9) \quad tr(TM) = ltr(TM) \oplus_{orth} S(TM^\perp),$$

$$(2.10) \quad T\bar{M}|_M = TM \oplus_{orth} tr(TM),$$

$$(2.11) \quad T\bar{M}|_M = S(TM) \oplus_{orth} (Rad(TM) \oplus_{orth} ltr(TM)) \oplus_{orth} S(TM^\perp).$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM|_{Rad(TM)}$. We say that a submanifold $(M, g, S(TM), S(TM^\perp))$ of M is

- (i) r -lightlike if $r < \min(m, n)$,
- (ii) coisotropic if $r = n < m, S(TM^\perp) = 0$,
- (iii) isotropic if $r = m < n, S(TM) = 0$,
- (iv) totally lightlike if $r = m = n, S(TM) = 0 = S(TM^\perp)$.

The Gauss and Weingarten equations are

$$(2.12) \quad \bar{\nabla}_X Y = \hat{\nabla}_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$(2.13) \quad \bar{\nabla}_X V = \hat{\nabla}'_X V - A_V X, \quad \forall V \in \Gamma(tr(TM)),$$

where $(\hat{\nabla}_X Y, A_V X)$ and $(h(X, Y), \hat{\nabla}_X^t V)$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. $\hat{\nabla}$ and $\hat{\nabla}^t$ are linear connection on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is symmetric $F(M)$ - bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is bilinear endomorphism of $\Gamma(TM)$. From (2.12) and (2.13), we have

$$(2.14) \quad \tilde{\nabla}_X Y = \hat{\nabla}_X Y + h^l(X, Y) + h^s(X, Y), \forall X, Y \in \Gamma(TM),$$

$$(2.15) \quad \tilde{\nabla}_X N = -A_N X + \hat{\nabla}_X^t(N) + D^s(X, N), \forall N \in \Gamma(tr(TM)),$$

$$(2.16) \quad \tilde{\nabla}_X W = -A_W X + \hat{\nabla}_X^s(W) + D^l(X, W), \forall W \in \Gamma(tr(TM)),$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\hat{\nabla}_X^t W)$, $D^s(X, N) = S(\hat{\nabla}_X^t N)$. L and S are the projection morphism of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ called the lightlike connection and screen transversal connection on M respectively. For any vector field X tangent to M , we put

$$(2.17) \quad \phi_* X = PX + FX,$$

where PX and FX are tangential and transversal parts of $\phi_* X$ respectively. Now, by using (2.12),(2.14)- (2.16) and metric connection $\tilde{\nabla}$, we obtain

$$(2.18) \quad \bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, Y)) = \bar{g}(A_W X, Y),$$

$$(2.19) \quad \bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X).$$

Denote the projection of TM on $S(TM)$ by \bar{P} . Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$(2.20) \quad \hat{\nabla}_X \bar{P}Y = \hat{\nabla}_X^* \bar{P}Y + h^*(X, \bar{P}Y),$$

for all $X, Y \in \Gamma(TM)$.

$$(2.21) \quad \hat{\nabla}_X \xi = -A_\xi^* X + \hat{\nabla}_X^{*t} \xi,$$

where $\xi \in \Gamma(\text{Rad}(TM))$.

Using above equation , we get

$$(2.22) \quad \bar{g}(h^l(X, \bar{P}Y), \xi) = g(A_{\xi}^*X, \bar{P}Y),$$

$$(2.23) \quad \bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y),$$

$$(2.24) \quad \bar{g}(h^l(X, \xi), \xi) = 0, A_{\xi}^* \xi = 0.$$

Note that in general $\hat{\nabla}$ is not a metric connection. Since $\tilde{\nabla}$ is metric connection, by using (2.14), we have

$$(2.25) \quad (\hat{\nabla}_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y).$$

3. SEMI-INCLINE LIGHTLIKE SUBMANIFOLDS OF INDEFINITE COSYMPLECTIC MANIFOLDS

This new class of lightlike submanifolds of an indefinite Cosymplectic manifold includes slant, contact Cauchy-Riemann lightlike submanifolds as its subcases which have been studied in [16], [5]. In this section, we introduce the definition of semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds and some properties. At first, we state the following lemmas for later use:

Lemma 3.1. *Let M be an r -lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} of index $2q$ with structure vector field tangent to M . Suppose that $\text{Rad}(TM)$ is a distribution on M such that $\text{Rad}(TM) \cap \phi_*\text{Rad}(TM) = 0$. Then $\phi_*\text{ltr}(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi_*\text{Rad}TM \cap \phi_*\text{ltr}(TM) = \{0\}$.*

Lemma 3.2. *Let M be a q -lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} of index $2q$ with structure vector field tangent to M . Suppose $\phi_*\text{Rad}(TM)$ is a distribution on M such that $\text{Rad}(TM) \cap \phi_*\text{Rad}(TM) = \{0\}$. Then any complementary distribution to $\phi_*\text{Rad}TM \oplus \phi_*\text{ltr}(TM)$ in $S(TM)$ is Riemannian.*

The proofs of lemma 3.1 and lemma 3.2 follow as in lemma 3.1 and lemma 3.2 respectively of [5], so we omit here them.

As mentioned in the introduction, the purpose of this paper is to define semi-incline lightlike submanifolds of an indefinite Cosymplectic manifold. To define this notion, one needs to consider angle between two vector fields. As we can see from Section one, a lightlike submanifold has two distributions viz. radical and screen. The radical distribution is totally lightlike and therefore, it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Thus one way to define semi-incline lightlike submanifolds is to choose a Riemannian screen distribution on lightlike submanifolds, for which we use lemma 3.2.

