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THE FIRST ENTIRE ZAGREB INDEX OF VARIOUS CORONA PRODUCTS AND THEIR BOUNDS

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Abstract. The First Entire Zagreb Index (FEZI) of a (molecular) graph was introduced by Alwardi et al. [2] as the sum of the squares of degree of all the vertices and edges of the given graph. In this paper, the exact expressions for the FEZI of two graphs of several types of Corona products are established. Finally, the obtained results are applied to compute the bounds for the FEZI of two graphs.

Keywords: degree (of vertex or edge); degree-based topological indices; first entire Zagreb index; corona product; graph operation.

2010 AMS Subject Classification: 05C07, 05C12, 05C50, 05C76.

1. Introduction

In this paper, we use only molecular graphs. Molecular graphs [16] are simple, connected graphs and in which nodes and edges are assumed to be atoms and chemical bonds compounds, respectively. we consider the notations $V(G)$ and $E(G)$ as the node and edge sets of a graph G, respectively. The degree of a vertex $u \in V(G)$, $d(u/G)$, is the cardinality of the set of edges which are incident to u. In chemical graph theory, a topological index of a graph could be represented by a single numerical number that characterizes the some properties of the corresponding

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molecular graph. Relationships like quantitative structure-property relationship (QSPR), quantitative structure-activity relationship (QSAR) of molecules or the biological activity with their structure are to be predicted by using the topological indices. The two most popular and extensively studied vertex-degree-based topological indices are the first and second Zagreb indices introduced by Gutmam et al. [6] in 1972, denoted as $M_1(G)$ and $M_2(G)$, respectively and are defined as follows

$$M_1(G) = \sum_{u \in V(G)} d^2(u/G) = \sum_{uv \in E(G)} [d(u/G) + d(v/G)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d(u/G)d(v/G).$$

In the paper [7], another vertex-degree-based topological index was introduced. It was denoted as $F(G)$ and was defined by

$$F(G) = \sum_{u \in V(G)} d^3(u/G) = \sum_{uv \in E(G)} [d^2(u/G) + d^2(v/G)].$$

This index was not further studied for a long time but it was studied by Furtula et al. in 2015 [4] in which this index was named as forgotten topological index or F-index.

In 2004, Milicevic et al. [15] introduced the first reformulated Zagreb index in terms of edge degrees instead of vertex degrees. The first reformulated Zagreb index of a graph G is defined by

$$EM_1(G) = \sum_{e \in E(G)} d^2(e/G) = \sum_{e=uv \in E(G)} \Big(d(u/G) + d(v/G) - 2\Big)^2,$$

where the degree of the edge $e = uv$ is defined as $d(e/G) = d(u/G) + d(v/G) - 2$.

We refer our interested readers to [3, 8, 12, 13, 17] for some more study of the topological indices of graph operations. The readers interested in more information on bounds for various topological indices can be referred to [1, 5].

2. PRELIMINARIES

The chemical relations (forces) of the inter-molecular forces of molecules subsist between the atoms as well as the atoms and bonds in the corresponding molecular graphs. Using this chemical reason, Alwardi et al. [2] introduced a new graph invariant, namely, the First Entire Zagreb Index (FEZI).

Definition 1. [2] *The FEZI of a graph* $G=(V,E)$ *is defined by* $M_1^{\varepsilon}(G)=\sum_{u\in V(G)\cup E(G)} d^2(u/G)$, *where* $d(u/G)$ *is the degree of a vertex or an edge* u *in* G.

The following Proposition is very important for computing the expressions of the FEZI.

Proposition 1. [2, 5] *For the graph G, the following formulas are*
(i) $M_1^{\varepsilon}(G)=M_1(G)+EM_1(G)$
(ii) $M_1^{\varepsilon}(G)=4|E(G)|-3M_1(G)+2M_2(G)+F(G)$.

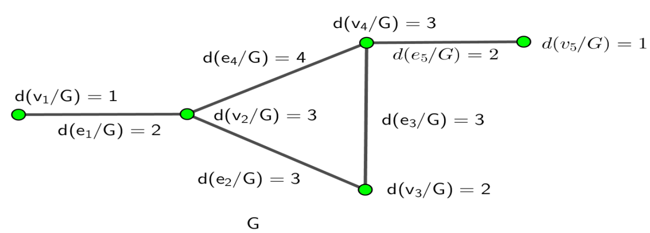

FIGURE 1. A graph(G) with $M_1^{\varepsilon}(G)=66$

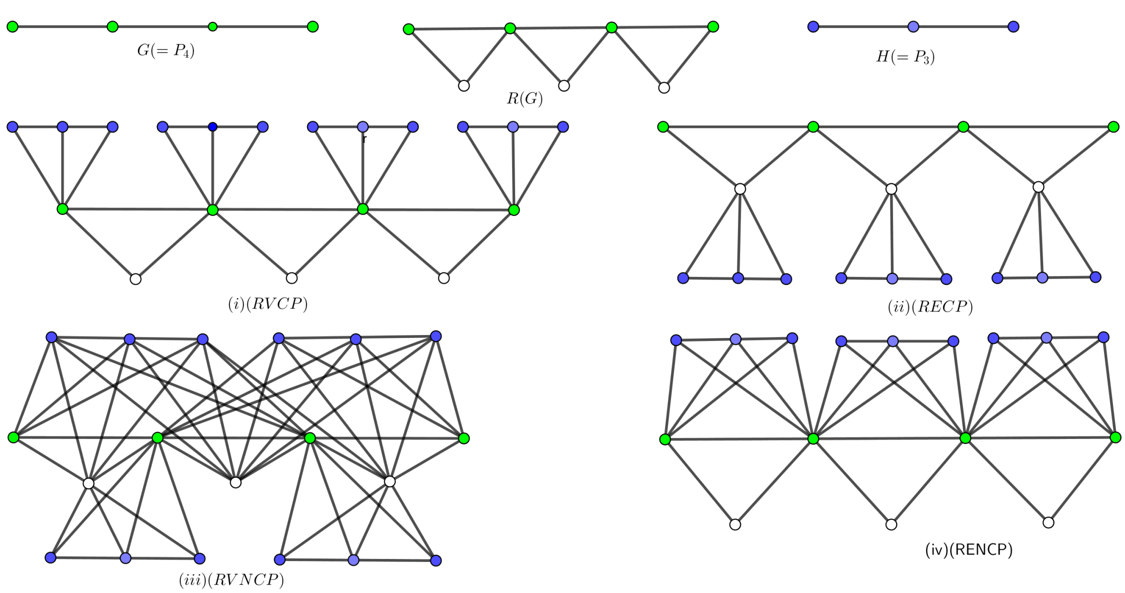

FIGURE 2. An example of various types of Corona products based on R-graphs of two graphs P_4 and P_3 such as (i) RVCP, (ii) RECP, (iii) RVNCP and (iv) RENCP

3. Main Results and Discussions

Throughout this section, we present some explicit expressions of the FEZI of two graphs for some graph operations in which one is based on R-graphs (triangle parallel graphs) such as R-Vertex Corona product (RVCP), R-Edge Corona product (RECP), R-Vertex Neighborhood Corona product (RVNCP) and R-Edge Neighborhood Corona product (RENCP) and another one is based on S-graphs (subdivision graphs) such as S-Vertex Corona product (SVCP), S-Edge Corona product (SECP), S-Vertex Neighborhood Corona product (SVNCP) and S-Edge Neighborhood Corona product (SENCP). We obtain the expression of FEZI for the Vertex-Edge Corona product (VECP) of two graphs. Also we compute some bounds for the FEZI of the nine different types of Corona product of graphs. It is to be noted that the subdivision graph $S(G)$ of a graph is the graph obtained from G by inserting a new vertex into every edge of G and the triangle parallel graph $R(G)$ is constructed from G by adding a new vertex v_e on each edge of G and then joining every newly inserted vertex to the end vertices of the corresponding $e \in E(G)$, respectively. There are two simple, connected graphs G and H having n_1, n_2 vertices and m_1, m_2 edges, respectively. To illustrate, we assume the familiar notations P_n and C_n as a path and cycle graph with n number of vertices, respectively. The maximum and minimum vertex degree of G and H are denoted by Δ_1, δ_1 and Δ_2, δ_2, respectively. For each $u \in V(G)$ and $v \in V(H)$, we have

$$\left.\begin{array}{c} \Delta_1 \geq d(u/G), \delta_1 \leq d(u/G), \\ \Delta_2 \geq d(v/H) \text{ and } \delta_2 \leq d(v/H). \end{array}\right\} \tag{1}$$

The equality holds if and only if G and H are regular graphs.

3.1. The R-Vertex Corona product(RVCP).

The RVCP of two graphs is a new graph operation based on R-graphs and it was introduced by Lan et al. [9] in 2015.

Definition 2. [9] *The RVCP of G and H, denoted by $G * H$, is the new graph obtained from vertex disjoint $R(G)$ and $|V(G)|$ copies of H by joining the ith vertex of $V(G)$ to every vertex in the ith copy of H.*

It has the number of vertices $\Big(|V(G)| + |E(G)| + |V(G)||V(H)|\Big)$ and the number of edges $\Big(3|E(G)| + |V(G)||E(H)| + |V(G)||V(H)|\Big)$, (see Fig. 2). Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, $I(G) =$

$V(R(G))\setminus V(G)=\{u_{e_1},u_{e_2},\dots,u_{e_{m_1}}\}$ and $V(H)=\{v_1,v_2,\dots,v_{n_2}\}$, so that $V(R(G))=V(G)\cup I(G)$. Let $V(H^i)=\{v_1^i,v_2^i,\dots,v_{n_2}^i\}$ be the vertex set of the i^{th} copy of H for $i=1,2,\dots,n_1$. Thus the vertex set and edge set of $R(G)*H$ are given by $V(G*H)=V(G)\cup I(G)\cup\Big(\cup_{i=1}^{n_1}V(H^i)\Big)$ and $E(G*H)=E_1^*\cup E_2^*\cup E_3^*\cup E_4^*$, where $E_1^*=\{u_iu_k\in E\Big(G*H\Big)|u_i,u_k\in V(G); i\neq k$ and $i,k=1,2,\dots,n_1\}$, $E_2^*=\{u_iu_{e_l}\in E\Big(G*H\Big)|u_i\in V(G),u_{e_l}\in I(G)$ for $i=1,2,\dots,n_1$ and $l=1,2,\dots,m_1\}$, $E_3^*=\{u_iv_j^i\in E\Big(G*H\Big)|u_i\in V(G),v_j^i\in V(H^i)$ for $i=1,2,\dots,n_1$ and $j=1,2,\dots,n_2\}$ and $E_4^*=\{v_p^iv_q^i\in E\Big(G*H\Big)|v_p^i,v_q^i\in V(H^i)$ for $i=1,2,\dots,n_1$ and $p,q=1,2,\dots,n_2\}$. The degrees of the vertices of $G*H$ are given by

$$
\left.\begin{aligned}
d(u_i/G*H)&=2d(u_i/G)+n_2 \textit{ for } i=1,2,\dots,n_1,\\
d(u_{e_l}/G*H)&=2 \textit{ for } l=1,2,\dots,m_1,\\
d(v_j^i/G*H)&=d(v_j/H)+1 \textit{ for } i=1,2,\dots,n_1 \textit{ and } j=1,2,\dots,n_2.
\end{aligned}\right\} \tag{2}
$$

We establish a formula of the FEZI for RVCP of two graphs.

Theorem 1. *The FEZI of $G*H$ is given by*
$M_1^{\varepsilon}\Big(G*H\Big)=8F(G)+n_1F(H)+4(4n_2-1)M_1(G)+2n_1M_1(H)+8M_2(G)+2n_1M_2(H)+10m_1n_2^2+4m_2n_1n_2+n_1n_2(n_2+1)+(4m_1+n_1n_2)(n_2-1)^2+4m_1(4m_2+1)$.

Proof. Applying the Definition 1, we have

$$M_1^{\varepsilon}\Big(G*H\Big)=\sum_{u\in V(G*H)\cup E(G*H)}d^2(u/G*H)$$

$$=\sum_{uv\in E(G*H)}\Big(d(u/G*H)+d(v/G*H)\Big)+\sum_{uv\in E(G*H)}\big(d(u/G*H)+d(v/G*H)-2\big)^2$$

$=A_1+A_2$ (say), where A_1 and A_2 denote the sums of the above terms in order.

