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D-HOMOTHETIC DEFORMATION OF (κ,μ) MANIFOLD

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Abstract. In this paper, we study the invariance of certain curvature conditions in (κ,μ)-contact metric manifold under D-homothetic deformation. Finally we give an example to verify the results.

Keywords: (κ,μ)-manifold; D-homothetic deformation; extended Ricci-recurrent; η-parallel.

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1. Introduction

The class of (κ,μ)-contact metric manifolds encases both Sasakian and non-Sasakian structures. This class of manifolds are invariant under D-homothetic transformation. It is noted that the class of spaces acquired through D-homothetic deformation [13] is a contact metric manifold whose curvature satisfies $R(X,Y)\xi = 0$. In [13], [14], the authors used D-homothetic deformation on Sasakian and K-contact structures to get results on the first Betti number , second Betti number and harmonic forms. A plane section in the tangent space $T_p(M)$ is called a ϕ-section if there exist a unit vector X in $T_p(M)$ orthogonal to ξ such that $\{X,\phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature $K(X,\phi X) = g(R(X,\phi X)X,\phi X)$ is called a ϕ-sectional curvature. A contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be of constant ϕ-sectional curvature if at any point $p\in M$, the sectional curvature $K(X,\phi X)$ is independent of

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choice of non-zero $X \in \mathscr{D}_p$, where $\mathscr{D}$ denotes the contact distribution of the contact metric manifold defined by $\eta = 0$.

The Riemannian curvature tensor R of Sasakian manifold of constant ϕ-sectional curvature is determined by Ogiue [9]. The geometry of contact Riemannian manfolds of constant ϕ-sectional curvature is obtained by Tanno [15]. If the ϕ-sectional curvature H is constant on K-contact Riemannian manifold $M(\phi,\xi,\eta,g)$ then H can be deformed by a D-homothetic deformation of the structure tensors [16]. An extensive research about D-homothetic deformation on contact geometry is carried out in recent years. The D-homothetic deformation is related to the following tensor structures. In other words, it means that the changing of the tensor form

$$\eta' = a\eta,\ \xi' = (\frac{1}{a})\xi,\ \phi' = \phi,\ g' = ag + (a-1)\eta\otimes\eta, \tag{1.1}$$

where a is a positive constant. In particular, some authors (Carriazo et al [3]), (De et al [4]) studied D-homothetic deformations of certain structures . An almost contact metric manaifold is said to be η-Einstein if its Ricci tensor S is of the form

$$S = \alpha g + \beta\eta\otimes\eta, \tag{1.2}$$

where α and β are smooth functions on the manifold.

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to different structures. As a weaker version of local symmetry Takahashi [12] introduced the notion of a local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of a local ϕ-symmetry of Takahashi [12]. De et al. [6] introduced the idea of ϕ-recurrent Sasakian manifolds. The notion of a generalized recurrent manifold has been introduced by Dubey [7] and studied by others. Again, the notion of a generalized Ricci recurrent manifold has been introduced and studied by De et. al. [5]. The properties of the extended generalized ϕ-recurrent β-Kenmotsu, Sasakian and $(LCS)_{2n+1}$-manifolds have been studied in [11], [10] and [18] respectively. Motivated by the above studies, in this paper we characterize the (κ,μ)-contact metric manifolds under D-homothetic deformation. We study the invariance properties of extended generalized ϕ-recurrent, locally ϕ-Ricci symmetric (κ,μ) manifolds under D-homothetic deformation. Also η-parallel Ricci tensor is considered in (κ,μ)-contact metric manifolds. Finally, we give an example of such manifold.