Definition. [17] Let M be a q -lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} of index $2q$ such that $2q < \dim(M)$ with structure vector field tangent to M . Then we say that M is a semi-incline lightlike submanifold of \bar{M} if the following conditions are satisfied:

- (i). ϕ_*RadTM is a distribution on M such that $RadTM \cap \phi_*RadTM = \{0\}$,
- (ii). there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = (\phi_*RadTM \oplus \phi_*ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,
- (iii). the distribution D_1 is an invariant distribution, i.e. $\phi_*D_1 = D_1$,
- (iv). the distribution D_2 is incline with angle $\theta (\neq 0)$, i.e. for each $p \in M$ and each non-zero vector $X \in (D_2)_p$, the angle θ between ϕ_*X and the vector subspace $(D_2)_p$ is a non-zero constant, which is independent of the choice of $p \in M$ and $X \in (D_2)_p$,

This constant angle θ is called the incline angle of distribution D_2 . A semi-incline lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$(3.1) \quad TM = RadTM \oplus_{orth} (\phi_*RadTM \oplus \phi_*ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}.$$

In particular, we have

- (i) if $D_1 = 0$, then M is a incline lightlike submanifold,
- (ii) if $D_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is a contact CR-lightlike submanifold.

Proposition 3.3. *There exist no proper semi-incline totally lightlike or isotropic submanifolds M in indefinite Cosymplectic manifolds \bar{M} with structure vector field tangent to M .*

Now, for any vector field X tangent to M , we put $\phi_*X = PX + FX$, where PX and FX are tangential and transversal parts of ϕ_*X respectively. We denote the projections on $RadTM$, ϕ_*RadTM , $\phi_*ltr(TM)$, D_1 and D_2 in TM by P_1 , P_2 , P_3 , P_4 , and P_5 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2 respectively. Then, for any $X \in \Gamma(TM)$, we get

$$(3.2) \quad X = P_1X + P_2X + P_3X + P_4X + P_5X + \eta(X)V.$$

Now, applying ϕ_* to (3.2), we have

$$(3.3) \quad \phi_*X = \phi_*P_1X + \phi_*P_2X + \phi_*P_3X + \phi_*P_4X + \phi_*P_5X,$$

By above, we have

$$(3.4) \quad \phi_*X = \phi_*P_1X + \phi_*P_2X + \phi_*P_3X + \phi_*P_4X + fP_5X + FP_5X,$$

we obtain that $\phi_*P_1X \in \Gamma(\phi_*RadTM)$, $\phi_*P_2X \in \Gamma(RadTM)$, $\phi_*P_3X \in \Gamma(ltr(TM))$, $\phi_*P_4X \in \Gamma(D_1)$, $fP_5X \in \Gamma(D_2)$ and $FP_5X \in \Gamma(S(TM^\perp))$. Where fP_5X and FP_5X are the tangential and transversal component of ϕ_*P_5X .

For any $W \in \Gamma(tr(TM))$, we have

$$(3.5) \quad W = Q_1W + Q_2W.$$

Operating ϕ_* in (3.5), we have

$$(3.6) \quad \phi_*W = \phi_*Q_1W + \phi_*Q_2W.$$

Now, we obtain using (3.6), we have

$$(3.7) \quad \phi_*W = \phi_*Q_1W + BQ_2W + CQ_2W.$$

We get $\phi_*Q_1W \in \Gamma(\phi_*ltr(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^\perp))$. where BQ_2W and CQ_2W are the tangential and transversal component of ϕ_*Q_2W . Now, by using (2.6), (3.4), (3.7)

and (2.14)-(2.16) and identifying the components on $RadTM$, ϕ_*RadTM , $\phi_*ltr(TM)$, D_1 , D_2 , $ltr(TM)$, $S(TM^\perp)$ and V , we have some expression given as follows:

$$(3.8) \quad P_1(\hat{\nabla}_X\phi_*P_1Y) + P_1(\hat{\nabla}_X\phi_*P_2Y) + P_1(\hat{\nabla}_X\phi_*P_4Y) + P_1(\hat{\nabla}_XfP_5Y) \\ = P_1(A_{FP_5Y}X) + P_1(A_{\phi_*P_3Y}X) + \phi_*P_2\hat{\nabla}_XY,$$

$$(3.9) \quad P_2(\hat{\nabla}_X\phi_*P_1Y) + P_2(\hat{\nabla}_X\phi_*P_2Y) + P_2(\hat{\nabla}_X\phi_*P_4Y) + P_2(\hat{\nabla}_XfP_5Y) \\ = P_2(A_{FP_5Y}X) + P_2(A_{\phi_*P_3Y}X) + \phi_*P_1\hat{\nabla}_XY,$$

$$(3.10) \quad P_3(\hat{\nabla}_X\phi_*P_1Y) + P_3(\hat{\nabla}_X\phi_*P_2Y) + P_3(\hat{\nabla}_X\phi_*P_4Y) + P_3(\hat{\nabla}_XfP_5Y) \\ = P_1(A_{FP_5Y}X) + P_1(A_{\phi_*P_3Y}X) + \phi_*h^l(X, Y),$$