Now
$$A_1=\sum_{u_iu_k\in E_1^*}\Big(d(u_i/G*H)+d(u_k/G*H)\Big)+\sum_{u_iu_{e_l}\in E_2^*}\Big(d(u_i/G*H)+d(u_{e_l}/G*H)\Big)+\sum_{u_iv_j^i\in E_3^*}\Big(d(u_i/G*H)+d(v_j^i/G*H)\Big)+\sum_{v_p^iv_q^i\in E_4^*}\Big(d(v_p^i/G*H)+d(v_q^i/G*H)\Big)$$

$$=\sum_{u_iu_k\in E(G)}\Big\{2\Big(d(u_i/G)+d(u_k/G)\Big)+2n_2\Big\}+\sum_{u_i\in V(G)}\Big\{2d(u_i/G)+(n_2+2)\Big\}d(u_i/G)$$

$$+\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big(2d(u_i/G)+d(v_j/H)+(n_2+1)\Big)+\sum_{i=1}^{n_1}\sum_{v_pv_q\in E(H)}\Big(\big(d(v_p/H)+d(v_q/H)\big)+2\Big)$$

$$=2M_1(G)+2n_2m_1+2M_1(G)+2(n_2+2)m_1+4m_1n_2+2n_1m_2+n_1n_2(n_2+1)+n_1\Big(M_1(H)+2m_2\Big).$$

$$A_2 = \sum_{u_iu_k\in E_1^*}\Big(d(u_i/G*H)+d(u_k/G*H)-2\Big)^2+\sum_{u_iu_{e_l}\in E_2^*}\Big(d(u_i/G*H)+d(u_{e_l}/G*H)-2\Big)^2+\sum_{u_iv_j^i\in E_3^*}\Big(d(u_i/G*H)+d(v_j^i/G*H)-2\Big)^2+\sum_{v_p^iv_q^i\in E_4^*}\Big(d(v_p^i/G*H)+d(v_q^i/G*H)-2\Big)^2$$

$$=\sum_{u_iu_k\in E(G)}\Big\{2\Big(d(u_i/G)+d(u_k/G)\Big)+2(n_2-1)\Big\}^2+\sum_{u_i\in V(G)}\Big(2d(u_i/G)+n_2\Big)^2d(u_i/G)$$

$$+\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big(2d(u_i/G)+d(v_j/H)+n_2-1\Big)^2+\sum_{i=1}^{n_1}\sum_{v_pv_q\in E(H)}\Big(d(v_p/H)+d(v_q/H)\Big)^2$$

$$=\sum_{u_iu_k\in E(G)}4\Big\{(d(u_i/G)+d(u_k/G))^2+(n_2-1)^2+2(n_2-1)(d(u_i/G)+d(u_k/G))\Big\}+\sum_{u_i\in V(G)}\Big(4d^3(u_i/G)+4n_2d^2(u_i/G)+n_2{}^2d(u_i/G)\Big)+\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big\{4d^2(u_i/G)+d^2(v_j/H)+(n_2-1)^2+4(n_2-1)d(u_i/G)+2(n_2-1)d(v_j/H)+4d(u_i/G)d(v_j/H)\Big\}+\sum_{i=1}^{n_1}\sum_{v_pv_q\in E(H)}\Big(d^2(u_p/H)+d^2(v_q/H)+2d(u_p/H)d(v_q/H)\Big)$$

$$=4\Big(F(G)+2M_2(G)+m_1(n_2-1)^2+2(n_2-1)M_1(G)\Big)+4F(G)+4n_2M_1(G)+2m_1n_2^2+4n_2M_1(G)+n_1M_1(H)+n_1n_2(n_2-1)^2+8m_1(n_2-1)n_2+4n_1(n_2-1)m_2+16m_1m_2++n_1\Big(F(H)+2M_2(H)\Big).$$

By summing A_1 and A_2, we have the required result. □

Applying the Theorem 1, we illustrate some examples below.

Example 1. *(i)* $M_1^\varepsilon(P_n*P_m)=(m^3n+17m^2n+94mn-14m^2-104m+30n-144)$.
(ii) $M_1^\varepsilon(C_n*C_m)=n\big(m^3+17m^2+98m+88\big)$.
(iii) $M_1^\varepsilon(C_n*P_m)=n\big(m^3+17m^2+94m+46\big)$.

We compute the bounds of the FEZI for RVCP of two graphs.

Theorem 2. *The bounds for the FEZI of $G*H$ are given by* $A\geq M_1^\varepsilon(G*H)\geq B$, *where* $A=2m_1\big(2\Delta_1+n_2\big)\big(6\Delta_1+3n_2-2\big)+2m_2n_1\big(2\Delta_2^2+\Delta_2+1\big)+n_1n_2\big(2\Delta_1+\Delta_2+n_2\big)\big(2\Delta_1+\Delta_2+n_2-1\big)+2n_1n_2+8m_1$ *and* $B=2m_1\big(2\delta_1+n_2\big)\big(6\delta_1+3n_2-2\big)+2m_2n_1\big(2\delta_2^2+\delta_2+1\big)+n_1n_2\big(2\delta_1+\delta_2+n_2\big)\big(2\delta_1+\delta_2+n_2-1\big)+2n_1n_2+8m_1$. *The equality holds if and only if G and H are regular graphs.*

Proof. From the Proposition 1 and using the Equation 2, we have

$$M_1^\varepsilon\big(G*H\big)=M_1(G*H)+EM_1(G*H)$$

$$=\sum_{u_iu_k\in E(G)}\Big\{2\Big(d(u_i/G)+d(u_k/G)\Big)+2n_2\Big\}+\sum_{u_i\in V(G)}\Big\{2d(u_i/G)+(n_2+2)\Big\}d(u_i/G)$$

$+\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big(2d(u_i/G)+d(v_j/H)+(n_2+1)\Big)+\sum_{i=1}^{n_1}\sum_{v_pv_q\in E(H)}\Big((d(v_p/H)+d(v_q/H))+$ $2\Big)+\sum_{u_iu_k\in E(G)}\Big\{2\Big(d(u_i/G)+d(u_k/G)\Big)+2(n_2-1)\Big\}^2+\sum_{u_i\in V(G)}\Big(2d(u_i/G)+n_2\Big)^2d(u_i/G)$ $+\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big(2d(u_i/G)+d(v_j/H)+n_2-1\Big)^2+\sum_{i=1}^{n_1}\sum_{v_pv_q\in E(H)}\Big(d(v_p/H)+d(v_q/H)\Big)^2.$

Also, from the Equation 1, we can write

$\leq 2m_1\Big(2\Delta_1+n_2\Big)+2m_1\Big(2\Delta_1+n_2+2\Big)+n_1n_2\Big(2\Delta_1+\Delta_2+n_2+1\Big)+2n_1m_2\Big(\Delta_2+1\Big)+$ $4m_1\Big(2\Delta_1+n_2-1\Big)^2+2m_1\Big(2\Delta_1+n_2\Big)^2+n_1n_2\Big(2\Delta_1+\Delta_2+n_2-1\Big)^2+4n_1m_2\Delta_2^2$ $=2m_1\Big(2\Delta_1+n_2\Big)+2m_1\Big(2\Delta_1+n_2\Big)+4m_1+n_1n_2\Big(2\Delta_1+\Delta_2+n_2\Big)+n_1n_2+$ $2n_1m_2\Big(\Delta_2+1\Big)+4m_1\Big(2\Delta_1+n_2\Big)^2-8m_1\Big(2\Delta_1+n_2\Big)+4m_1+2m_1\Big(2\Delta_1+n_2\Big)^2+$ $n_1n_2\Big(2\Delta_1+\Delta_2+n_2\Big)^2-2n_1n_2\Big(2\Delta_1+\Delta_2+n_2\Big)+n_1n_2+4n_1m_2\Delta_2^2$ $=2m_1\big(2\Delta_1+n_2\big)\big(6\Delta_1+3n_2-2\big)+2m_2n_1\big(2\Delta_2^2+\Delta_2+1\big)+n_1n_2\big(2\Delta_1+\Delta_2+n_2\big)\big(2\Delta_1+\Delta_2+$ $n_2-1\big)+2n_1n_2+8m_1.$

Similarly, for the reverse bound we can do that $M_1^{\varepsilon}\big(G*H\big)\geq 2m_1\Big(2\delta_1+n_2\Big)+2m_1\Big(2\delta_1+$ $n_2+2\Big)+n_1n_2\Big(2\delta_1+\delta_2+n_2+1\Big)+2n_1m_2\Big(\delta_2+1\Big)+4m_1\Big(2\delta_1+n_2-1\Big)^2+2m_1\Big(2\delta_1+n_2\Big)^2+$ $n_1n_2\Big(2\delta_1+\delta_2+n_2-1\Big)^2+4n_1m_2\delta_2^2$. After simplification we get the desired result. □

Corollary 1. *If G and H are r_1 and r_2-regular graphs (i.e. $\Delta_1=\delta_1=d(u/G)=r_1$ and $\Delta_2=\delta_2=d(v/H)=r_2$) with the orders n_1, n_2 and the sizes m_1, m_2 respectively, then*
$M_1^{\varepsilon}(G*H)=2m_1\big(2r_1+n_2\big)\big(6r_1+3n_2-2\big)+2m_2n_1\big(2r_2^2+r_2+1\big)+n_1n_2\big(2r_1+r_2+n_2\big)\big(2r_1+$ $r_2+n_2-1\big)+2n_1n_2+8m_1.$

3.2. The R-Edge Corona product(RECP).

The RECP of two graphs is a one kind of graph operation based on R-graphs. It was introduced by Lan et al. [9] in 2015.

Definition 3. [9] *The RECP of two vertex-disjoint graphs G and H, denoted by $G\star H$, is a new graph obtained from one copy of the semi-total point graph $R(G)$ and also connected $|I(G)|$ copies of graph H by joining the l^{th} vertex of $I(G)$ to every vertex in the l^{th} copy of H.*

The graph $G\star H$ has $(|V(G)|+|E(G)|+|E(G)||V(H)|)$ vertices and $(3|E(G)|+|E(G)||E(H)|+|E(G)||V(H)|)$ edges (see Fig. 2). Let $V(G)=\{u_1,u_2,...u_{n_1}\}$, $I(G)=\{u_{e_1},u_{e_2},\ldots,u_{e_{m_1}}\}$ and $V(H)=\{v_1,v_2,\ldots,v_{n_2}\}$. Also, for $l=1,2,\ldots,m_1$ let $V(H^l)=$

$\{v_1^l, v_2^l, \ldots, v_{n_2}^l\}$ *be the vertex set of the* l^{th} *copy of* H. *So,* $V(R(G)) = V(G) \cup I(G)$ *and* $V(G\star H) = V(G) \cup I(G) \cup \Big(V(H^1) \cup V(H^2) \cup \ldots \cup V(H^{m_1})\Big)$ *and* $E\Big(G\star H\Big) = E_1^\star \cup E_2^\star \cup E_3^\star \cup E_4^\star$, *where* $E_1^\star = \{u_iu_k \in E\Big(G\star H\Big) | u_i, u_k \in V(G); i \neq k$ *and* $i,k = 1,2,\ldots,n_1\}$, $E_2^\star = \{u_iu_{e_l} \in E\Big(G\star H\Big) | u_i \in V(G), u_{e_l} \in I(G)$ *for* $i = 1,2,\ldots,n_1$ *and* $l = 1,2,\ldots,m_1\}$, $E_3^\star = \{u_{e_l}v_{j^l} \in E\Big(G\star H\Big) | u_{e_l} \in I(G), v_{j^l} \in V(H^l)$ *for* $l = 1,2,\ldots,m_1$ *and* $j = 1,2,\ldots,n_2\}$ *and* $E_4^\star = \{v_{p^l}v_{q^l} \in E\Big(G\star H\Big) | v_{p^l}, v_{q^l} \in V(H^l)$ *for* $l = 1,2,\ldots,m_1$
and $p,q = 1,..,n_2\}$. *The degrees of the vertices of* $G\star H$ *are given by*

$$\left.\begin{aligned} d(u_i/G\star H) &= 2d(u_i/G) \text{ for } i = 1,2,\ldots,n_1,\\ d(u_{e_l}/G\star H) &= 2 + n_2 \text{ for } l = 1,2,\ldots,m_1,\\ d(v_j^l/G\star H) &= d(v_j/H) + 1 \text{ for } l = 1,2,\ldots,m_1 \text{ and } j = 1,2,\ldots,n_2. \end{aligned}\right\} \tag{3}$$

In the following Theorem, we determine the FEZI for RECP of two graphs.