2. Preliminaries

Let M be $(2n+1)$-dimensional almost contact metric manifold. Then it carries two fields ϕ and ξ and a 1-form η. The field ϕ represents the endomorphism of the tangent spaces, the field ξ is called characteristic vector field and η is a 1-form satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ g(X,\xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = 1,\ \ \phi\xi = 0,\ \ \eta \circ \phi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), g(X, \phi Y) = d\eta(X,Y), \tag{2.4}$$

for any vector fields $X, Y \in \chi(M)$. In a contact metric manifold, we characterize a $(1,1)$ tensor field h by $h = \frac{1}{2}\mathscr{L}_\xi \phi$, where $\mathscr{L}$ denotes the Lie differentiation. At this point h is symmetric and satisfies $h\phi = -\phi h$. Also we have $Trh = Tr\phi h = 0$ and $h\xi = 0$. The (κ,μ)-nullity distribution of a Riemannian manifold (M,g) is a distribution

$$\begin{aligned} N(\kappa,\mu) : p \mapsto N_p(\kappa,\mu) = \{Z \in \chi_p(M) : R(X,Y)Z = \kappa[g(Y,Z)X - g(X,Z)Y] \\ + \mu[g(Y,Z)hX - g(X,Z)hY]\} \end{aligned} \tag{2.5}$$

for any $X,Y,Z \in \chi_p(M)$ and κ and μ being constants, where R denotes the Riemannian curvature tensor and $\chi_p(M)$ denotes the tangent vector space of M at any point $p \in M$. If the characteristic vector field of a contact metric manifold belongs to the (κ,μ) nullity distribution, then the relation

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2.6}$$

holds. A contact metric manifold with $\xi \in N(\kappa,\mu)$ is called a (κ,μ)-contact metric manifold [1]. In a (κ,μ)-contact metric manifold M the following relations hold [1], [2]:

$$h^2 = (\kappa - 1)\phi^2, \tag{2.7}$$

$$\nabla_X \xi = -\phi X - \phi hX, \tag{2.8}$$

$$(\nabla_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX), \tag{2.9}$$

$$(\nabla_X\eta)Y = g(X+hX,\phi Y), \tag{2.10}$$

$$R(\xi,X)Y = \kappa(g(X,Y)\xi - \eta(Y)X) + \mu(g(hX,Y)\xi - \eta(Y)hX), \tag{2.11}$$

(2.12)

$$S(X,Y) = [2(n-1)-n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)+[2(1-n)+n(2\kappa+\mu)]\eta(X)\eta(Y), n\geq 1$$

$$S(X,\xi) = 2n\kappa\eta(X), \tag{2.13}$$

$$r = 2n[2n-2+\kappa-n\mu], \tag{2.14}$$

$$(\nabla_X h)(Y) = [(1-\kappa)g(X,\phi Y)+g(X,h\phi Y)]\xi + \eta(Y)[(1-\kappa)\phi X+\phi hX] - \mu\eta(X)\phi hY, \tag{2.15}$$

where S and r are the Ricci tensor and scalar curvature respectively and Q is the Ricci opertor, i.e., $g(QX,Y) = S(X,Y)$.

3. The D-homothetic Deformation in (κ,μ) Contact Metric Manifold

Let (M,ϕ,ξ,η,g) be $(2n+1)$ dimensional (κ,μ)-contact metric manifold and $(M,\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ be obtained from (M,ϕ,ξ,η,g) by homothetic deformation (1.1). Throught the paper the quantity with bar denote quantities in $(M,\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ and the quantity without bar are for (M,ϕ,ξ,η,g). The relation between $\bar{R}$ and R of $(M,\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ as follows: [8] .

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + (1-a)[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2\eta(X)\eta(Z)hY \\ & - 2\eta(Y)\eta(Z)hX + 2g(\phi Y,X)\phi Z + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi] \\ & + \frac{(1-a)}{a}[2\eta(Y)g(hX,Z)\xi - 2\eta(X)g(hY,Z)\xi + (1-\kappa)\{\eta(Y)g(X,Z)\xi \\ & - \eta(X)g(Y,Z)\xi\} + g(\phi hX,Z)\phi Y - g(\phi hY,Z)\phi hX] + (a^2-1)[\eta(Y)\eta(Z)X \\ & - \eta(X)\eta(Z)Y],\end{aligned} \tag{3.1}$$

for any vector fields X, Y, Z on M.