$$(3.11) \quad P_4(\hat{\nabla}_X\phi_*P_1Y) + P_4(\hat{\nabla}_X\phi_*P_2Y) + P_4(\hat{\nabla}_X\phi_*P_4Y) + P_4(\hat{\nabla}_XfP_5Y) \\ = P_4(A_{FP_5Y}X) + P_4(A_{\phi_*P_3Y}X) + \phi_*P_4\hat{\nabla}_XY,$$

$$(3.12) \quad P_5(\hat{\nabla}_X\phi_*P_1Y) + P_5(\hat{\nabla}_X\phi_*P_2Y) + P_5(\hat{\nabla}_X\phi_*P_4Y) + P_5(\hat{\nabla}_XfP_5Y) \\ = P_5(A_{FP_5Y}X) + P_5(A_{\phi_*P_3Y}X) + f\phi_*P_5\hat{\nabla}_XY + Bh^s(X, Y),$$

$$(3.13) \quad h^l(X, \phi_*P_1Y) + h^l(X, \phi_*P_2Y) + h^l(X, \phi_*P_4Y) + h^l(X, fP_5Y) \\ = \phi_*P_3\hat{\nabla}_XY - \hat{\nabla}_X^l\phi_*P_3Y - D^l(X, FP_5Y),$$

$$(3.14) \quad h^s(X, \phi_*P_1Y) + h^s(X, \phi_*P_2Y) + h^s(X, \phi_*P_4Y) + h^s(X, fP_5Y) \\ = Ch^s(X, Y) - \hat{\nabla}_X^sFP_5Y - D^l(X, \phi_*P_3Y) + FP_5\hat{\nabla}_XY,$$

$$(3.15) \quad \eta(\hat{\nabla}_X\phi_*P_1Y) + \eta(\hat{\nabla}_X\phi_*P_2Y) + \eta(\hat{\nabla}_X\phi_*P_4Y) + \eta(\hat{\nabla}_XfP_5Y) \\ = \eta(A_{\phi_*P_3Y}X) + \eta(A_{FP_5Y}X).$$

Theorem 3.4. *Let M be a q -lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} of index $2q$ with structure vector field tangent to M . Then M is a semi-incline lightlike submanifold if and only if*

- (i) ϕ_*RadTM is a distribution on M such that $RadTM \cap \phi_*RadTM = \{0\}$,

(ii) the distribution D_1 is an invariant distribution, i.e. $\phi_*D_1 = D_1$,

(iii) there exists a constant $\delta \in [0, 1)$ such that $P^2X = -\delta X$.

Moreover, there also exists a constant $\mu \in (0, 1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = (\phi_*RadTM \oplus \phi_*ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\delta = \cos^2\theta$, where θ is a incline angle of D_2 .

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Then distribution D_1 is invariant with respect to ϕ_* and ϕ_*RadTM is a distribution on M such that $RadTM \cap \phi_*RadTM = \{0\}$. Now for any $X \in \Gamma(D_2)$ we have $|PX| = |\phi_*X|\cos\theta$, which implies

$$(3.16) \quad \cos\theta = \frac{|PX|}{|\phi_*X|}.$$

Using equation (3.16), we obtain

$$\cos^2\theta = \frac{|PX|^2}{|\phi_*X|^2} = \frac{g(PX, PX)}{g(\phi_*X, \phi_*X)} = \frac{g(X, P^2X)}{g(X, \phi_*^2X)}$$

Using above, we get

$$(3.17) \quad g(X, P^2X) = \cos^2\theta g(X, \phi_*^2X).$$

Since M is semi-incline lightlike submanifold $\cos^2\theta = \delta \in [0, 1)$ and therefore from (3.17), we have

$$g(X, P^2X) = \lambda g(X, \phi_*^2X) = g(X, \lambda \phi_*^2X),$$

which gives

$$(3.18) \quad g(X, (P^2 - \delta \phi_*^2)X) = 0.$$

Since $(P^2 - \delta \phi_*^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2 \times D_2}$ is non degenerate, from (3.18), we get $(P^2 - \delta \phi_*^2)X = 0$ which gives

$$(3.19) \quad P^2X = \delta \phi_*^2X = -\delta X.$$

Now, for any vector field $X \in \Gamma(D_2)$, we have

$$(3.20) \quad \phi_*X = PX + FX,$$

where PX and FX are tangential and transversal parts of ϕ_*X respectively. Applying ϕ_* to (3.20) and taking tangential component, we get

$$(3.21) \quad -X = P^2X + BFX.$$

From (3.19) and (3.21), we get

$$(3.22) \quad BFX = -\mu X,$$

$$1 - \delta = \mu \in [0, 1).$$

Which proves (iii)

Conversely. Suppose that conditions (i), (ii) and (iii) are satisfied. From (3.21), for any $X \in \Gamma(D_2)$, we get

$$(3.23) \quad -X = P^2X - \mu X,$$

which implies

$$(3.24) \quad P^2X = -\delta X,$$

$1 - \mu = \delta \in [0, 1)$. Now, we have

$$\cos\theta = \frac{g(\phi_*X, PX)}{|\phi_*X||PX|} = \frac{g(X, \phi_*PX)}{|\phi_*X||PX|} = \frac{g(X, P^2X)}{|\phi_*X||PX|} = -\delta \frac{g(X, \phi_*^2X)}{|\phi_*X||PX|} = \frac{g(\phi_*X, \phi_*X)}{|\phi_*X||PX|}.$$

From above equation, we obtain

$$(3.25) \quad \cos\theta = \delta \frac{|\phi_*X|}{|PX|}.$$

Therefore using (3.16) and (3.25), we obtain $\cos^2\theta = \delta$.