Theorem 3. *The FEZI of* $G\star H$ *is given by*
$$M_1^\varepsilon\Big(G\star H\Big) = 4EM_1(G) + 4F(G) + m_1F(H) + 4(n_2+3)M_1(G) + 2m_1M_1(H) + 2m_1M_2(H)$$
$$+ m_1n_2(n_2+1)^2 + 3m_1n_2^2 + 4m_1m_2(n_2+1) + 5m_1n_2 + 4m_1m_2 - 8m_1.$$

Proof. To calculate the FEZI of $G\star H$, we follow the Definition 1 and the Equations 1 and 3. Let us consider T_1 = The contribution of $M_1(G\star H)$ and $EM_1(G\star H)$ in $E_1^\star$
$$= \sum_{u_iu_k \in E_1^\star}\Big(d(u_i/G\star H) + d(u_k/G\star H)\Big) + \sum_{u_iu_k \in E_1^\star}\Big(d(u_i/G\star H) + d(u_k/G\star H) - 2\Big)^2$$
$$= \sum_{u_iu_k \in E(G)}\Big(2d(u_i/G) + 2d(u_k/G)\Big) + \sum_{u_iu_k \in E(G)}\Big\{2\Big(d(u_i/G) + d(u_k/G) - 2\Big) + 2\Big\}^2$$
$$= 2M_1(G) + 4EM_1(G) + 8M_1(G) - 12m_1.$$

Next, let T_2 = The contribution of $M_1(G\star H)$ and $EM_1(G\star H)$ in $E_2^\star$
$$= \sum_{u_iu_{e_l} \in E_2^\star}\Big(d(u_i/G\star H) + d(u_{e_l}/G\star H)\Big) + \sum_{u_iu_{e_l} \in E_2^\star}\Big(d(u_i/G\star H) + d(u_{e_l}/G\star H) - 2\Big)^2$$
$$= \sum_{u_i \in V(G), u_{e_l} \in I(G)}\Big(2d(u_i/G) + n_2 + 2\Big) + \sum_{u_i \in V(G), u_{e_l} \in I(G)}\Big(2d(u_i/G) + n_2\Big)^2$$
$$= \sum_{u_i \in V(G)}\Big(2d(u_i/G) + n_2 + 2\Big)d(u_i/G) + \sum_{u_i \in V(G)}\Big(2d(u_i/G) + n_2\Big)^2 d(u_i/G)$$
$$= 2M_1(G) + 2m_1(n_2+2) + 4F(G) + 4n_2M_1(G) + 2m_1n_2^2.$$

Similarly, let us consider T_3 = The contribution of $M_1(G\star H)$ and $EM_1(G\star H)$ in $E_3^\star$
$$= \sum_{u_{e_l}v_j^l \in E_3^\star}\Big(d(u_{e_l}/G\star H) + d(v_j^l/G\star H)\Big) + \sum_{u_{e_l}v_j^l \in E_3^\star}\Big(d(u_{e_l}/G\star H) + d(v_j^l/G\star H) - 2\Big)^2$$

$$= \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \Big(d(v_j/H) + n_2 + 3\Big) + \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \Big(d(v_j/H) + (n_2+1)\Big)^2$$
$$= 2m_1m_2 + m_1n_2{}^2 + 3m_1n_2 + m_1M_1(H) + 4m_1m_2(n_2+1) + m_1n_2(n_2+1)^2.$$

Finally, let T_4 = The contribution of $M_1(G \star H)$ and $EM_1(G \star H)$ in $E_4^\star$

$$= \sum_{v_p^l v_q^l \in E_4^\star} \Big(d(v_p^l/G\star H) + d(v_q^l/G\star H)\Big) + \sum_{v_p^l v_q^l \in E_4^\star} \Big(d(v_p^l/G\star H) + d(v_q^l/G\star H) - 2\Big)^2$$
$$= \sum_{l=1}^{m_1} \sum_{v_pv_q \in E(H)} (d(v_p/H) + d(v_q/H) + 2) + \sum_{l=1}^{m_1} \sum_{v_pv_q \in E(H)} \Big(d(v_p/H) + d(v_q/H)\Big)^2$$
$$= m_1M_1(H) + 2m_1m_2 + m_1F(H) + 2m_1M_2(H).$$

We get the desired result by summing the above four expressions. □

Applying Theorem 3, we have the following results.

Example 2. *(i)* $M_1^\varepsilon(P_n \star P_m) = (m^3n + 9m^2n + 50mn - m^3 - 9m^2 - 58m + 38n - 110)$

(ii) $M_1^\varepsilon(C_n \star C_m) = n\big(m^3 + 9m^2 + 54m + 88\big)$

(iii) $M_1^\varepsilon(C_n \star P_m) = n\big(m^3 + 9m^2 + 50m + 38\big)$.

In the following Theorem, we compute the bounds on the FEZI for RECP of two graphs.

Theorem 4. *The bounds for the FEZI of $G\star H$ are computed as $C \geq M_1^\varepsilon(G\star H) \geq D$, where $C = 16m_1\Delta_1^2 + 2m_1(2\Delta_1+n_2)(2\Delta_1+n_2+1) + m_1n_2(\Delta_2+n_2)(\Delta_2+n_2+3) + 2m_1m_2(2\Delta_2^2+\Delta_2+1) - 12m_1\Delta_1 + 8m_1 + 4m_1n_2$ and $D = 16m_1\delta_1^2 + 2m_1(2\delta_1+n_2)(2\delta_1+n_2+1) + m_1n_2(\delta_2+n_2)(\delta_2+n_2+3) + 2m_1m_2(2\delta_2^2+\delta_2+1) - 12m_1\delta_1 + 8m_1 + 4m_1n_2$. The equality holds if and only if G and H are regular graphs.*

Proof. Using the Definition 1 and the Equations 1 and 3, we have $M_1^\varepsilon\Big(G\star H\Big)$

$$= \sum_{u_iu_k \in E(G)} \Big(2d(u_i/G) + 2d(u_k/G)\Big) + \sum_{u_iu_k \in E(G)} \Big\{2\Big(d(u_i/G) + d(u_k/G) - 2\Big) + 2\Big\}^2$$
$$+ \sum_{u_i \in V(G), u_{e_l} \in I(G)} \Big(2d(u_i/G) + n_2 + 2\Big) + \sum_{u_i \in V(G), u_{e_l} \in I(G)} \Big(2d(u_i/G) + n_2\Big)^2$$
$$+ \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \Big(d(v_j/H) + n_2 + 3\Big) + \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \Big(d(v_j/H) + (n_2+1)\Big)^2$$
$$+ \sum_{l=1}^{m_1} \sum_{v_pv_q \in E(H)} (d(v_p/H) + d(v_q/H) + 2) + \sum_{l=1}^{m_1} \sum_{v_pv_q \in E(H)} \Big(d(v_p/H) + d(v_q/H)\Big)^2$$
$$\leq \sum_{u_iu_k \in E(G)} \Big(4\Delta_1\Big) + \sum_{u_iu_k \in E(G)} \{2(\Delta_1 - 1)\}^2$$
$$+ \sum_{u_i \in V(G), u_{e_l} \in I(G)} \Big(2\Delta_1 + n_2 + 2\Big) + \sum_{u_i \in V(G), u_{e_l} \in I(G)} \Big(2\Delta_1 + n_2\Big)^2$$

$$+\sum_{u_{e_l}\in I(G)}\sum_{v_j\in V(H)}(\Delta_2+n_2+3)+\sum_{u_{e_l}\in I(G)}\sum_{v_j\in V(H)}\Big(\Delta_2+(n_2+1)\Big)^2$$

$$+\sum_{u_{e_l}\in I(G)}\sum_{v_pv_q\in E(H)}(2\Delta_2+2)+\sum_{u_{e_l}\in I(G)}\sum_{v_pv_q\in E(H)}\Big(2\Delta_2\Big)^2$$

$$=4m_1(4\Delta_1^2-3\Delta_1+1)+2m_1(2\Delta_1+n_2+2)+2m_1(2\Delta_1+n_2)^2$$

$$+m_1n_2(\Delta_2+n_2+3)+m_1n_2(\Delta_2+n_2+1)^2+2m_1m_2(\Delta_2+1)+4m_1m_2\Delta_2^2$$

$$=16m_1\Delta_1^2+2m_1(2\Delta_1+n_2)(2\Delta_1+n_2+1)+m_1n_2(\Delta_2+n_2)(\Delta_2+n_2+3)+$$

$$2m_1m_2(2\Delta_2^2+\Delta_2+1)-12m_1\Delta_1+8m_1+4m_1n_2=\text{C (say)}.$$

Analogously, using the equations 1 and 3, one can calculate the following $M_1^{\varepsilon}(G\star H)\geq D$. The equality holds if and only if G and H are regular graphs. □

3.3. The R-Vertex Neighborhood Corona product(RVNCP).

The RVNCP introduced by Lan et al. [9] is a one type of Corona product of two graphs based on R-graphs.

Definition 4. [9] *The RVNCP of G and H, denoted by $G\oplus H$, is a novel graph made of one copy of $R(G)$ graph and connects n_1 copies of H, all vertex-disjoint, and joining the neighbors of the i^{th} vertex of G in $R(G)$ to every vertex in the i^{th} copy of H.*

Let G and H be two simple connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively and the vertex sets $V(G)=\{u_1,u_2,\dots,u_{n_1}\}$, $I(G)=\{u_{e_1},u_{e_2},\dots,u_{e_{m_1}}\}$ and $V(H)=\{v_1,v_2,\dots,v_{n_2}\}$. Also, let $V(H^i)=\{v_1^i,v_2^i,\dots,v_{n_2}^i\}$, $i=1,2,\dots,n_1$ be the vertex set of the i^{th} copy of H. Then $V(G\oplus H)=V(G)\cup I(G)\cup\Big(V(H^1)\cup V(H^2)\cup\ldots\cup V(H^{n_1})\Big)$ and $E\Big(G\oplus H\Big)=E_1^{\oplus}\cup E_2^{\oplus}\cup E_3^{\oplus}\cup E_4^{\oplus}\cup E_5^{\oplus}$, where $E_1^{\oplus}=\{u_iu_k\in E\Big(G\oplus H\Big)|u_i,u_k\in V(G)\}$, $E_2^{\oplus}=\{u_iu_{e_k}\in E\Big(G\oplus H\Big)|u_i\in V(G),u_{e_k}\in I(G)\}$, $E_3^{\oplus}=\{v_p^iv_q^i\in E\Big(G\oplus H\Big)|v_p^i,v_q^i\in V(H^i)\}$, $E_4^{\oplus}=\{u_kv_j^i\in E\Big(G\oplus H\Big)|u_k\in V(G),v_j^i\in V(H^i)\}$ and $E_5^{\oplus}=\{u_{e_k}v_j^i\in E\Big(G\oplus H\Big)|u_{e_k}\in I(G),v_j^i\in V(H^i)\}$. The graph $G\oplus H$ has $(|V(G)|+|E(G)|+|V(G)||V(H)|)$ vertices and $(3|E(G)|+|V(G)||E(H)|+4|E(G)||V(H)|)$ edges, (see Fig. 2). The degrees of the vertices of $G\oplus H$ are given by

$$\left.\begin{aligned}d(u_i/G\oplus H)&=(n_2+2)d(u_i/G)\,\textit{for } i=1,2,\dots,n_1,\\ d(u_{e_k}/G\oplus H)&=2(1+n_2)\,\textit{for } k=1,2,\dots,m_1,\\ d(v_j^i/G\oplus H)&=d(v_j/H)+2d(u_i/G)\,\textit{for } i=1,2,\dots,n_1 \textit{ and } j=1,2,\dots,n_2.\end{aligned}\right\}\tag{4}$$

In the following Theorem, we obtain the FEZI for RVNCP of two graphs.

Theorem 5. *The FEZI of $G \oplus H$ is given by*

$M_1^{\varepsilon}(G \oplus H) = (n_2^3 + 6n_2^2 + 20n_2 + 8)F(G) + n_1F(H) + \left(9n_2^2 - 4n_2 + 40m_2 + 4m_2n_2 - 4\right)M_1(G) + (20m_1 - 3n_1)M_1(H) + 2(n_2+2)(5n_2+2)M_2(G) + 2n_1M_2(H) + 4m_1n_2^2(2n_2+3) + 16m_1n_2(m_2 + 1) + 4m_2n_1 - 32m_1m_2 + 8m_1.$

Proof. Applying the Proposition 1 and the Equation 4, we have

$M_1^{\varepsilon}(G \oplus H) = M_1(G \oplus H) + EM_1(G \oplus H)$
$= \sum\limits_{u_i \in V(G \oplus H)} d^2(u_i/G \oplus H) + \Big\{ \sum\limits_{u_iu_k \in E_1^{\oplus}} \Big(d(u_i/G \oplus H)) + d(u_k/G \oplus H) - 2\Big)^2 + \sum\limits_{u_iu_{e_k} \in E_2^{\oplus}} \Big(d(u_i/G \oplus H)) + d(u_{e_k}/G \oplus H) - 2\Big)^2 \Big\} + \sum\limits_{v_p^iv_q^i \in E_3^{\oplus}} \Big(d(v_p^i/G \oplus H) + d(v_q^i/G \oplus H) - 2\Big)^2 + \Big\{ \sum\limits_{u_kv_j^i \in E_4^{\oplus}} \Big(d(u_k/G \oplus H) + d(v_j^i/G \oplus H) - 2\Big)^2 + \sum\limits_{u_{e_k}v_j^i \in E_5^{\oplus}} \Big(d(u_{e_k}/G \oplus H) + d(v_j^i/G \oplus H) - 2\Big)^2 \Big\} = C_1 + C_2 + C_3 + C_4$ (say), where C_1, C_2, C_3 and C_4 denote the sums of the above terms in order.