Using (3.1), we derive

$$\bar{S}(Y,Z) = aS(Y,Z) + (a-1)[(a^2-2a-\kappa+1)g(Y,Z) + (2na^2+2na+2a-a^2+\kappa-1) \eta(Y)\eta(Z) + a(2+\mu)g(hY,Z)]. \tag{3.2}$$

Theorem 3.1. *Under a D-homothetic deformation the expression $Q\phi - \phi Q$ of a $(2n+1)$-dimensional (κ,μ)-contact metric manifold is invariant, provided $\mu = -2$.*

Proof: From (3.1) we have

$$\bar{Q}X = QX + \frac{a-1}{a}[(a^2-2a-\kappa+1)X + (2na^2+2na+2a-a^2+\kappa-1)\eta(X)\xi + a(2+\mu)hX]. \tag{3.3}$$

Operating $\bar{\phi} = \phi$ on both sides of above equation from the left, we have,

$$\bar{\phi}\bar{Q}X = \phi QX + \frac{a-1}{a}[(a^2-2a-\kappa+1)\phi X + a(2+\mu)\phi hX]. \tag{3.4}$$

Again, putting $\bar{\phi}X = \phi X$ in (3.2) we have

$$\bar{Q}\bar{\phi}X = Q\phi X + \frac{a-1}{a}[(a^2-2a-\kappa+1)\phi X + a(2+\mu)h\phi X]. \tag{3.5}$$

From (3.3) and (3.5) we get

$$(\bar{\phi}\bar{Q} - \bar{Q}\bar{\phi})X = (\phi Q - Q\phi)X + 2a(a-1)\{2+\mu\}\phi hX. \tag{3.6}$$

Hence the proof.

Lemma 3.1. *In a $(2n+1)$-dimensional η-Einstein (κ,μ) manifold $M(\phi,\xi,\eta,g)$, the Ricci tensor is expressed as*

$$S(X,Y) = (\frac{r}{2n} - \kappa)g(X,Y) - (\frac{r}{2n} - 2n\kappa - \kappa)\eta(X)\eta(Y). \tag{3.7}$$

Proof: On contracting (1.2) we have

$$r = (2n+1)\alpha + \beta, \tag{3.8}$$

where r is the scalar curvature of the manifold. Again putting $X = \xi$ in (2.13) we obtain,

$$\alpha + \beta = 2n\kappa. \tag{3.9}$$

Solving (3.8) and (3.9) we obtain values for $\alpha = \frac{r}{2n} - \kappa$ and $\beta = -\frac{r}{2n} + (2n+1)\kappa$. Putting the values of α and β in (1.2), we get (3.7).

Theorem 3.2. *Under D-homothetic deformation, a $(2n+1)$- dimensional η-Einstein (κ,μ)-contact metric manifold transforms to η-Einstein (κ,μ)-contact metric manifold provided $\mu = -2$.*

Proof: Let $M(\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional η-Einstein (κ,μ)-contact metric manifold which becomes $M(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$ under a D-homothetic deformation. Then from (3.1) it follows by virtue of (3.7) that

$$\bar{S}(X,Y) = \bar{A}\bar{g}(X,Y) = \bar{B}\bar{\eta}(X)\bar{\eta}(Y) + (\frac{2+\mu}{a})\bar{g}(hX,Y), \tag{3.10}$$

where $\bar{A}$ and $\bar{B}$ are smooth functions given by

$$\bar{A} = \frac{1}{a}(\frac{r}{2n} - \kappa + (\frac{a-1}{a})(a^2 - 2a - \kappa - 1)) \tag{3.11}$$

and

$$\begin{aligned}\bar{B} = &-(\frac{a-1}{a})(\frac{r}{2n} - \kappa + (\frac{a-1}{a})(a^2 - 2a - \kappa + 1)) - \frac{1}{a^2}(\frac{r}{2n} - 2n\kappa - \kappa - (\frac{a-1}{a})) \\ &[2na^2 + 2na + 2a - a^2 + \kappa - 1]).\end{aligned} \tag{3.12}$$

The Proof follows by (3.10).