So M is a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold.

Corollary 3.5. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with incline angle θ , then for any $X, Y \in \Gamma(D_2)$, we have*

- (i) $g(PX, PX) = \cos^2\theta g(X, Y),$
- (ii) $g(FX, FX) = \sin^2\theta g(X, Y).$

Proof. The proof of above Corollary follows as in proof of Corollary 3.2 of [17].

Lemma 3.6. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Then for any $X, Y \in \Gamma(TM - \{V\})$, we have*

- (i) $g(\hat{\nabla}_X Y, V) = \bar{g}(Y, \phi_* X)$,
- (ii) $g([X, Y], V) = 2\bar{g}(X, \phi_* Y)$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Since $\hat{\nabla}$ is a metric connection, from (2.7) and (2.14), for any $X, Y \in \Gamma(TM - \{V\})$, we have

$$(3.26) \quad g(\hat{\nabla}_X Y, V) = \bar{g}(Y, \phi_* X),$$

using (2.5) and (3.26), for any $X, Y \in \Gamma(TM - \{V\})$, we get

$$(3.27) \quad g([X, Y], V) = 2\bar{g}(X, \phi_* Y).$$

Theorem 3.7. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $RadTM$ is integrable if and only if*

$$P_1(\hat{\nabla}_X \phi_* P_1 Y) = P_1(\hat{\nabla}_Y \phi_* P_1 X)$$

$$P_4(\hat{\nabla}_X \phi_* P_1 Y) = P_4(\hat{\nabla}_Y \phi_* P_1 X)$$

$$P_5(\hat{\nabla}_X \phi_* P_1 Y) = P_5(\hat{\nabla}_Y \phi_* P_1 X),$$

and

$$h^l(Y, \phi_* P_1 X) = h^l(X, \phi_* P_1 Y)$$

$$h^s(Y, \phi_* P_1 X) = h^s(X, \phi_* P_1 Y),$$

for all $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Let $X, Y \in \Gamma(RadTM)$. From equation (3.8), we have

$$P_1(\hat{\nabla}_X \phi_* P_1 Y) = \phi_* P_2 \hat{\nabla}_X Y,$$

which gives

$$P_1(\hat{\nabla}_X \phi_* P_1 Y) - P_1(\hat{\nabla}_Y \phi_* P_1 X) = \phi_* P_2[X, Y],$$

using (3.11), we obtain

$$P_4(\hat{\nabla}_X \phi_* P_1 Y) = \phi_* P_4 \hat{\nabla}_X Y,$$

which gives

$$P_4(\hat{\nabla}_X \phi_* P_1 Y) - P_4(\hat{\nabla}_Y \phi_* P_1 X) = \phi_* P_4[X, Y],$$

now, using (3.12), we get

$$P_5(\hat{\nabla}_X \phi_* P_1 Y) = f P_5 \hat{\nabla}_X Y + B h^s(X, Y),$$

which gives

$$P_5(\hat{\nabla}_X \phi_* P_1 Y) - P_5(\hat{\nabla}_Y \phi_* P_1 X) = f P_5[X, Y],$$

in view of (3.13), we have

$$h^l(X, \phi_* P_1 Y) = \phi_* P_3 \hat{\nabla}_X Y,$$

which gives

$$h^l(X, \phi_* P_1 Y) - h^l(Y, \phi_* P_1 X) = \phi_* P_3[X, Y],$$

also, using (3.14), we get

$$h^s(X, \phi_* P_1 Y) = C h^s(X, Y) + F P_5 \hat{\nabla}_X Y,$$

which gives

$$h^s(X, \phi_* P_1 Y) - h^s(Y, \phi_* P_1 X) = F P_5[X, Y],$$

which completes the prove of the Theorem.

Theorem 3.8. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $D_1 \oplus \{V\}$ is integrable if and only if*

$$P_1(\hat{\nabla}_X \phi_* P_4 Y) = P_1(\hat{\nabla}_Y \phi_* P_4 X),$$

$$P_4(\hat{\nabla}_X \phi_* P_4 Y) = P_4(\hat{\nabla}_Y \phi_* P_4 X),$$

$$P_5(\hat{\nabla}_X \phi_* P_4 Y) = P_5(\hat{\nabla}_Y \phi_* P_4 X),$$

and

$$h^l(Y, \phi_* P_4 X) = h^l(X, \phi_* P_4 Y),$$

$$h^s(Y, \phi_* P_4 X) = h^s(X, \phi_* P_4 Y),$$

for all $X, Y \in \Gamma(D_1 \oplus \{V\})$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Let $X, Y \in \Gamma(D_1 \oplus \{V\})$. From (3.8), we have

$$P_1(\hat{\nabla}_X \phi_* P_4 Y) = \phi_* P_2 \hat{\nabla}_X Y,$$

which gives

$$P_1(\hat{\nabla}_X \phi_* P_4 Y) - P_1(\hat{\nabla}_Y \phi_* P_4 X) = \phi_* P_2[X, Y],$$

using (3.9), we get

$$P_2(\hat{\nabla}_X \phi_* P_4 Y) = \phi_* P_1 \hat{\nabla}_X Y,$$

which gives

$$P_2(\hat{\nabla}_X \phi_* P_4 Y) - P_2(\hat{\nabla}_Y \phi_* P_4 X) = \phi_* P_4[X, Y],$$