Next, we compute the above terms separately.

Firstly, $C_1 = \sum\limits_{u \in V(G \oplus H)} d^2(u/G \oplus H)$
$= \sum\limits_{u_i \in V(G)} \Big((n_2+2)d(u_i/G)\Big)^2 + \sum\limits_{u_{e_k} \in I(G)} \Big(2(n_2+1)\Big)^2 + \sum\limits_{u_i \in V(G)} \sum\limits_{v_j \in V(H)} \Big(2d(u_i/G) + d(v_j/H)\Big)^2$
$= (n_2+2)^2M_1(G) + 4(n_2+1)^2m_1 + 4n_2M_1(G) + n_1M_1(H) + 16m_1m_2.$

$C_2 = \sum\limits_{u_iu_k \in E_1^{\oplus}} \Big(d(u_i/G \oplus H)) + d(u_k/G \oplus H) - 2\Big)^2 + \sum\limits_{u_iu_{e_k} \in E_2^{\oplus}} \Big(d(u_i/G \oplus H)) + d(u_{e_k}/G \oplus H) - 2\Big)^2$
$= \sum\limits_{u_iu_k \in E(G)} \Big\{(n_2+2)\Big(d(u_i/G) + d(u_k/G)\Big) - 2\Big\}^2 + \sum\limits_{\substack{uv \in E(G \oplus H) \\ u = u_i \in V(G), v = u_{e_k} \in I(G)}} \Big\{(n_2+2)d(u_i/G) + 2n_2\Big\}^2$
$= \sum\limits_{u_iu_k \in E(G)} \Big\{(n_2+2)^2\Big(d^2(u_i/G) + d^2(u_k/G)\Big) + 2(n_2+2)^2d(u_i/G)d(u_k/G) - 4(n_2+2)\Big(d(u_i/G) + d(u_k/G)\Big) + 4\Big\} + \sum\limits_{u_i \in V(G)} \Big((n_2+2)^2d^2(u_i/G) + 4n_2(n_2+2)d(u_i/G) + 4n_2^2\Big)d(u_i/G)$
$= (n_2+2)^2F(G) + 2(n_2+2)^2M_2(G) - 4(n_2+2)M_1(G) + 4m_1 + (n_2+2)^2F(G) + 4n_2(n_2+2)M_1(G) + 8m_1n_2^2.$

Also, $C_3 = \sum_{v_p^i v_q^i \in E_3^{\oplus}} \Big(d(v_p^i/G\oplus H)+d(v_q^i/G\oplus H)-2\Big)^2$

$= \sum_{i=1}^{n_1} \sum_{v_p v_q \in E(H)} \Big(\big(d(v_p/H)+d(v_q/H)-2\big)+4d(u_i/G)\Big)^2$

$= n_1 EM_1(H) + 16m_2M_1(G) + 16m_1M_1(H) - 32m_1m_2 = n_1\{F(H) - 4M_1(H) + 2M_2(H) + 4m_2\} + 16m_2M_1(G) + 16m_1M_1(H) - 32m_1m_2.$

$C_4 = \sum_{u_k v_j^i \in E_4^{\oplus}} \Big(d(u_k/G\oplus H)+d(v_j^i/G\oplus H)-2\Big)^2 + \sum_{u_{e_k} v_j^i \in E_5^{\oplus}} \Big(d(u_{e_k}/G\oplus H)+d(v_j^i/G\oplus H)-2\Big)^2$

$= \sum_{\substack{u_i \in V(G) \\ w_i \in N_G(u_i), w_i \in V(G)}} \sum_{v_J \in V(H)} \Big\{(n_2+2)d(u_i/G)+2d(w_i/G)+d(v_j/H)-2\Big\}^2$

$+ \sum_{\substack{u_i \in V(G) \\ w_i \in N_G(u_i), w_i \in I(G)}} \sum_{v_j \in V(H)} \Big\{2d(u_i/G)+d(v_j/H)+2n_2\Big\}^2$

$= \sum_{u_i \in V(G)} \sum_{w_i \in N_G(u_i)} \sum_{v_j \in V(H)} \Big\{(n_2+2)^2d^2(u_i/G) + 4d^2(w_i/G) + d^2(v_j/H) + 4 + 4(n_2+2)d(u_i/G)d(w_i/G) + 2(n_2+2)d(u_i/G)d(v_j/H) - 4(n_2+2)d(u_i/G) + 4d(w_i/G)d(v_j/H) - 8d(w_i/G) - 4d(v_j/H)\Big\}$

$+ \sum_{v_j \in V(H)} \sum_{u_i \in V(G)} \Big(4d^2(u_i/G) + d^2(v_j/H) + 4n_2^2 + 4d(u_i/G)d(v_j/H) + 8n_2d(u_i/G) + 4n_2d(v_j/H)\Big)d(u_i/G)$

$= n_2(n_2+2)^2F(G) + 4n_2F(G) + 2m_1M_1(H) + 8m_1n_2 + 8n_2(n_2+2)M_2(G) + 4m_2(n_2+2)M_1(G) - 4n_2(n_2+2)M_1(G) + 8m_2M_1(G) - 8n_2M_1(G) - 16m_1m_2 + 4n_2F(G) + 2m_1M_1(H) + 8m_1n_2^3 + 8m_2M_1(G) + 8n_2^2M_1(G) + 16m_1m_2n_2.$

Adding C_1, C_2, C_3 and C_4 and taking simple calculation, we get the desired result. □

The following results are direct consequence of the Theorem 5.

Example 3. (i) $M_1^{\varepsilon}(P_n \oplus P_m) = (16m^3n + 168m^2n + 440mn - 22m^3 - 270m^2 - 716m - 272n + 168)$

(ii) $M_1^{\varepsilon}(C_n \oplus C_m) = n\big(16m^3 + 168m^2 + 472m + 88\big)$

(iii) $M_1^{\varepsilon}(C_n \oplus P_m) = n\big(16m^3 + 168m^2 + 440m - 176\big).$

Theorem 6. *The bounds for the FEZI of $(G\oplus H)$ are given by*

$K \geq M_1^{\varepsilon}(G\oplus H) \geq L,$ *where* $K = n_1n_2\big(2\Delta_1+\Delta_2\big)^2 + 4m_2n_1\big(2\Delta_1+\Delta_2-1\big)^2 + \big(n_1+6m_1\big)\big(n_2+2\big)^2\Delta_1^2 + 8m_1(n_2-1)(n_2+2)\Delta_1 + 4m_1n_2(3n_2+2) + 2m_1n_2\Big((n_2+4)\Delta_1+\Delta_2-2)^2 +$

$(2\Delta_1+\Delta_2+2n_2)^2\Big)+8m_1$, *and* $L=n_1n_2\big(2\delta_1+\delta_2\big)^2+4m_2n_1\big(2\delta_1+\delta_2-1\big)^2+\big(n_1+6m_1\big)\big(n_2+2\big)^2\delta_1^2+8m_1(n_2-1)(n_2+2)\delta_1+4m_1n_2(3n_2+2)+2m_1n_2\Big((n_2+4)\delta_1+\delta_2-2)^2+(2\delta_1+\delta_2+2n_2)^2\Big)+8m_1$.

The equality holds if and only if G and H are regular graphs.

Proof. The proof is similar to the Theorem 2. □

3.4. The R-Edge Neighborhood Corona product(RENCP).

Lan et al.[9] defined four new graph operations based on R-graphs. The RENCP is a one of the four new graph operations.

Definition 5. [9] *The RENCP of G and H, denoted by $G\otimes H$, is a new graph which is achieved from one copy of $R(G)$ graph and also it adjoins $|I(G)|$ copies of H and joining the neighbors of the i^{th} vertex of $I(G)$ in $R(G)$ to every vertex in the i^{th} copy of H.*

Let $V(G)=\{u_1,u_2,\dots,u_{n_1}\}$, $I(G)=\{u_{e_1},u_{e_2},\dots,u_{e_{m_1}}\}$ *and* $V(H)=\{v_1,v_2,\dots,v_{n_2}\}$. *For* $i=1,2,\dots,m_1$, *let* $V(H^i)=\{v_1^i,\dots,v_{n_2}^i\}$ *be the vertex set of the i^{th} copy of H. So,* $V(G)\cup I(G)\cup\Big(\cup_{i=1}^{m_1}V(H^i)\Big)$ *is the partition of* $V(R(G)\otimes H)$ *and* $E\Big(G\otimes H\Big)=E_1^{\otimes}\cup E_2^{\otimes}\cup E_3^{\otimes}\cup E_4^{\otimes}$, *where* $E_1^{\otimes}=\{u_iu_k\in E\Big(G\otimes H\Big)|u_i,u_k\in V(G)\}$, $E_2^{\otimes}=\{u_iu_{e_k}\in E\Big(G\otimes H\Big)|u_i\in V(G),u_{e_k}\in I(G)\}$, $E_3^{\otimes}=\{v_p^iv_q^i\in E\Big(G\otimes H\Big)|v_p^i,v_q^i\in V(H^i)\}$ *and* $E_4^{\otimes}=\{u_kv_j^i\in E\Big(G\otimes H\Big)|u_k\in V(G),v_j^i\in V(H^i)\}$. *Thus the graph $G\otimes H$ has* $(|V(G)|+|E(G)|+|E(G)||V(H)|)$ *vertices and* $(3|E(G)|+|E(G)||E(H)|+2|E(G)||V(H)|)$ *edges, respectively (see Fig. 2).*

From definition, the degrees of the vertices of $G\otimes H$ are follows as

$$\left.\begin{aligned} d(u_i/G\otimes H)&=(n_2+2)d(u_i/G)\text{ for }i=1,2,\dots,n_1,\\ d(u_{e_k}/G\otimes H)&=2\text{ for }k=1,2,\dots,m_1,\\ d(v_j^i/G\otimes H)&=d(v_j/H)+2\text{ for }i=1,2,\dots,m_1\text{ and }j=1,2,\dots,n_2.\end{aligned}\right\}\tag{5}$$

Here we calculate the FEZI for RENCP of two graphs.

Theorem 7. *The FEZI for $G\otimes H$ is given by*

$$M_1^{\varepsilon}(G\otimes H)=(n_2+2)^3F(G)+m_1F(H)+(n_2+2)(n_2+4m_2-2)M_1(G)$$
$$+7m_1M_1(H)+2(n_2+2)^2M_2(G)+2m_1M_2(H)+12m_1m_2+4m_1n_2+8m_1.$$

Proof. With the help of the Definition 1 as well as the Proposition 1 and also the Equation 5, we have

$$M_1^{\varepsilon}(G\otimes H)$$
$$= \sum_{u_iu_k\in E_1^{\otimes}} \Big(d(u_i/G\otimes H)+d(u_k/G\otimes H)\Big) + \sum_{u_iu_{e_k}\in E_2^{\otimes}} \Big(d(u_i/G\otimes H)+d(u_{e_k}/G\otimes H)\Big) +$$
$$\sum_{v_p^iv_q^i\in E_3^{\otimes}} \Big(d(v_p^i/G\otimes H)+d(v_q^i/G\otimes H)\Big) + \sum_{u_kv_j^i\in E_4^{\otimes}} \Big(d(u_k/G\otimes H)+d(v_j^i/G\otimes H)\Big) +$$
$$\sum_{u_iu_k\in E_1^{\otimes}} \Big(d(u_i/G\otimes H)+d(u_k/G\otimes H)-2\Big)^2 + \sum_{u_iu_{e_k}\in E_2^{\otimes}} \Big(d(u_i/G\otimes H)+d(u_{e_k}/G\otimes H)-2\Big)^2 +$$
$$\sum_{v_p^iv_q^i\in E_3^{\otimes}} \Big(d(v_p^i/G\otimes H)+d(v_q^i/G\otimes H)-2\Big)^2 + \sum_{u_kv_j^i\in E_4^{\otimes}} \Big(d(u_k/G\otimes H)+d(v_j^i/G\otimes H)-2\Big)^2$$
$$= \sum_{u_iu_k\in E(G)} \Big((n_2+2)(d(u_i/G)+d(u_k/G))\Big) + \sum_{u_i\in V(G)} \Big((n_2+2)d(u_i/G)+2\Big)d(u_i/G) +$$
$$\sum_{i=1}^{m_1} \sum_{v_pv_q\in E(H)} \Big(d(v_p/H) + d(v_q/H) + 4\Big) + \sum_{u_i\in V(G)} \sum_{v_j\in V(H)} \Big((n_2 + 2)d(u_i/G) + d(v_j/H) +$$
$$2\Big)d(u_i/G) +$$
$$\sum_{u_iu_k\in E(G)} \{(n_2+2)\big(d(u_i/G)+d(u_k/G)\big)-2\}^2 + \sum_{u_i\in V(G)} (n_2+2)^2 d^3(u_i/G)$$
$$+ \sum_{v_j\in V(H)} \sum_{u_i\in V(G)} \Big((n_2+2)d(u_i/G)+d(v_j/H)\Big)^2 d(u_i/G) + \sum_{i=1}^{m_1} \sum_{v_pv_q\in E(H)} \Big(d(v_p/H)+d(v_q/H)+$$
$$2\Big)^2$$
$$= m_1M_1(H)+8m_1m_2+4m_1n_2+4m_1+(n_2+2)^2M_1(G)+(n_2+2)^2F(G)+2(n_2+2)^2M_2(G)+$$
$$4m_1-4(n_2+2)M_1(G)+m_1F(H)+2m_1M_2(H)+4m_1m_2+4m_1M_1(H)+(n_2+2)^2F(G)+$$
$$n_2(n_2+2)^2F(G)+2m_1M_1(H)+4m_2(n_2+2)M_1(G).$$

After simple calculation, we get the desired result. □

From the Theorem 4, we have the following results.