Theorem 3.3. *Under D-homothetic deformation, the ϕ-sectional curvature of a $(2n+1)$-dimensional (κ,μ)-contact metric manifold is invariant, provided $\kappa = (1-3a)$.*

Proof: Here we consider the ϕ-sectional curvature on a $(2n+1)$-dimensional (κ,μ)-contact metric manifold. From (3.1) it can be easily seen that

$$\bar{K}(X,\phi X) - K(X,\phi X) = -(1-a)(3a + \kappa - 1). \tag{3.13}$$

Hence we have the proof of the theorem.

4. EXTENDED GENERALIZED ϕ-RECCURRENT, LOCALLY ϕ-RICCI SYMMETRY AND η-PARALLEL (κ,μ)-MANIFOLD

Firstly, we study the properties of the extended generalized ϕ-reccurrent (κ,μ)- manifolds under D-homothetic deformation.

Definition 4.1. *A (κ,μ)-manifold M (ϕ,ξ,η,g), is said to be an extended generalized ϕ-recurrent manifold under D-homothetic deformation if its curvature tensor $\bar{R}$ satisfies*

$$\phi^2((\nabla_W\bar{R})(X,Y)Z = A(W)\phi^2(\bar{R}(X,Y)Z) + B(W)\phi^2(G(X,Y)Z), \tag{4.1}$$

for X, Y, Z, W $\in \chi(M)$, where A and B are non-vanishing 1-forms such that $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$ and G is a tensor field of type $(1,3)$ defined as

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

The 1-forms A and B are called the associated 1-forms of the manifold.

Definition 4.2. *A (κ,μ) manifold M (ϕ,ξ,η,g) is said to be generalized Ricci-recurrent manifold under D-homothetic deformation if its non-vanishing Ricci tensor $\bar{S}$ satisfies the relation*

$$(\nabla_W\bar{S})(Y,Z) = A(W)\bar{S}(Y,Z) + B(W)g(Y,Z), \tag{4.2}$$

for all vector fields W, X, Y $\in \chi(M)$.

Theorem 4.1. *If an extended generalized ϕ-recurrent (κ,μ)-manifold M under D-homothetic deformation is a generalized Ricci-recurrent manifold, then the 1-forms A and B are related as* $2n(1-a^2-\kappa a)A(W) + (4n^2-2n-1)B(W) = 0$.

Proof: Let us suppose that the manifold $M(\phi,\xi,\eta,g)$, is an extended generalized ϕ-recurrent (κ,μ)-manifold under D-homothetic deformation. Then from (2.1), (2.2), (2.3) and (4.1), we have

$$\begin{aligned} -(\nabla_W\bar{R})(X,Y)Z + \eta((\nabla_W\bar{R})(X,Y)Z)\xi &= A(W)[-\bar{R}(X,Y)Z + \eta(\bar{R}(X,Y)Z)\xi] \\ &\quad + B(W)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi], \end{aligned} \tag{4.3}$$

from which it follows that

$$
\begin{aligned}
&-g((\nabla_W\bar{R})(X,Y)Z,U)+\eta((\nabla_W\bar{R})(X,Y)Z)\eta(U)=A(W)[-g(\bar{R}(X,Y)Z,U)\\
&\qquad +\eta(\bar{R}(X,Y)Z)\eta(U)]+B(W)[-g(G(X,Y)Z,U)\\
&\qquad +\eta(G(X,Y)Z)\eta(U)].
\end{aligned}
\tag{4.4}
$$