now, using (3.12), we obtain

$$P_5(\hat{\nabla}_X \phi_* P_4 Y) = f P_5 \hat{\nabla}_X Y + B h^s(X, Y),$$

which gives

$$P_5(\hat{\nabla}_X \phi_* P_4 Y) - P_5(\hat{\nabla}_Y \phi_* P_4 X) = f P_5[X, Y],$$

in view of (3.13), we have

$$h^l(X, \phi_* P_4 Y) = \phi_* P_3 \hat{\nabla}_X Y,$$

which gives

$$h^l(X, \phi_* P_4 Y) - h^l(Y, \phi_* P_4 X) = \phi_* P_3[X, Y],$$

also, using (3.14), we get

$$h^s(X, \phi_* P_4 Y) = C h^s(X, Y) + F P_5 \hat{\nabla}_X Y,$$

which gives

$$h^s(X, \phi_* P_4 Y) - h^s(Y, \phi_* P_4 X) = \phi_* P_5[X, Y],$$

which completes the prove of the Theorem.

Theorem 3.9. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $D_2 \oplus \{V\}$ is integrable if and only if*

$$\begin{aligned} P_1(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) &= P_1, (A_{FP_5Y}X - A_{FP_5X}Y), \\ P_2(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) &= P_2, (A_{FP_5Y}X - A_{FP_5X}Y), \\ P_4(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) &= P_4, (A_{FP_5Y}X - A_{FP_5X}Y), \\ h^l(X, fP_5Y) - h^l(Y, fP_5X) &= D^l(Y, FP_5X) - D^l(X, FP_5Y), \end{aligned}$$

for all $X, Y \in \Gamma(D_2 \oplus \{V\})$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Let $X, Y \in \Gamma(D_2 \oplus \{V\})$. From (3.8), we have

$$P_1(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) = \phi_* P_2 \hat{\nabla}_X Y,$$

which gives

$$P_1(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) - P_1(A_{FP_5Y}X - A_{FP_5X}Y) = \phi_* P_2[X, Y],$$

from (3.9), we have

$$P_2(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) - P_2(A_{FP_5Y}X - A_{FP_5X}Y) = \phi_* P_1 \hat{\nabla}_X Y,$$

which gives

$$P_2(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) - P_2(A_{FP_5Y}X - A_{FP_5X}Y) = \phi_* P_1[X, Y],$$

in view of (3.11), we have

$$P_4(\hat{\nabla}_X fP_5Y) - P_4(A_{FP_5Y}X) = \phi_* P_4 \hat{\nabla}_X Y,$$

which gives

$$P_4(\hat{\nabla}_X fP_5Y - \hat{\nabla}_Y fP_5X) - P_4(A_{FP_5Y}X - A_{FP_5X}Y) = \phi_* P_4[X, Y],$$

now, from (3.13), we obtain

$$h^l(X, fP_5Y) + D^l(X, FP_5Y) = \phi_* P_3 \hat{\nabla}_X Y,$$

which implies

$$h^l(X, fP_5Y) - h^l(Y, fP_5X) - D^l(Y, FP_5X) + D^l(X, FP_5Y) = \phi_*P_3[X, Y],$$

which prove all the required results of the Theorem.

Theorem 3.10. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field V tangent to M . Then induced connection $\hat{\nabla}$ is not a metric connection.*

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Suppose that the induced connection is a metric connection. Then, for all $X, Y \in \Gamma(TM)$, we have

$$\hat{\nabla}_X P_2 Y \in \Gamma(RadTM) \text{ and } h^l(X, Y) = 0.$$

Thus from (2.6), for any $Z \in \Gamma(\phi_*RadTM)$ and $W \in \Gamma(\phi_*ltr(TM))$, we get

$$(3.28) \quad \tilde{\nabla}_W \phi_* Z - \phi_* \tilde{\nabla}_W Z = \bar{g}(Z, W)V.$$

Using (2.14) and (3.28) and taking tangential components, we have

$$(3.29) \quad \begin{aligned} \hat{\nabla}_W \phi_* Z - \phi_* P_1 \hat{\nabla}_W Z - \phi_* P_2 \hat{\nabla}_W Z - \phi_* P_4 \hat{\nabla}_W Z \\ = fP_5 \hat{\nabla}_W Z + Bh^s(Z, W) + \bar{g}(Z, W)V. \end{aligned}$$

Since $TM = RadTM \oplus_{orth} (\phi_*RadTM \oplus \phi_*ltr(TM)) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$, and using

(3.29)

$$(3.30) \quad \hat{\nabla}_W \phi_* Z - \phi_* P_2 \hat{\nabla}_W Z = 0, \phi_* P_1 \hat{\nabla}_W Z = 0, \phi_* P_4 \hat{\nabla}_W Z = 0,$$

and

$$(3.31) \quad fP_5 \hat{\nabla}_W Z - Bh^s(Z, W) = 0, \bar{g}(Z, W)V = 0.$$

Now, taking $W = \phi_*N$ and $Z = \phi_*\xi$ in (3.31), we have

$$\bar{g}(N, \xi)V = 0.$$

which gives $V = 0$.

By above, we can see this is a contradiction. Hence M does not have a metric connection.

4. FOLIATIONS ON SEMI-INCLINE LIGHTLIKE SUBMANIFOLDS OF INDEFINITE COSYMPLECTIC MANIFOLDS

In this section, we proceed to obtain necessary and sufficient conditions for foliations determined by distributions $D_1 \oplus \{V\}$ and $D_2 \oplus \{V\}$ on a semi-incline lightlike submanifolds of indefinite Cosymplectic manifolds to be totally geodesic.