Example 4. *(i)* $M_1^{\varepsilon}(P_n\otimes P_m) = 2\big(4m^3n+38m^2n+102mn-7m^3-65m^2-158m-14n-14\big)$

(ii) $M_1^{\varepsilon}(C_n\otimes C_m) = 4n\big(2m^3+19m^2+55m+22\big)$

(iii) $M_1^{\varepsilon}(C_n\otimes P_m) = 4n\big(2m^3+19m^2+51m-7\big)$.

Theorem 8. *The bounds for the FEZI of $G\otimes H$ are given by*

$U \geq M_1^{\varepsilon}(G\otimes H) \geq V$, where $U = 2m_1\Delta_1(n_2+2)\Big(3(n_2+2)\Delta_1-2\Big)+2m_1m_2(2\Delta_2^2+5\Delta_2+4)+2m_1n_2\Big(\big((n_2+2)\Delta_1+\Delta_2\big)^2+(n_2+2)\Delta_1+\Delta_2+2\Big)+8m_1$ and $V = 2m_1\delta_1(n_2+2)\Big(3(n_2+2)\delta_1-2\Big)+2m_1m_2(2\delta_2^2+5\delta_2+4)+2m_1n_2\Big(\big((n_2+2)\delta_1+\delta_2\big)^2+(n_2+2)\delta_1+\delta_2+2\Big)+8m_1$.

The equality holds if and only if G and H are regular graphs.

Proof. The proof is similar to the Theorem 4. □

3.5. The Subdivision-Vertex Corona product(SVCP). The SVCP of two graphs was introduced by Lu et al. [11].

Definition 6. [11] *The SVCP of G and H, denoted by $G\odot H$, is a new graph. It is obtained from $S(G)$ and n_1 copies of H, all vertex-disjoint, by joining the i^{th} vertex of $V(G)$ to every vertex in the i^{th} copy of H.*

From definition it is clear that the $G\odot H$ has $(m_1+n_1+n_1n_2)$ vertices and $(2m_1+n_1n_2+n_1m_2)$ edges, (see Fig.3). Also, let $V(G)=\{u_1,u_2,\dots,u_{m_1}\}$, $I(G)=V(S(G))\setminus V(G)=\{u_{e_1},u_{e_2},\dots,u_{e_{m_1}}\}$ and $V(H)=\{v_1,v_2,\dots,v_{n_2}\}$, so that $V(S(G))=V(G)\cup I(G)$. Let $V(H^i)=\{v_1^i,v_2^i,\dots,v_{n_2}^i\}$ be the vertex set of the i^{th} copy of H, $i=1,2,\dots,n_1$, so that $V(G\odot H)=V(G)\cup I(G)\cup\Big(\cup_{i=1}^{n_1}V(H^i)\Big)$ and $E\Big(G\odot H\Big)=E_1^{\odot}\cup E_2^{\odot}\cup E_3^{\odot}$ where $E_1^{\odot}=\{u_iv_j^i\in E\Big(G\odot H\Big)|u_i\in V(G),v_j^i\in V(H^i)$ for $i=1,2,\dots,n_1$ and $j=1,2,\dots,n_2\}$,
$E_2^{\odot}=\{u_iu_{e_k}\in E\Big(G\odot H\Big)|u_i\in V(G),u_{e_k}\in I(G)$ for $i=1,2,\dots,n_1$ and $k=1,2,\dots,m_1\}$ and $E_3^{\odot}=\{v_l^iv_m^i\in E\Big(G\odot H\Big)|v_l^i,v_m^i\in V(H^i)$ for $i=1,2,\dots,n_1$ and $l,m=1,2,\dots,n_2\}$. The degrees of the vertices of $G\odot H$ are:

$$\left.\begin{aligned} d(u_i/G\odot H)&=d(u_i/G)+n_2 \text{ for } i=1,2,\dots,n_1,\\ d(u_{e_k}/G\odot H)&=2 \text{ for } k=1,2,\dots,m_1,\\ d(v_j^i/G\odot H)&=d(v_j/H)+1 \text{ for } i=1,2,\dots,n_1 \text{ and } j=1,2,\dots,n_2.\end{aligned}\right\}\tag{6}$$

In the following theorem, the FEZI for SVCP of two graphs is computed.

Theorem 9. *The FEZI of $G\odot H$ is given by*

$M_1^{\varepsilon}(G\odot H)=F(G)+n_1F(H)+(3n_2+1)M_1(G)+2n_1M_1(H)+2n_1M_2(H)+4n_2(m_1n_2+m_2n_1)+n_1n_2(n_2^2-n_2+2)+2m_1n_2^2+8m_1m_2+4m_1.$

Proof. Using the Definition 1 and the Equation 6, the FEZI for $G\odot H$ is

$M_1^{\varepsilon}(G\odot H)=M_1(G\odot H)+EM_1(G\odot H)=S_1+S_2$ (say), respectively.

Now, $S_1=M_1(G\odot H)$

$$=\sum_{u_iv_j^i\in E_1^{\odot}}\Big(d(u_i/(G\odot H))+d(v_j^i/(G\odot H))\Big)+\sum_{u_iu_{e_k}\in E_2^{\odot}}\Big(d(u_i/(G\odot H))+d(u_{e_k}/(G\odot H))\Big)$$
$$+\sum_{v_l^iv_m^i\in E_3^{\odot}}\Big(d(v_l^i/(G\odot H))+d(v_m^i/(G\odot H))\Big)$$
$$=\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big(d(u_i/G)+n_2+d(v_j/H)+1\Big)+\sum_{u_i\in V(G)}\Big(d(u_i/G)+n_2+2)\Big)d(u_i/G)$$

$$+\sum_{i=1}^{n_1}\sum_{v_lv_m\in E(H)}\Big(d(u_l/H)+d(v_m/H)+2\Big)$$
$$=2(m_1n_2+m_2n_1)+n_1n_2(n_2+1)+M_1(G)+2m_1(n_2+2)+n_1M_1(H)+2n_1m_2.$$

Finally, $S_2=EM_1(G\odot H)$

$$=\sum_{u_iv_j^i\in E_1^{\odot}}\Big(d(u_i/(G\odot H))+d(v_j^i/(G\odot H))-2\Big)^2+\sum_{u_iu_{e_k}\in E_2^{\odot}}\Big(d(u_i/(G\odot H))+d(u_{e_k}/(G\odot H))-2\Big)^2+\sum_{v_l^iv_m^i\in E_3^{\odot}}\Big(d(v_l^i/(G\odot H))+d(v_m^i/(G\odot H))-2\Big)^2$$
$$=\sum_{u_i\in V(G)}\sum_{v_j\in V(H)}\Big(d(u_i/G)+d(v_j/H)+(n_2-1)\Big)^2+\sum_{u_i\in V(G)}\Big(d(u_i/G)+n_2\Big)^2d(u_i/G)$$
$$+\sum_{i=1}^{n_1}\sum_{v_lv_m\in E(H)}\Big(d(u_l/H)+d(v_m/H)\Big)^2$$
$$=n_2M_1(G)+n_1M_1(H)+n_1n_2(n_2-1)^2+8m_1m_2+4(n_2-1)(m_1n_2+m_2n_1)+F(G)+2n_2M_1(G)+2m_1{n_2}^2+n_1F(H)+2n_1M_2(H).$$

By adding S_1 and S_2 with simple calculation, we have the desired result. □

Using Theorem 9, we calculate the FEZI of several chemically interesting molecular graphs

Example 5. (i) $M_1^{\varepsilon}(P_n\odot P_m)=(m^3n+9m^2n+42mn-6m^2-26m-34n-16)$.
(ii) $M_1^{\varepsilon}(C_n\odot C_m)=n\big(m^3+9m^2+46m+16\big)$.
(iii) $M_1^{\varepsilon}(C_n\odot P_m)=n\big(m^3+9m^2+42m-34\big)$.

Theorem 10. *The bounds of the FEZI for $G\odot H$ are determined as*

$W_1\geq M_1^{\varepsilon}(G\odot H)\geq W_2$, *where* $W_1=n_1n_2(\Delta_1+\Delta_2+n_2)(\Delta_1+\Delta_2+n_2-1)+2m_1(\Delta_1+n_2)(\Delta_1+n_2+1)+2m_2n_1(2\Delta_2^2+\Delta_2+1)+2n_1n_2+4m_1$ *and* $W_2=n_1n_2(\delta_1+\delta_2+n_2)(\delta_1+\delta_2+n_2-1)+2m_1(\delta_1+n_2)(\delta_1+n_2+1)+2m_2n_1(2\delta_2^2+\delta_2+1)+2n_1n_2+4m_1$.

The equality holds if and only if G and H are regular graphs.

Proof. The proof is analogous to that of Theorem 2. □

3.6. The Subdivision-Edge Corona product(SECP). The SECP of two graphs is a new type graph operation among different types of Corona product and it was introduced by Lu et al. [11].

Definition 7. [11] *The SECP for G and H, denoted by $G \ominus H$, is a novel graph based on the subdivision graph $S(G)$ and $|I(G)|$ copies of H and by connecting the i^{th} vertex of $I(G)$ to every vertex in the i^{th} copy of H.*

The graph $G \ominus H$ has $(|V(G)| + |E(G)||V(H)| + |E(G)|)$ vertices and $|E(G)|\big(|E(H)| + |V(H)| + 2\big)$ edges, (see Fig. 3). Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, $I(G) = V(S(G)) \setminus V(G) = \{u_{e_1}, u_{e_2}, \ldots, u_{e_{m_1}}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$, so that $V(G) = V(G) \cup I(G)$. Let $V(H^i) = \{v_1^i, v_2^i, \ldots, v_{n_2}^i\}$ be the vertex set of the i^{th} copy of H for $i = 1, 2, \ldots, n_2$, so that $V(G \ominus H) = V(G) \cup I(G) \cup \Big(\cup_{i=1}^{n_2} V(H^i)\Big)$ and $E\Big(G \ominus H\Big) = E(G) \cup_{i=1}^{m_1} E(H^i) \cup \{u_{e_i}v_j^i : u_{e_i} \in I(G), v_j^i \in V(H^i)$ for $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, n_2\}$. The degree distributions of the vertices of $G \ominus H$ are given by

$$\left.\begin{aligned} d(u_i/G \ominus H) &= d(u_i/G) \text{ for } i = 1, 2, \ldots, n_1, \\ d(u_{e_i}/G \ominus H) &= (2 + n_2) \text{ for } i = 1, 2, \ldots, m_1, \\ d(v_j^i/G \ominus H) &= d(v_j/H) + 1 \text{ for } i = 1, 2, \ldots, m_1 \text{ and } j = 1, 2, \ldots, n_2. \end{aligned}\right\} \tag{7}$$

Now we reckon the FEZI for SECP of two graphs.

Theorem 11. *The FEZI for $G \ominus H$ is given by*

$M_1^{\varepsilon}(G \ominus H) = F(G) + m_1F(H) + (2n_2 + 1)M_1(G) + 2m_1M_1(H) + 2m_1M_2(H) + m_1\Big(n_2^3 + 5n_2^2 + 4n_2m_2 + 6n_2 + 8m_2 + 4\Big)$.

Proof. From the Proposition 1, we get $M_1^{\varepsilon}(G \ominus H) = M_1(G \ominus H) + EM_1(G \ominus H) = R_1 + R_2$ (say), respectively.