Let $\{e_i, i=1,2,....,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Replacing $X=U=e_i$ in (4.4) and taking summation over i, $1\leq i\leq 2n+1$, we have

$$
\begin{aligned}
&(\nabla_W\bar{S})(Y,Z)-g((\nabla_W\bar{R})(\xi,Y)Z,\xi)=A(W)[\bar{S}(Y,Z)-\eta(\bar{R}(\xi,Y)Z)]\\
&\qquad +B(W)[(2n-1)g(Y,Z)+\eta(Y)\eta(Z)].
\end{aligned}
\tag{4.5}
$$

In consequence of (2.1), (2.2), (3.1) we have

$$
\eta(\bar{R}(\xi,Y)Z)=(\frac{a^2-1+\kappa}{a})[g(Y,Z)-\eta(Y)\eta(Z)]+(\mu-\frac{2(1-a)}{a})g(hY,Z).
\tag{4.6}
$$

The covariant derivative of the above equation along the vector field W gives

$$
\begin{aligned}
g((\nabla_W\bar{R})(\xi,Y)Z,\xi)&=[\frac{a^2-1+\kappa}{a}-\mu(1-\kappa)+\frac{2(1-\kappa)(1-a)}{a}]g(\phi W,Y)\eta(Z)\\
&+[\frac{a^2-1+\kappa}{a}-\mu+\frac{2(1-a)}{a}]g(\phi hW,Y)\eta(Z)-\mu(1-\kappa)\\
&g(\phi W,Z)\eta(Y)-\mu g(\phi hW,Z)\eta(Y)-\mu g(\phi hW,Z)\eta(Y)\\
&-\mu(\mu-\frac{2(1-a)}{a})g(\phi hY,Z)\eta(W).
\end{aligned}
\tag{4.7}
$$

In view of (4.6), (4.7), (4.5) becomes

(4.8)

$$
\begin{aligned}
&(\nabla_W\bar{S})(Y,Z)-[\frac{a^2-1+\kappa}{a}-\mu(1-\kappa)+\frac{2(1-\kappa)(1-a)}{a}]g(\phi W,Y)\eta(Z)\\
&-[\frac{a^2-1+\kappa}{a}-\mu+\frac{2(1-a)}{a}]g(\phi hW,Y)\eta(Z)+\mu(1-\kappa)g(\phi W,Z)\eta(Y)+\mu g(\phi hW,Z)\eta(Y)\\
&+\mu g(\phi hW,Z)\eta(Y)+\mu(\mu-\frac{2(1-a)}{a})g(\phi hY,Z)\eta(W)\\
&=A(W)[\bar{S}(Y,Z)-(\frac{a^2-1+\kappa}{a})[g(Y,Z)-\eta(Y)\eta(Z)]+(\mu-\frac{2(1-a)}{a})g(hY,Z)]\\
&+B(W)[(2n-1)g(Y,Z)+\eta(Y)\eta(Z)].
\end{aligned}
$$

From (4.8) and the definition (4.2), it follows that an extended generalized ϕ-recurrent (κ,μ)-manifold under D-homothetic deformation is a generalized Ricci-recurrent manifold if and only if

$$\begin{aligned}
&[\frac{a^2-1+\kappa}{a}-\mu(1-\kappa)+\frac{2(1-\kappa)(1-a)}{a}]g(\phi W,Y)\eta(Z)+[\frac{a^2-1+\kappa}{a}-\mu\\
&+\frac{2(1-a)}{a}]g(\phi hW,Y)\eta(Z)-\mu(1-\kappa)g(\phi W,Z)\eta(Y)-\mu g(\phi hW,Z)\eta(Y)\\
&-\mu g(\phi hW,Z)\eta(Y)-\mu(\mu-\frac{2(1-a)}{a})g(\phi hY,Z)\eta(W)-(\frac{a^2-1+\kappa}{a})A(W)\\
&[g(Y,Z)-\eta(Y)\eta(Z)]-(\mu-\frac{2(1-a)}{a})A(W)g(hY,Z)]+B(W)[2(n-1)g(Y,Z)\\
&+\eta(Y)\eta(Z)]=0.
\end{aligned} \tag{4.9}$$