Definition. A semi-incline lightlike submanifold M of an indefinite Cosymplectic manifold \bar{M} is said to be mixed geodesic if its second fundamental form h satisfies $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic semi- incline lightlike submanifold if $h^l(X, Y) = 0$ and $h^s(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $RadTM$ defines a totally geodesic foliation if and only if*

$$\bar{g}(\hat{\nabla}_X \phi_* P_2 Z + \hat{\nabla}_X \phi_* P_4 Z + \hat{\nabla}_X f P_5 Z, \phi_* Y) = \bar{g}(A_{\phi_* P_3 Z} X + A_{F P_5 Z} X, \phi_* Y),$$

for all $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . To prove $RadTM$ defines a totally geodesic foliation it is sufficient to show that $\hat{\nabla}_X Y \in \Gamma(RadTM)$, for all $X, Y \in \Gamma(RadTM)$. Since $\tilde{\nabla}$ is metric connection, using (2.2), (2.6), (2.14) and (3.4), for any $X, Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we obtain

$$\bar{g}(\hat{\nabla}_X Y, Z) = -\bar{g}(\tilde{\nabla}_X (\phi_* P_2 Z + \phi_* P_3 Z + \phi_* P_4 Z + f P_5 Z + F P_5 Z), \phi_* Y),$$

which implies

$$\bar{g}(\hat{\nabla}_X Y, Z) = -\bar{g}(A_{\phi_* P_3 Z} X + A_{F P_5 Z} X - \hat{\nabla}_X \phi_* P_2 Z - \hat{\nabla}_X \phi_* P_4 Z - \hat{\nabla}_X f P_5 Z, \phi_* Y),$$

which completes the prove of the Theorem.

Theorem 4.2. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

- (1) $\bar{g}(A_{FZ}X, \phi_*Y) = \bar{g}(\hat{\nabla}_X fZ, \phi_*Y)$,
- (ii) $A_{\phi_*W}X$ and $\hat{\nabla}_X \phi_*N$ have no component $D_1 \oplus \{V\}$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$, $Z \in \Gamma(D_2)$, $W \in \Gamma(\phi_*ltr(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . The distribution $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\hat{\nabla}_X Y \in \Gamma(D_1 \oplus \{V\})$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$. Since $\tilde{\nabla}$ is metric connection, from (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $Z \in \Gamma(D_2)$, we obtain

$$\bar{g}(\hat{\nabla}_X Y, Z) = -\bar{g}(\tilde{\nabla}_X \phi_*Z, \phi_*Y),$$

which gives us

$$\bar{g}(\hat{\nabla}_X Y, Z) = \bar{g}(A_{FZ}X - \hat{\nabla}_X fZ, \phi_*Y),$$

using equations (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we have

$$\bar{g}(\hat{\nabla}_X Y, N) = -\bar{g}(\phi_*Y, \tilde{\nabla}_X \phi_*N),$$

by which we obtain

$$\bar{g}(\hat{\nabla}_X Y, N) = -\bar{g}(\phi_*Y, \hat{\nabla}_X \phi_*N),$$

now, using (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $W \in \Gamma(\phi_*ltr(TM))$, we have

$$\bar{g}(\hat{\nabla}_X Y, W) = -\bar{g}(\phi_*Y, \tilde{\nabla}_X \phi_*W),$$

which implies

$$\bar{g}(\hat{\nabla}_X Y, W) = -\bar{g}(\phi_*Y, A_{\phi_*W}X),$$

which completes the prove of the Theorem.

Theorem 4.3. *Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $(D_2 \oplus \{V\})$ defines a totally geodesic foliation if and only if*

$$\bar{g}(\hat{\nabla}_X \phi_*Z, fY) = -\bar{g}(h^s(X, \phi_*Z), FY),$$

$$\bar{g}(fY, \hat{\nabla}_X \phi_*N) = -\bar{g}(FY, h^s(X, \phi_*N)),$$

$$\bar{g}(fY, A_{\phi_*W}X) = -\bar{g}(h^s(\phi_*Y, D^s(X, \phi_*W)),$$

for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(\phi_*\text{ltr}(TM))$.

Proof. Let M be a semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\hat{\nabla}_X Y \in \Gamma(D_2 \oplus \{V\})$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$. Since $\tilde{\nabla}$ is metric connection, from (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we obtain

$$\bar{g}(\hat{\nabla}_X Y, Z) = -\bar{g}(\tilde{\nabla}_X \phi_* Z, \phi_* Y),$$

which gives us

$$\bar{g}(\hat{\nabla}_X Y, N) = -\bar{g}(\hat{\nabla}_X \phi_* Z, fY) - \bar{g}(h^s(X, \phi_* Z), FY),$$

using (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(\text{ltr}(TM))$, we have

$$\bar{g}(\hat{\nabla}_X Y, N) = -\bar{g}(\phi_* Y, \tilde{\nabla}_X \phi_* N),$$

by which we obtain

$$\bar{g}(\hat{\nabla}_X Y, N) = -\bar{g}(fY, \hat{\nabla}_X \phi_* N) - \bar{g}(FY, h^s(X, \phi_* N)),$$

now, using (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $W \in \Gamma(\phi_*\text{ltr}(TM))$, we have

$$\bar{g}(\hat{\nabla}_X Y, W) = -\bar{g}(\phi_* Y, \tilde{\nabla}_X \phi_* W),$$

which implies

$$\bar{g}(\hat{\nabla}_X Y, W) = -\bar{g}(fY, A_{\phi_* W} X) - \bar{g}(FY, D^s(X, \phi_* W)),$$

which completes the prove of the Theorem.