Now, $R_1 = M_1(G \ominus H)$

$= \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \Big(d(u_j/H) + n_2 + 3\Big) + \sum_{i=1}^{m_1} \sum_{v_jv_k \in E(H)} \Big(d(v_j/H) + d(v_k/H) + 2\Big)$

$+ \sum_{u_i \in V(G)} \Big(d(u_i/G) + n_2 + 2\Big) d(u_i/G)$ (Using the Equation 7)

$= M_1(G) + m_1\Big(M_1(H) + 4m_2 + n_2\Big) + m_1(n_2 + 2)^2.$

Finally, $R_2 = EM_1(G \ominus H)$

$= \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} (d(v_j/H) + 1 + n_2)^2 + \sum_{i=1}^{m_1} \sum_{v_jv_k \in E(H)} \Big(d(v_j/H) + d(v_k/H)\Big)^2 + \sum_{u_i \in V(G)} (d(u_i/G) + n_2)^2 d(u_i/G)$

$= m_1\Big(\sum_{j=1}^{n_2} (d^2(v_j/H) + (n_2 + 1)^2 + 2(n_2 + 1)d(v_j/H)) + \sum_{v_jv_k \in E(H)} \{(d^2(v_j/H) + d^2(v_k/H)) +$

$$2d(v_j/H)d(v_k/H)\}\Big) + \sum_{u_i \in V(G)} (d^3(u_i/G) + n_2^2 d(u_i/G) + 2n_2 d^2(u_i/G)).$$
$$= m_1\Big(M_1(H) + n_2(n_2+1)^2 + 4m_2(n_2+1) + F(H) + 2M_2(H)\Big) + F(G) + 2m_1n_2^2 + 2n_2M_1(G).$$

Adding R_1 and R_2, we get the desired result. □

Using Theorem 11, we obtain the following results.

Example 6. *(i)* $M_1^{\varepsilon}(P_n \ominus P_m) = (m^3n + 13m^2n + 38mn - m^3 - 13m^2 - 42m - 30n + 22)$

(ii) $M_1^{\varepsilon}(C_n \ominus C_m) = n\big(m^3 + 9m^2 + 46m + 16\big)$

(iii) $M_1^{\varepsilon}(C_n \ominus P_m) = n\big(m^3 + 9m^2 + 42m - 42\big)$.

Theorem 12. *The upper and lower bound of the FEZI for $G \ominus H$ are given by*

$X_1 \geq M_1^{\varepsilon}(G \ominus H) \geq X_2$, *where* $X_1 = m_1n_2(\Delta_2 + n_2)(\Delta_2 + n_2 + 3) + 2m_1m_2(2\Delta_2^2 + \Delta_2 + 1) + 2m_1(\Delta_1 + n_2)(\Delta_1 + n_2 + 1) + 4m_1(n_2+1)$, *and* $X_2 = m_1n_2(\delta_2 + n_2)(\delta_2 + n_2 + 3) + 2m_1m_2(2\delta_2^2 + \delta_2 + 1) + 2m_1(\delta_1 + n_2)(\delta_1 + n_2 + 1) + 4m_1(n_2+1)$.

The equality holds if and only if G and H are regular graphs.

Proof. Similar proof to the Theorem 4. □

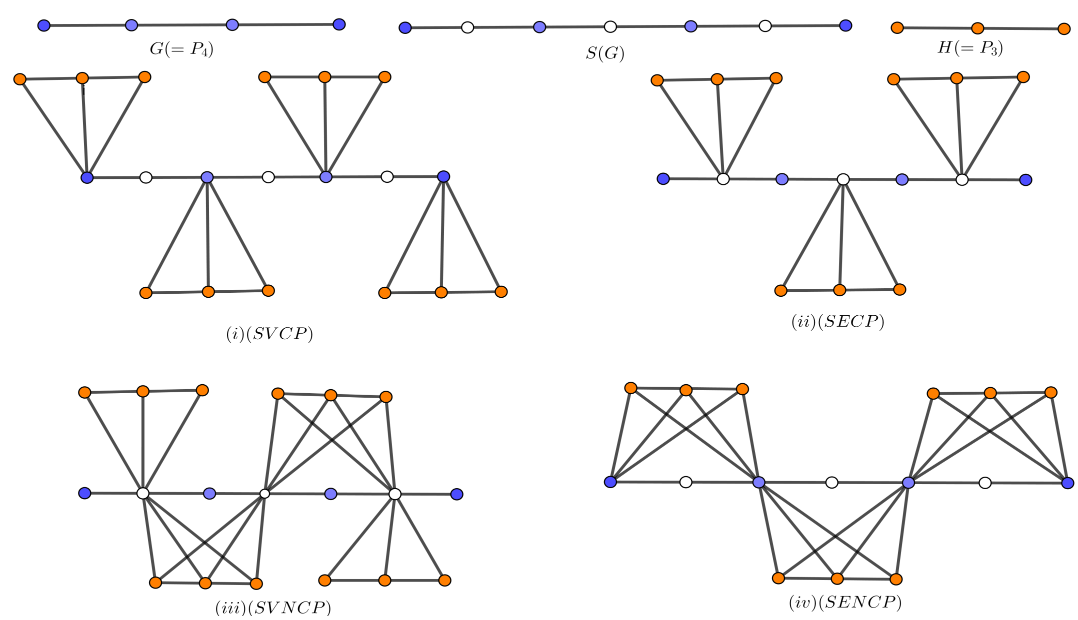

FIGURE 3. An example of various types of Corona products based on S-graphs of two graphs P_4 and P_3 like as (i) SVCP, (ii) SECP, (iii) SVNCP and (iv) SENCP.

3.7. The Subdivision-Vertex Neighborhood Corona product (SVNCP). Liu et al. [10] introduced the SVNCP of two vertex-disjoint graphs.

Definition 8. [10] *The SVNCP of G and H, denoted by $G \boxdot H$, is the graph obtained from $S(G)$ and n_1 copies of H, all vertex-disjoint, and joining the neighbors of the i^{th} vertex of $V(G)$ to every vertex in the i^{th} copy of H.*

The graph $G \boxdot H$ has $(n_1+m_1+n_1n_2)$ vertices and $(2m_1+n_1m_2+2m_1n_2)$ edges (see Fig. 3).

Let $V(G)=\{u_1,u_2,\dots,u_{n_1}\}, I(G)=\{u_{e_1},u_{e_2},\dots,u_{e_{m_1}}\}$ and $V(H)=\{v_1,v_2,\dots,v_{n_2}\}$. Also, let $V(H^i)=\{v_1^i,v_2^i,\dots,v_{n_2}^i\}$ be the vertex set of the i^{th} copy of H, for $i=1,2,\dots,n_1$.

So, $V(G)\boxdot H)=V(G)\cup I(G)\cup\Big(V(H^1)\cup V(H^2)\cup\ldots\cup V(H^{n_1})\Big)$ and $E\Big(G\boxdot H\Big)=E_1^{\boxdot}\cup E_2^{\boxdot}\cup E_3^{\boxdot}$, where $E_1^{\boxdot}=\{v_j^iv_k^i\in E\Big(G\boxdot H\Big)|v_j^i,v_k^i\in V(H^i)\}$, $E_2^{\boxdot}=\{u_iu_{e_k}\in E\Big(G\boxdot H\Big)|u_i\in V(G),u_{e_k}\in I(G)\}$ and $E_3^{\boxdot}=\{u_{e_k}v_j^i\in E\Big(G\boxdot H\Big)|u_{e_k}\in I(G),v_j^i\in V(H^i)\}$.

The degrees of the vertices of $G\boxdot H$ are given by

$$\left.\begin{aligned} d(u_i/G\boxdot H)&=d(u_i/G) \text{ for } i=1,2,\dots,n_1,\\ d(u_{e_i}/G\boxdot H)&=2(n_2+1) \text{ for } i=1,2,\dots,m_1,\\ d(v_j^i/G\boxdot H)&=d(v_j/H)+d(u_i/G) \text{ for } i=1,2,\dots,n_1 \text{ and } j=1,2,\dots,n_2.\end{aligned}\right\}\tag{8}$$

In the following Theorem, we obtain the FEZI for SVNCP of two graphs.

Theorem 13. *The FEZI of $G\boxdot H$ is given by*

$M_1^{\varepsilon}(G\boxdot H)=n_1EM_1(H)+(n_2+1)F(G)+(4n_2^2+12m_2+n_2+1)M_1(G)+(10m_1+n_1)M_1(H)+4m_1n_2^2(2n_2+3)+4m_1(2n_2+1)+8m_1m_2(2n_2-1)$.

Proof. From the Definition 1 and the Equation 8, the first entire Zagreb index of $G\boxdot H$ is

$$\begin{aligned} &M_1^{\varepsilon}(G\boxdot H)=M_1(G\boxdot H)+EM_1(G\boxdot H)\\ &=\sum_{v_j^iv_k^i\in E_1^{\boxdot}}\Big(d(v_j^i/(G\boxdot H))+d(v_k^i/(G\boxdot H))\Big)+\sum_{u_iu_{e_k}\in E_2^{\boxdot}}\Big(d(u_i/(G\boxdot H))+d(u_{e_k}/(G\boxdot H))\Big)\\ &+\sum_{u_{e_k}v_j^i\in E_3^{\boxdot}}\Big(d(u_{e_k}/(G\boxdot H))+d(v_j^i/(G\boxdot H))\Big)+\sum_{v_j^iv_k^i\in E_1^{\boxdot}}\Big(d(v_j^i/(G\boxdot H))+d(v_k^i/(G\boxdot H))-2\Big)^2\\ &+\sum_{u_iu_{e_k}\in E_2^{\boxdot}}\Big(d(u_i/(G\boxdot H))+d(u_{e_k}/(G\boxdot H))-2\Big)^2+\sum_{u_{e_k}v_j^i\in E_3^{\boxdot}}\Big(d(u_{e_k}/(G\boxdot H))+d(v_j^i/(G\boxdot H))-2\Big)^2\end{aligned}$$

$$= \sum_{i=1}^{n_1} \sum_{u_j v_k \in E(H)} \Big(2d(u_k/G) + d(v_j/H) + d(v_k/H)\Big)$$
$$+ \sum_{i=1}^{n_1} \Big(d(u_i/G) + 2n_2 + 2\Big) d(u_j/G) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \Big(d(u_i/G) + d(v_j/H) + 2n_2 + 2\Big) d(u_i/G)$$
$$+ \sum_{i=1}^{n_1} \sum_{u_j v_k \in E(H)} \Big(2d(u_i/G) + (d(v_j/H) + d(v_k/H) - 2)\Big)^2 + \sum_{i=1}^{n_1} (d(u_i/G) + 2n_2)^2 d(u_j/G) +$$
$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \Big(d(u_i/G) + d(v_j/H) + 2n_2\Big)^2 d(u_i/G)$$
$$= 4m_1m_2 + n_1M_1(H) + M_1(G) + 4m_1(n_2+1) + n_2M_1(G) + 4m_1m_2 + 4m_1n_2(n_2+1) + 4m_2M_1(G) + n_1EM_1(H) + 8m_1M_1(H) - 16m_1m_2 + F(G) + 4m_2M_1(G) + 8m_1n_2^2 + n_2F(G) + 2m_1M_1(H) + 8m_1n_2^3 + 4m_2M_1(G) + 4n_2^2M_1(G) + 16m_1m_2n_2$$
$$= n_1EM_1(H) + (n_2+1)F(G) + (4n_2^2 + 12m_2 + n_2 + 1)M_1(G) + (10m_1 + n_1)M_1(H) + 4m_1n_2^2(2n_2+3) + 4m_1(2n_2+1) + 8m_1m_2(2n_2-1).$$ □

Using the Theorem 13, we determine the following results.

Example 7. *(i)* $M_1^{\varepsilon}(P_n \boxdot P_m) = (8m^3n + 44m^2n + 92mn - 8m^3 - 52m^2 - 116m - 100n + 100)$
(ii) $M_1^{\varepsilon}(C_n \boxdot C_m) = 2n\big(4m^3 + 22m^2 + 54m + 8\big)$
(iii) $M_1^{\varepsilon}(C_n \boxdot P_m) = 2n\big(4m^3 + 22m^2 + 46m - 50\big)$.

Theorem 14. *The bounds for the FEZI of $G \boxdot H$ are given by $Y_1 \geq M_1^{\varepsilon}(G \boxdot H) \geq Y_2$, where*
$Y_1 = 2m_2n_1(\Delta_1 + \Delta_2)\Big(2(\Delta_1 + \Delta_2) - 3\Big) + 2m_1n_2(\Delta_1 + \Delta_2 + 2n_2)(\Delta_1 + \Delta_2 + 2n_2 + 1) + 2m_1(\Delta_1 + 2n_2)(\Delta_1 + 2n_2 + 1) + 4(m_1n_2 + m_2n_1) + 4m_1$ *and* $Y_2 = 2m_2n_1(\delta_1 + \delta_2)\Big(2(\delta_1 + \delta_2) - 3\Big) + 2m_1n_2(\delta_1 + \delta_2 + 2n_2)(\delta_1 + \delta_2 + 2n_2 + 1) + 2m_1(\delta_1 + 2n_2)(\delta_1 + 2n_2 + 1) + 4(m_1n_2 + m_2n_1) + 4m_1$.

Proof. In a manner analogous to the proof of the Theorem 4, we can establish the above Theorem. □

3.8. The Subdivision-Edge Neighborhood Corona product(SENCP).

Liu et al. [10] defined four new graph operations based on S-graphs. The SENCP of two graphs is one out of four operations.