Let $\{e_i : i = 1,2,...,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $Y = Z = e_i$ in (4.9) and taking summation over i, $1 \leq i \leq 2n+1$, we have

$$2n(1-a^2-\kappa a)A(W)+(4n^2-2n-1)B(W)=0. \tag{4.10}$$

Next, we deal with the study of locally ϕ-Ricci symmetric (κ,μ)-manifolds under D-homothetic deformation.

Theorem 4.2. *The property of locally ϕ-Ricci symmetry on an (κ,μ)-manifold is invariant under the D-homothetic deformation provided $\mu = -2$.*

Proof: Differentiating (3.2) covariantly with respect to W we have

$$\begin{aligned}
(\nabla_W\bar{Q})X=(\nabla_W Q)X+(\frac{a-1}{a})(2na^2+2na+2a-a^2+\kappa-1)((\nabla_W\eta)(X)\xi\\
+\eta(X)(-\phi W-\phi hW))+(2+\mu)(\nabla_W h)X.
\end{aligned} \tag{4.11}$$

Simplifying by using (2.10) and (2.15) and operating ϕ^2 on both sides and suppose that X is orthogonal to ξ, we find that

$$\bar{\phi}^2(\nabla_W\bar{Q})(X)=\phi^2(\nabla_W Q)(X)+(2+\mu)\mu\eta(W)\phi hX. \tag{4.12}$$

Hence the proof.

Now, we deal with the study of η-parallel (κ,μ)-manifolds under D-homothetic deformation.

Theorem 4.3. *Under D-homothetic deformation, an η-Parallel Ricci tensor in a (κ,μ)- manifold remains η-parallel, provided $\mu=-2$.*

Proof: Differentiating (3.1) covariantly with respect to W and then using (2.10) and (2.15) we have

(4.13)
$$(\nabla_W\bar{S})(X,Y)=(\nabla_W S)(X,Y)+(\frac{a-1}{a})(2na^2+2na+2a-a^2+\kappa-1)(\eta(Y)(\nabla_W\eta)(X)$$
$$+\eta(X)(\nabla_W\eta)(Y))+(a-1)(2+\mu)[(1-\kappa)g(W,\phi X)\eta(Y)-g(W,\phi hX)\eta(Y)$$
$$-(1-\kappa)\eta(X)g(\phi W,Y)-\eta(X)g(\phi hW,Y)-\mu\eta(W)g(\phi hX,Y)].$$

Replacing the vector fields X by ϕX and Y by ϕY in (4.13) and then by using (2.1) and (2.2) we obtain

$$(\nabla_W\bar{S})(X,Y)=(\nabla_W S)(X,Y)-(a-1)(2+\mu)\mu\eta(W)g(X,\phi Y). \tag{4.14}$$

Hence the Proof.

5. Example

We consider 3-dimensional manifold $M=\{(x,y,z)\in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let $\{E_1,E_2,E_3\}$ be linearly independent global frame on M given by $E_1=\frac{\partial}{\partial x}$, $E_2=\frac{\partial}{\partial y}$ and $E_3=2y\frac{\partial}{\partial x}+2x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}$. $[E_1,E_2]=0$, $[E_2,E_3]=2E_1$, $[E_1,E_3]=2E_2$. Let g be a metric defined by $g(E_1,E_2)=g(E_2,E_3)=g(E_1,E_3)=0$, $g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1$. Let η be the 1-form defined by $\eta(V)=g(V,E_1)$ for any $V\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi E_1=0$, $\phi E_2=E_3$, $\phi E_3=-E_2$ and $hE_1=0$, $hE_2=E_2$ and $hE_3=-E_3$. Using the linearity of ϕ and g, we have $\eta(E_1)=1$, $\phi^2V=-V+\eta(V)\xi$ and $g(\phi V,\phi W)=g(V,W)-\eta(V)\eta(W)$, for any $V,W\in\chi(M)$.