Theorem 4.4. *Let M be a mixed geodesic semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} with structure vector field tangent to M . Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if*

- (i) $\hat{\nabla}_X \phi_* Z$ has no component in then $D_2 \oplus \{V\}$,
- (ii) $\bar{g}(fY, \hat{\nabla}_X \phi_* N) = -\bar{g}(FY, h^s(X, \phi_* N))$,
- (iii) $\bar{g}(fY, A_{\phi_* W} X) = \bar{g}(FY, D^s(X, \phi_* W))$,

for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in (D_1)$, $N \in \Gamma(\text{ltr}(TM))$ and $W \in \Gamma(\phi_*\text{ltr}(TM))$.

Proof. Let M be a mixed geodesic semi-incline lightlike submanifold of an indefinite Cosymplectic manifold \bar{M} . Then

$$h(X, Y) = 0,$$

for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. The distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\hat{\nabla}_X Y \in \Gamma(D_2 \oplus \{V\})$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\bar{\nabla}$ is metric connection. Using (2.2), (2.6) and (2.14), for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$, we have

$$\bar{g}(\hat{\nabla}_X Y, Z) = \bar{g}(\bar{\nabla}_X \phi_* Z, \phi_* Y),$$

which implies

$$\bar{g}(\hat{\nabla}_X Y, Z) = -\bar{g}(\hat{\nabla}_X \phi_* Z, fY) - \bar{g}(h^s(X, \phi_* Z), FY),$$

now, using (2.2), (2.6) and (2.14) for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(\text{ltr}(TM))$, we get

$$\bar{g}(\hat{\nabla}_X Y, N) = -g(\phi_* Y, \bar{\nabla}_X \phi_* N)$$

the above equation gives

$$\bar{g}(\hat{\nabla}_X Y, N) = -\bar{g}(\phi_* Y, \hat{\nabla}_X \phi_* N) - \bar{g}(FY, h^s(X, \phi_* N)),$$

now, using (2.2), (2.6) and (2.14) for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $W \in \Gamma(\phi_* \text{ltr}(TM))$, we obtain

$$\bar{g}(\hat{\nabla}_X Y, W) = -g(\phi_* Y, \bar{\nabla}_X \phi_* W),$$

which gives

$$\bar{g}(\hat{\nabla}_X Y, W) = -\bar{g}(\phi_* Y, A_{\phi_* W} X) - \bar{g}(FY, D^s(X, \phi_* W)),$$

which completes the prove of the Theorem.

5. EXAMPLES

Example 1. Let $(R_q^{2m+1}, \bar{g}, \phi_*, \eta, V)$ denote the manifold R_q^{2m+1} with its usual Cosymplectic structure given by

$$\eta = dz, V = \partial z,$$

$$\bar{g} = \eta \otimes \eta - \sum_{i=1}^{\frac{q}{2}} (dx^i \otimes dx^i + dy^i \otimes dy^i) + \sum_{i=q+1}^m (dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi_*\left(\sum_{i=1}^m (X_i\partial x_i + Y_i\partial y_i) + Z\partial z\right) = \sum_{i=1}^m (Y_i\partial x_i - X_i\partial y_i)$$

where (x_i, y_i, z) are the cartesian coordinates on R_q^{2m+1} .

Now, we construct some examples of semi-incline lightlike submanifold of an indefinite Cosymplectic manifold.

Example 2. Let $(R_2^{13}, \bar{g}, \phi_*, \eta, V)$ be an indefinite Cosymplectic manifold, where \bar{g} is of signature $(-, +, +, +, +, +, -, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of R_2^{13} given by $-x^1 = y^2 = u_1, x^2 = u_2, y^1 = u_3, x^3 = -y^4 = u_4, x^4 = y^3 = u_5, x^5 = u_6 \sin u_7, y^5 = u_6 \cos u_7, x^6 = \sin u_6, y^6 = \cos u_6, z = u_8$.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where

$$Z_1 = (-\partial x_1 + \partial y_2), Z_2 = \partial x_2$$

$$Z_3 = \partial y_1, Z_4 = (\partial x_3 - \partial y_4)$$

$$Z_5 = (\partial x_4 + \partial y_3), Z_6 = (\sin u_7 \partial x_5 + \cos u_7 \partial y_5 + \cos u_6 \partial x_6 - \sin u_6 \partial y_6)$$

$$Z_7 = (u_6 \cos u_7 \partial x_5 - u_6 \sin u_7 \partial y_5), Z_8 = V = \partial z$$

Hence $RadTM = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, V\}$.

Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$$W_1 = (\partial x_3 + \partial y_4),$$

$$W_2 = (\partial x_4 - \partial y_3)$$

$$W_3 = (\sin u_7 \partial x_5 + \cos u_7 \partial y_5 - \cos u_6 \partial x_6 + \sin u_6 \partial y_6)$$

$$W_4 = (u_6 \sin u_6 \partial x_6 - u_6 \cos u_6 \partial y_6)$$

It follows that $\phi_*Z_1 = Z_2 + Z_3$ and $\phi_*N = \frac{1}{2}(Z_2 - Z_3)$, which implies that $\phi_*Rad(TM)$ and $\phi_*ltr(TM)$ are distribution on M . On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\phi_*Z_4 = -Z_5, \phi_*Z_5 = Z_4$, which implies that D_1 is invariant with respect to ϕ_* and $D_2 = span\{Z_6, Z_7\}$ is a incline distribution with incline angle $\frac{\pi}{4}$. Hence M is a semi-incline 2-lightlike submanifold of R_2^{13} .