Definition 9. [10] *The SENCP of G and H, denoted by $G \boxminus H$, is a new graph achieved from $S(G)$ and $|I(G)|$ copies of H, all vertex-disjoint, and by adjoining the neighbors of the i^{th} vertex of $I(G)$ to every vertex in the i^{th} copy of H.*

It is clear that the graph $G \boxminus H$ *has* $(|V(G)|+|E(G)|+|E(G)||V(H)|)$ *vertices and* $(|E(G)||E(H)|+2|E(G)|+2|E(G)||V(H)|)$ *edges, (see Fig. 3).*

Let $V(G)=\{u_1,u_2,\ldots,u_{n_1}\}, I(G)=\{e_1,e_2,\ldots,e_{m_1}\}$ *and* $V(H)=\{v_1,v_2,\ldots,v_{n_2}\}$. *Also, for* $i=1,2,\ldots,m_1$ *let* $V(H^i)=\{v_1^i,v_2^i,\ldots,v_{n_2}^i\}$ *be the vertex set of the* i^{th} *copy of* H.

So, $V(G)=V(G)\cup I(G)$ *and* $V(G\boxminus H)=V(G)\cup I(G)\cup\Big(V(H^1)\cup V(H^2)\cup\ldots\cup V(H^{m_1})\Big)$ *and* $E\Big(G\boxminus H\Big)=E_1^{\boxminus}\cup E_2^{\boxminus}\cup E_3^{\boxminus}$ *where* $E_1^{\boxminus}=\{v_j^iv_k^i\in E\Big(G\boxminus H\Big)|v_j^i,v_k^i\in V(H^i)$ *for* $i=1,2,\ldots,m_1$ *and* $j,k=1,2,\ldots,n_2\}$,
$E_2^{\boxminus}=\{u_iu_{e_k}\in E\Big(G\boxminus H\Big)|u_i\in V(G),u_{e_k}\in I(G)$ *for* $i=1,2,\ldots,n_1$ *and* $k=1,2,\ldots,m_1\}$ *and* $E_3^{\boxminus}=\{u_kv_j^i\in E\Big(G\boxminus H\Big)|u_k\in V(G),v_j^i\in V(H^i)$ *for* $i=1,2,\ldots,n_1$ *and* $j=1,2,\ldots,n_2\}$.

The degrees of the vertices of $G\boxminus H$ are given by

$$\left.\begin{array}{c} d(u_i/G\boxminus H)=(n_2+1)d(u_i/G) \text{ for } i=1,2,\ldots,n_1,\\ d(e_i/G\boxminus H)=2 \text{ for } i=1,2,\ldots,m_1,\\ d(v_j^i/G\boxminus H)=d(v_j/H)+2 \text{ for } i=1,2,\ldots,n_1 \text{ and } j=1,2,\ldots,n_2. \end{array}\right\} \tag{9}$$

We obtain the FEZI for the SENCP of two graphs.

Theorem 15. *The FEZI for the SENCP of G and H is given by*

$$M_1^{\varepsilon}(G\boxminus H)=(n_2+1)^3F(G)+m_1EM_1(H)+(n_2+1)(n_2+4m_2+1)M_1(G)+11m_1M_1(H)+8m_1m_2+4m_1n_2+4m_1.$$

Proof. Using the Proposition 1, we obtain

$$\begin{aligned}
&M_1^{\varepsilon}(G\boxminus H)=M_1(G\boxminus H)+EM_1(G\boxminus H)\\
&=\sum_{v_j^iv_k^i\in E_1^{\boxminus}}\Big(d(v_j^i/(G\boxminus H))+d(v_k^i/(G\boxminus H))\Big)+\sum_{u_iu_{e_k}\in E_2^{\boxminus}}\Big(d(u_i/(G\boxminus H))+d(u_{e_k}/(G\boxminus H))\Big)\\
&+\sum_{u_iu_{e_k}\in E_3^{\boxminus}}\Big(d(u_i/(G\boxminus H))+d(u_{e_k}/(G\boxminus H))\Big)+\sum_{v_j^iv_k^i\in E_1^{\boxminus}}\Big(d(v_j^i/(G\boxminus H))+d(v_k^i/(G\boxminus H))-2\Big)^2\\
&+\sum_{u_iu_{e_k}\in E_2^{\boxminus}}\Big(d(u_i/(G\boxminus H))+d(u_{e_k}/(G\boxminus H))-2\Big)^2+\sum_{u_iu_{e_k}\in E_3^{\boxminus}}\Big(d(u_i/(G\boxminus H))+d(u_{e_k}/(G\boxminus\\
&H))-2\Big)^2=\sum_{i=1}^{m_1}\sum_{v_jv_k\in E(H)}(d(v_j/H)+d(v_k/H)+4)+\sum_{i=1}^{n_1}\Big((n_2+1)d(u_i/G)+2\Big)d(u_i/G)\\
&+\sum_{k=1}^{n_1}\sum_{j=1}^{n_2}\Big((n_2+1)d(u_k/G)+d(v_j/H)+2\Big)d(u_k/G)
\end{aligned}$$

$+\sum_{i=1}^{m_1}\sum_{v_jv_k\in E(H)}(d(v_j/H)+d(v_k/H)+2)^2+(n_2+1)^2\sum_{i=1}^{n_1}d^3(v_i/G)$
$+\sum_{k=1}^{n_1}\sum_{j=1}^{n_2}\Big((n_2+1)d(u_k/G)+d(v_j/H)+2\Big)^2 d(u_k/G)$ (Using the equation 9)
$= m_1M_1(H) + 4m_1m_2 + (n_2 + 1)M_1(G) + 4m_1 + n_2(n_2 + 1)M_1(G) + 4m_1m_2 + 4m_1n_2 + m_1EM_1(H)+8m_1M_1(H)+(n_2+1)^2F(G)+$
$n_2(n_2+1)^2F(G)+2m_1M_1(H)+4m_2(n_2+1)M_1(G)$
$= (n_2 + 1)^3F(G) + m_1EM_1(H) + (n_2 + 1)(n_2 + 4m_2 + 1)M_1(G) + 11m_1M_1(H) + 8m_1m_2 + 4m_1n_2+4m_1.$ □

Applying the Theorem 15, we have the following results.

Example 8. *(i)* $M_1^\varepsilon(P_n\boxminus P_m)=(8m^3n+44m^2n+92mn-14m^3-72m^2-114m-84n+84)$
(ii) $M_1^\varepsilon(C_n\boxminus C_m)=4n\big(2m^3+11m^2+27m+4\big)$
(iii) $M_1^\varepsilon(C_n\boxminus P_m)=4n\big(2m^3+11m^2+23m-21\big)$.

Theorem 16. *The bounds for $M_1^\varepsilon(G\boxminus H)$ are calculated as*

$Z_1\geq M_1^\varepsilon(G\boxminus H)\geq Z_2$, *where* $Z_1=2m_1m_2(2\Delta_1^2+5\Delta_2+4)+2m_1(n_2+1)\Big((n_2+1)\Delta_1+1\Big)\Delta_1+2m_1n_2\Big((n_2+1)\Delta_1+\Delta_2\Big)\Big((n_2+1)\Delta_1+\Delta_2+1\Big)+4m_1(n_2+1)$, *and for the lower bound* $Z_2=2m_1m_2(2\delta_1^2+5\delta_2+4)+2m_1(n_2+1)\Big((n_2+1)\delta_1+1\Big)\delta_1+2m_1n_2\Big((n_2+1)\delta_1+\delta_2\Big)\Big((n_2+1)\delta_1+\delta_2+1\Big)+4m_1(n_2+1)$. *The equality hold if and only if G and H are regular graphs.*

Proof. The proof is analogous to that of the Theorem 2. □

3.9. Vertex Edge Corona product(VECP).

The VECP of two graphs was introduced by Malpashree [14] in 2016.

Definition 10. [14] *The VECP of two graphs G and H is denoted by $G\bullet H$, is a new graph created by taking one copy of G, $|V(G)|$ copies of H and $|E(G)|$ copies of H, along with joining i^{th} vertex of G to every vertex in the i^{th} vertex copy of H and also joining end vertices of j^{th} edge of G to every vertex in the j^{th} edge copy of H, where $i=1,2,\ldots,n_1$and $j=1,2,\ldots,m_1$.*

Let $V(G)=\{u_1,u_2,\ldots,u_{n_1}\},E(G)=\{e_1,e_2,\ldots,e_{m_1}\}$ *and* $V(H)=\{v_1,v_2,\ldots,v_{n_2}\}$ *and* $E(H)=\{e_1^*,e_2^*,\ldots,e_{m_2}^*\}$. *We denote, the vertex set of the i^{th} vertex copy of H by* $V_{v_i}(H)=\{v_{i1}^{u_i},v_{i2}^{u_i},\ldots,v_{in_2}^{n_2}\}$ *and the vertex set of the j^{th} edge copy of H by*

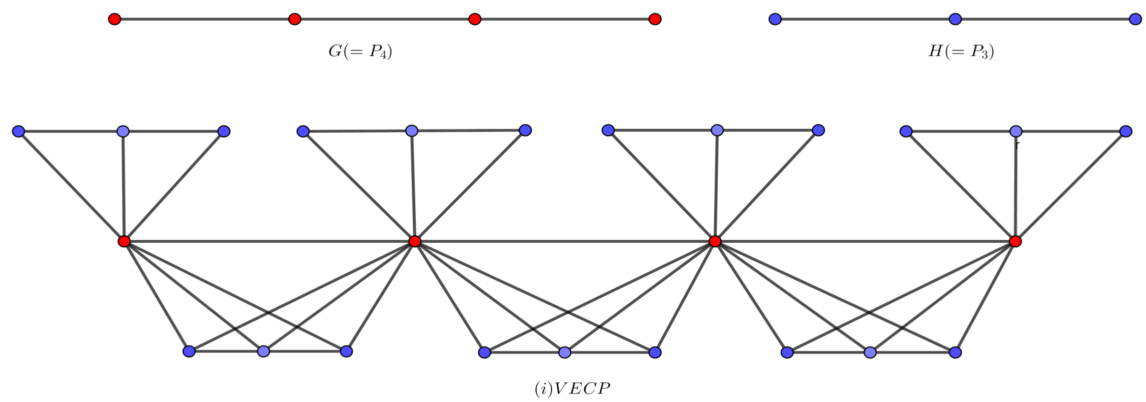

FIGURE 4. An example of various types of Corona products of two graphs P_4 and P_3 like (i) VECP.

$V_{e_j}(H)=\{v^{e_j}_{j1},v^{e_j}_{j2},\dots,v^{e_j}_{jn_2}\}$. *Also, we denote by* $E_{e_j}(H)$ *and* $E_{v_i}(H)$, *the edge set of* j^{th} *edge and* i^{th} *vertex copy of* H, *respectively. Then* $V(G\bullet H)=V(G)\cup_{i=1}^{n_1}V_{v_i}(H)\cup_{j=1}^{m_1}V_{e_j}(H)$ *and* $E(G\bullet H)=E^\bullet_1\cup E^\bullet_2\cup E^\bullet_3\cup E^\bullet_4\cup E^\bullet_5$, *where* $E^\bullet_1=\{u_iu_j\in E\Big(G\bullet H\Big)|u_i,u_j\in V(G);i\neq j$ *and* $i,j=1,2,\dots,n_1\}$, $E^\bullet_2=\{v^{e_j}_{jk}v^{e_j}_{jl}\in E\Big(G\bullet H\Big)|v^{e_j}_{jk},v^{e_j}_{jl}\in V_{e_j}(H)$ *or* $v^{e_j}_{jk}v^{e_j}_{jl}\in E_{e_j}(H),j=1,2,\dots,m_1$ *and* $k\neq l;k,l=1,2,\dots,n_2\}$, $E^\bullet_3=\{u_iv^{e_j}_{jk}\in E\Big(G\bullet H\Big)|u_i\in V(G),v^{e_j}_{jk}\in V_{e_j}(H),i=1,2,\dots,n_1;j=1,2,\dots,m_1$ *and* $k=1,2,\dots,n_2\}$, $E^\bullet_4=\{v^{u_i}_{ij}v^{u_i}_{ik}\in E\Big(G\bullet H\Big)|v^{u_i}_{ij},v^{u_i}_{ik}\in V_{v_i}(H)$ *or* $v^{u_i}_{ij}v^{u_i}_{ik}\in E_{v_i}(H),i=1,2,\dots,n_1$ *and* $j\neq k;j,k=1,2,\dots,n_2\}$ *and* $E^\bullet_5=\{u_iv^{u_i}_{ik}\in E\Big(G\bullet H\Big)|u_i\in V(G),v^{u_i}_{ik}\in V_{v_i}(H),i=1,2,\dots,n_1$ *and* $j=1,2,\dots,n_2\}$. *The graph* $G\bullet H$ *has* $\Big(|V(G)|+|E(G)||V(H)|+|V(G)||V(H)|\Big)$ *vertices and* $\Big(|E(G)|+|V(G)|(|E(H)|+|V(H)|)+|E(G)|(|E(H)|+2|V(H)|)\Big)$ *edges, (see Fig. 4).*

The degrees of the vertices of $G\bullet H$ are given by

(10)
$$\left.\begin{array}{c} d(u_i/G\bullet H)=(n_2+1)d(u_i/G)+n_2,\forall u_i\in V(G)\text{ for }i=1,2,\dots,n_1,\\ d(v^{e_j}_{jk}/G\bullet H)=d(v_k/H)+2,\forall v^{e_j}_{jk}\in V_{e_j}(H)\text{ for }j=1,2,\dots,m_1\text{ and }k=1,2,\dots,n_2,\\ d(v^{u_i}_{ik}/G\bullet H)=d(v_k/H)+1,\forall v^{u_i}_{ik}\in V_{v_i}(H)\text{ for }i=1,2,\dots,n_1\text{ and }k=1,2,\dots,n_2. \end{array}\right\}$$

In the following Theorem, the explicit expression of FEZI for $(G\bullet H)$ is computed.