The Riemannian connection ∇ of the metric tensor g is given by

$$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).$$

Using Koszul's formula, we get the following,

$$\begin{aligned}\nabla_{E_1}E_3=2E_2,\nabla_{E_1}E_2=-2E_3,\nabla_{E_1}E_1=0,\nabla_{E_2}E_3=0,\nabla_{E_2}E_2=0,\nabla_{E_2}E_1=-2E_3,\\ \nabla_{E_3}E_3=0,\nabla_{E_3}E_2=0,\nabla_{E_3}E_1=0.\end{aligned} \tag{5.1}$$

From (5.1) it can be easily seen that (ϕ,ξ,η,g) is a (κ,μ) manifold. Next we find the curvature tensor as follows:

$$
\begin{aligned}
&R(E_1,E_2)E_3=0, R(E_2,E_3)E_3=-4E_2, R(E_1,E_3)E_3=0,\\
&R(E_1,E_2)E_2=0, R(E_2,E_3)E_2=4E_3, R(E_1,E_3)E_2=0,\\
&R(E_1,E_2)E_1=-4E_2, R(E_2,E_3)E_1=0, R(E_1,E_3)E_1=4E_3.
\end{aligned} \tag{5.2}
$$

In view of the expression of the curvature tensor we find the Ricci tensor as follows:

$$S(E_1,E_1)=g(R(E_1,E_2)E_2,E_1)+g(R(E_1,E_3)E_3,E_1)=0. \tag{5.3}$$

Similarly we find $S(E_2,E_2)=-4=S(E_3,E_3)$. Hence $r=-8$.

It is well known that in a 3-dimensional manifold, the curvature tensor R satisfies the relation

$$R(X,Y)Z=S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y]. \tag{5.4}$$

From (2.12) we have

$$S(X,Y)=-\mu g(X,Y)+\mu g(hX,Y)+(2\kappa+\mu)\eta(X)\eta(Y). \tag{5.5}$$

From (5.5) we can find that

(5.6)

$$
\begin{aligned}
R(X,Y)Z=&\,2\mu[g(X,Z)Y-g(Y,Z)X]+\mu[g(hY,Z)X-g(hX,Z)Y+g(Y,Z)hX-g(X,Z)hY]\\
&+(2\kappa+\mu)[\eta(Y)X-\eta(X)Y]\eta(Z)+(2\kappa+\mu)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\xi\\
&-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y].
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
{}'R(X,Y,Z,W)=&\,\mu[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)]+\mu[g(hY,Z)g(X,W)\\
&-g(hX,Z)g(Y,W)+g(Y,Z)g(hX,W)-g(X,Z)g(hY,W)]\\
&+(2\kappa+\mu)[\eta(Y)g(X,W)-\eta(X)g(Y,W)]\eta(Z)\\
&+(2\kappa+\mu)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\eta(W)\\
&-\frac{r}{2}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)].
\end{aligned} \tag{5.7}
$$

In view of above relation we get

$K(E_1,\phi E_1)=0,$

$K(E_2,\phi E_2) = g(R(E_2,\phi E_2)E_2,\phi E_2) = g(R(E_1,E_3)E_2,E_3) = 2\mu + \frac{r}{2}$.
Similarly we have $K(E_3,\phi E_3) = 2\mu + \frac{r}{2}$. Again from (3.1) it can be easily shown that $\bar{K}(E_2,\phi E_2) - K(E_2,\phi E_2) = -(1-a)(3a-1)$. Similarly we have $\bar{K}(E_3,\phi E_3) - K(E_3,\phi E_3) = -(1-a)(3a-1)$ Therefore (κ,μ)-manifold satisfies the relation (3.13) and hence Theorem (3.3) is verified.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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