Example 3. Let $(R_2^{13}, \bar{g}, \phi_*, \eta, V)$ be an indefinite Cosymplectic manifold, where \bar{g} is of signature $(-, +, +, +, +, +, -, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of R_2^{13} given by

$x^1 = y^2 = u_1, x^2 = u_2, y^1 = u_3, x^3 = y^4 = u_4, x^4 = -y^3 = u_5, x^5 = u_6 \cos \theta, y^5 = u_7 \sin \theta, x^6 = u_7 \sin \theta, y^6 = u_6 \sin \theta, z = u_8.$

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where

$$Z_1 = (\partial x_1 + \partial y_2), Z_2 = \partial x_2$$

$$Z_3 = \partial y_1, Z_4 = (\partial x_3 + \partial y_4)$$

$$Z_5 = (\partial x_4 - \partial y_3), Z_6 = (\cos \theta \partial x_5 + \sin \theta \partial y_6)$$

$$Z_7 = (\sin \theta \partial x_6 - \cos \theta \partial y_5), Z_8 = V = \partial z$$

Hence $RadTM = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$.

Now $ltr(TM)$ is spanned by $N = -\partial x_1 + \partial y_2$ and $S(TM^\perp)$ is spanned by

$$W_1 = (\partial x_3 - \partial y_4),$$

$$W_2 = (\partial x_4 + \partial y_3)$$

$$W_3 = (\sin \theta \partial x_5 - \cos \theta \partial y_6)$$

$$W_4 = (\cos \theta \partial x_6 - \sin \theta \partial y_5)$$

It follows that $\phi_* Z_1 = Z_2 - Z_3$ and $\phi_* N = \frac{1}{2}(Z_2 + Z_3)$, which implies that $\phi_* Rad(TM)$ and $\phi_* ltr(TM)$ are distribution on M . On the other hand, we can see that $D_1 = span\{Z_4, Z_5\}$ such that $\phi_* Z_4 = Z_5$, $\phi_* Z_5 = -Z_4$, which implies that D_1 is invariant with respect to ϕ_* and $D_2 = span\{Z_6, Z_7\}$ is a incline distribution with incline angle 2θ . Hence M is a semi-incline 2-lightlike submanifold of R_2^{13} .

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] D.E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhauser, 2002.
- [2] B.Y. Chen, Geometry of Slant Submanifolds, Katholieke Universiteit, Leuven, 1990.
- [3] B.Y. Chen, Slant Immersions, Bull. Aust. Math. Soc. 41 (1990), 135-147.
- [4] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, Slant Sub-manifolds in Sasakian manifolds, Glasgow Math J. 42(2000), 125-138.
- [5] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, Semi-slant Submanifolds of a Sasakian manifold, Geom. Dedicata, 78(1999), 183-199.

- [6] K.L. Duggal and B. Sahin, Lightlike submanifolds of indefinite Sasakian manifolds , *Int. J. Math. Math. Sci.* 2007 (2007), Article ID 57585.
- [7] K.L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, vol. 364 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands 1996.
- [8] K.L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds*, Birkhäuser Verlag, Basel, 2010.
- [9] R.S. Gupta, S.M.K. Haider, A. Sharfuddin, *Slant submanifolds in Cosymplectic Manifolds*, *Colloq. Math.* 105 (2006), 207-219.
- [10] R.S. Gupta, A. Upadhyay, and A. Sharfuddin, *Slant lightlike submanifolds of indefinite Cosymplectic manifolds*, *Mediterr. J. Math.* 8 (2011), 215-227.
- [11] D.L. Johnson and L.B. Whitt, *Totally Geodesic Foliations*, *J. Differ. Geom.* 15(1980), 225-235.
- [12] T.H. Kang and S.K. Kim, *Lightlike hypersurfaces of indefinite Cosymplectic manifolds*, *Int. Math. Forum*, 2 (2007), 3303-3316.
- [13] A. Lotta, *Slant submanifolds in contact geometry*, *Bull. Math. Soc. Roumanie* 39 (1996), 183-198.
- [14] N. Papaghiuc, *Semi-slant submanifolds of a Kaehlerian manifold*, *An. Stiint. Univ. Al. I. Cuza Iasi Sect. I a Mat.* 40 (1994), 55-61.
- [15] B. Sahin, *Slant lightlike submanifolds of indefinite Hermitian manifolds*, *Balkan J. Geom. Appl.* 13 (2008), 107-119.
- [16] B. Sahin and C. Yildirim, *Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds*, *Filomat* 26 (2012), 277-287.
- [17] S.S. Shukla and A. Yadav, *Semi-slant lightlike submanifolds of indefinite kaehler manifolds*, *Rev. Union Math. Argentina*, 56 (2015), 21-37.
- [18] S.S. Shukla and A. Yadav, *Screen Semi-slant lightlike submanifolds of indefinite Sasakian manifolds*, *Hokkaido Math. J.* 45 (2016), 365-381.