Theorem 17. *The FEZI of $G\bullet H$ is given by*

$M_1^{\varepsilon}(G\bullet H) = (n_2+1)^3F(G)+(m_1+n_1)F(H)+(n_2+1)(3n_2{}^2+6n_2+4m_2-3)M_1(G)+(7m_1+2n_1)M_1(H)+2(n_2+1)^2M_2(G)+2(m_1+n_1)M_2(H)+8m_1n_2{}^2-n_1n_2^2+6m_1n_2{}^3+n_1n_2^3-4m_1n_2+2n_1n_2+4m_2n_1n_2+16m_1m_2n_2+20m_1m_2+4m_1.$

Proof. By the Proposition 1 and also from the Equation 10, we have

$$M_1^{\varepsilon}(G\bullet H) = M_1(G\bullet H)+EM_1(G\bullet H)$$
$$=\Big(\sum_{u_i\in V(G)} d^2(u_i/(G\bullet H))+\sum_{j=1}^{m_1}\sum_{v_{jk}^{e_j}\in V_{e_j}(H)} d^2(v_{jk}^{e_j}/(G\bullet H))+\sum_{u_i\in V(G)}\sum_{v_{ik}^{u_i}\in V_{v_i}(H)} d^2(v_{ik}^{u_i}/(G\bullet H))\Big)$$
$$+\sum_{u_iu_j\in E_1^{\bullet}}\Big\{d(u_i/(G\bullet H))+d(v_j/(G\bullet H)-2\Big\}^2+\sum_{v_{jk}^{e_j}v_{jl}^{e_j}\in E_2^{\bullet}}\Big\{d(v_{jk}^{e_j}/(G\bullet H))+d(v_{jl}^{e_j}/(G\bullet H))-2\Big\}^2+\sum_{u_iv_{jk}^{e_j}\in E_3^{\bullet}}\Big\{d(u_i/(G\bullet H)+d(v_{jk}^{e_j}/(G\bullet H))-2\Big\}^2+\sum_{v_{ij}^{u_i}v_{ik}^{u_i}\in E_4^{\bullet}}\Big\{d(v_{ij}^{u_i}/(G\bullet H))+d(v_{ik}^{u_i}/(G\bullet H))-2\Big\}^2+\sum_{u_iv_{ij}^{u_i}\in E_5^{\bullet}}\Big\{d(u_i/(G\bullet H))+d(v_{ij}^{u_i}/(G\bullet H))-2\Big\}^2$$

$=J_1+J_2+J_3+J_4+J_5+J_6$ are denoted as the sum of the above terms in order, respectively.

$$J_1=\Big(\sum_{u_i\in V(G)} d^2(u_i/(G\bullet H))+\sum_{j=1}^{m_1}\sum_{v_{jk}^{e_j}\in V_{e_j}(H)} d^2(v_{jk}^{e_j}/(G\bullet H))+\sum_{u_i\in V(G)}\sum_{v_{ik}^{u_i}\in V_{v_i}(H)} d^2(v_{ik}^{u_i}/(G\bullet H))\Big)$$
$$=\sum_{u_i\in V(G)}\Big((n_2+1)d(u_i/G)+n_2\Big)^2+\sum_{j=1}^{m_1}\sum_{v_k\in V_{e_j}(H)}\Big(d(v_k/H)+2\Big)^2+\sum_{u_i\in V(G)}\sum_{v_k\in V_{v_i}(H)}\Big(d(v_k/H)+1\Big)^2$$
$$=\sum_{u_i\in V(G)}\Big((n_2+1)^2d^2(u_i/G)+2n_2(n_2+1)d(u_i/G)+n_2^2\Big)+\sum_{j=1}^{m_1}\sum_{v_k\in V_{e_j}(H)}\Big(d^2(v_k/H)+4d(v_k/H)+4\Big)+\sum_{u_i\in V(G)}\sum_{v_k\in V_{v_i}(H)}\Big(d^2(v_k/H)+2d(v_k/H)+1\Big)$$
$$=(n_2+1)^2M_1(G)+4m_1n_2(n_2+1)+n_1n_2^2+m_1M_1(H)+4m_1n_2+8m_1m_2+n_1M_1(H)+n_1n_2+4m_2n_1.$$

$$J_2=\sum_{u_iu_j\in E_1^{\bullet}}\Big\{d(u_i/(G\bullet H))+d(v_j/(G\bullet H)-2\Big\}^2$$
$$=\sum_{u_iu_j\in E(G)}\Big\{(n_2+1)d(u_i/(G))+(n_2+1)d(v_j/(G)+2n_2-2\Big\}^2$$
$$=\sum_{u_iu_j\in E(G)}\Big\{(n_2+1)^2\big(d(u_i/G)+d(u_j/G)-2\big)^2+8n_2(n_2+1)\big(d(v_i/(G)+d_(v_j/G)-2\big)+$$

$16n_2^2\Big\}$

$= (n_2+1)^2F(G) - 4(n_2+1)^2M_1(G) + 2(n_2+1)^2M_2(G) + 4m_1(n_2+1)^2 + 8n_2(n_2+1)M_1(G) - 16m_1n_2.$

Also, $J_3 = \sum_{v_{jk}^{e_j}v_{jl}^{e_j}\in E_2^\bullet} \{d(v_{jk}^{e_j}/(G\bullet H)) + d(v_{jl}^{e_j}/(G\bullet H)) - 2\}^2$

$= \sum_{j=1}^{m_1}\sum_{v_kv_l\in E_{e_j}(H)} \Big\{d(v_k/H) + d(v_l/H) + 2\Big\}^2$

$= \sum_{j=1}^{m_1}\sum_{v_kv_l\in E(H)} \Big(\big(d^2(v_k/H) + d^2(v_l/H)\big) + 4\big(d(v_k/H) + d(v_l/H)\big) + 2d(v_k/H)d(v_l/H) + 4\Big)$

$= m_1F(H) + 2m_1M_2(H) + 4m_1M_1(H) + 4m_1m_2.$

Similarly, $J_4 = \sum_{u_iv_{jk}^{e_j}\in E_3^\bullet} \Big\{d(u_i/(G\bullet H) + d(v_{jk}^{e_j}/(G\bullet H)) - 2\Big\}^2$

$= \sum_{u_i\in V(G)}\sum_{v_{jk}^{e_j}\in V_{e_j}(H)} \Big\{d(u_i/(G\bullet H) + d(v_{jk}^{e_j}/(G\bullet H)) - 2\Big\}^2 d(u_i/G)$

$= \sum_{u_i\in V(G)}\sum_{v_k\in V_{e_j}(H)} \Big\{(n_2+1)d(u_i/G) + d(v_k/H) + n_2\Big\}^2 d(u_i/G)$

$= \sum_{u_i\in V(G)}\sum_{v_k\in V_{e_j}(H)} \Big\{(n_2 + 1)^2d^2(u_i/G) + d^2(v_k/H) + n_2^2 + 2n_2(n_2 + 1)d(u_i/G) + 2(n_2 + 1)d(u_i/G)(d(v_k/H) + 2n_2d(v_k/H)\Big\}d(u_i/G)$

$= n_2(n_2 + 1)^2F(G) + 2m_1n_2^3 + 2m_1M_1(H) + 2n_2^2(n_2 + 1)M_1(G) + 8m_1m_2n_2 + 4m_2(n_2 + 1)M_1(G).$

Next, $J_5 = \sum_{v_{ij}^{u_i}v_{ik}^{u_i}\in E_4^\bullet} \Big\{d(v_{ij}^{u_i}/(G\bullet H)) + d(v_{ik}^{u_i}/(G\bullet H)) - 2\Big\}^2$

$= \sum_{u_i\in V(G)}\sum_{v_jv_k\in E_{v_i}(H)} \Big\{d(v_j/H) + d(v_k/H)\Big\}^2$

$= n_1F(H) + 2n_1M_2(H).$

Finally, $J_6 = \sum_{u_iv_{ij}^{u_i}\in E_5^\bullet} \Big\{d(u_i/(G\bullet H)) + d(v_{ij}^{u_i}/(G\bullet H)) - 2\Big\}^2$

$= \sum_{u_i\in V(G)}\sum_{v_j\in V_{v_i}(H)} \Big\{(n_2+1)d(u_i/G) + d(v_j/H) + n_2 - 1\Big\}^2$

$= \sum_{u_i\in V(G)}\sum_{v_j\in V_{v_i}(H)} \Big\{(n_2+1)^2d^2(u_i/G) + d^2(v_j/H) + (n_2-1)^2 + 2(n_2+1)d(u_i/G)d(v_j/H) + 2(n_2^2-1)d(u_i/G) + 2(n_2-1)d(v_j/H)\Big\}$

$= n_2(n_2 + 1)^2M_1(G) + n_1M_1(H) + n_1n_2(n_2 - 1)^2 + 8m_1m_2(n_2 + 1) + 4m_1n_2(n_2^2 - 1) + 4n_1m_2(n_2 - 1).$

By adding J_1, J_2, J_3, J_4, J_5 and J_6, we get the desired result. □

From the above Theorem 17, we get the following results.

Example 9. *(i)* $M_1^{\varepsilon}(C_n \bullet C_m) = n\big(27m^3 + 111m^2 + 154m + 8\big)$
(ii) $M_1^{\varepsilon}(P_n \bullet P_m) = (27m^3n + 111m^2n + 118mn - 38m^3 - 160m^2 - 136m - 142n + 100)$
(iii) $M_1^{\varepsilon}(C_n \bullet P_m) = n\big(27m^3 + 111m^2 + 118m - 142\big)$.

Theorem 18. *The bounds for the FEZI of $G \bullet H$ are given by*
$\alpha \geq M_1^{\varepsilon}(G \bullet H) \geq \beta$, *where* $\alpha = 2m_1\Big((n_2+1)\Delta_1 + n_2\Big)\Big(2(n_2+1)\Delta_1 + 2n_2 - 3\Big) + 2m_1m_2\Big(2\Delta_2^2 + 5\Delta_2 + 4\Big) + 2m_1n_2\Big((n_2+1)\Delta_1 + \Delta_2 + n_2\Big)\Big((n_2+1)\Delta_1 + \Delta_2 + n_2 + 1\Big) + 2m_2n_1\Big(2\Delta_2^2 + \Delta_2 + 1\Big) + n_1n_2\Big((n_2+1)\Delta_1 + \Delta_2 + n_2\Big)\Big((n_2+1)\Delta_1 + \Delta_2 + n_2 - 1\Big) + 2n_1n_2 + 4m_1(n_2+1)$, *and* $\beta = 2m_1\Big((n_2+1)\delta_1 + n_2\Big)\Big(2(n_2+1)\delta_1 + 2n_2 - 3\Big) + 2m_1m_2\Big(2\delta_2^2 + 5\delta_2 + 4\Big) + 2m_1n_2\Big((n_2+1)\delta_1 + \delta_2 + n_2\Big)\Big((n_2+1)\delta_1 + \delta_2 + n_2 + 1\Big) + 2m_2n_1\Big(2\delta_2^2 + \delta_2 + 1\Big) + n_1n_2\Big((n_2+1)\delta_1 + \delta_2 + n_2\Big)\Big((n_2+1)\delta_1 + \delta_2 + n_2 - 1\Big) + 2n_1n_2 + 4m_1(n_2+1)$.

The equality holds if and only if G and H are regular graphs.

4. Conclusion

In this paper, we have established some exact formulas with examples for the FEZI of several types of Corona product of two graphs based on *R*-graphs and *S*-graphs such as RVCP, RECP, RVNCP, RENCP and SVCP, SECP, SVNCP and SENCP, respectively. Additionally, we have determined the FEZI of VECP of two graphs. As an application, we have obtained the lower and upper bounds for the FEZI of each Corona product of the two graphs. Also, we have applied our results to find the FEZI of several chemically interesting molecular graphs. We have proposed some possible directions for future research.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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