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CHARACTERIZATION OF EXPONENTIATED FAMILY OF DISTRIBUTIONS BASED ON RECURRENCE RELATIONS FOR GENERALIZED ORDER STATISTICS

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Abstract. In this paper, we derive recurrence relations for moment, conditional moment generating functions and product moments of generalized order statistics based on exponentiated family of distributions. Recurrence relations for moment, conditional moment generating functions and product moments of ordinary order statistics and ordinary record values are obtained as special cases. These recurrence relations are also used to characterize this family.

Keywords: Generalized order statistics, Order statistics, Characterizations, Conditional moments.

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1. Introduction

A concept of generalized order statistics (gOSs) was introduced by Kamps [10,11]. Ordinary order statistics (oOSs), ordinary record values (oRVs), sequential order statistics, ordering via truncated distributions and censoring schemes are special cases of the gOSs. Keseling [12] characterized some continuous distributions based on conditional distributions of gOSs. Ahsanullah [4] characterized the exponential distribution based on independence of functions of gOSs and presented the estimators of its parameters. Cramer

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and Kamps [8] derived relations for expectations of functions of gOSs within a class of continuous distributions. Pawlas and Szynal [14] derived recurrence relations for single and product moments of gOSs from Pareto, generalized Pareto and Burr distributions.

In recent years, a large number of publications have dealt with recurrence relations for single and product moments of gOSs. Ahmad and Fawzy [3] derived recurrence relations for moments of gOSs within a class of doubly truncated distributions. Athar and Islam [7] obtained recurrence relations for single and product moments of gOSs from a general class of distribution. AL-Hussaini et al. [5] obtained recurrence relations for moment and conditional moment generating functions of gOSs based on random samples drawn from a population whose distribution is a member of a doubly truncated class of distributions. Ahmad [2] derived recurrence relations for single and product moments of gOSs from doubly truncated Burr type XII distribution. Abdul-Moniem [1] obtained recurrence relations for moments of lower gOSs form exponentiated Lomax distribution and its characterization.

Consider the cumulative distribution function (cdf) $F(x)$ as

$$F(x) = (1 - e^{-\lambda(x)})^{\theta}, \qquad x \geq 0, \tag{1}$$

where $\lambda(x)$ is a non-negative, continuous, monotone increasing, differentiable function of x such that $\lambda(x) \to 0$ as $x \to 0^+$ and $\lambda(x) \to \infty$ as $x \to \infty$ and the parameter $\theta > 0$. We call this class the exponentiated family of distributions. This family contains many exponentiated distributions such as exponentiated Weibull, exponentiated exponential, exponentiated Rayleigh, exponentiated Pareto distributions,...etc.

The probability density function (pdf) corresponding to (1) is given by

$$f(x) = \theta\, \lambda^{'}(x)\, e^{-\lambda(x)}\, [1 - e^{-\lambda(x)}]^{\theta - 1}. \tag{2}$$

Eq. (2) can be rewritten as

$$\bar{F}(x) = 1 + \frac{(1 - e^{\lambda(x)})}{\theta\, \lambda^{'}(x)}\, f(x). \tag{3}$$

2. Characterizations Based on Recurrence Relations for Single Moment Generating Functions of gOSs

Let $X_{1;\,n;\,m;\,k}, X_{2;\,n;\,m;\,k}, ..., X_{n;\,n;\,m;\,k}$ be n gOSs from the pdf (2), (m and k are real numbers, $n > 1$ and $k \geq 1$). The pdf of $X_{r;\,n;\,m;\,k}, 1 \leq r \leq n$, is given by Kamps [10] as follows:

$$f_{X_{r,n,m,k}}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \quad x \in \chi, \tag{4}$$

where χ is the domain on which $f_{X_{r,n,m,k}}(x)$ is positive,

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \;\; \gamma_i = k + (n-i)(m+1),$$

and for $z \in (0,1)$,

$$g_m(z) = \begin{cases} \frac{\left[1-(1-z)^{m+1}\right]}{m+1}, & \text{m} \neq \text{-1}, \\ \\ -\ln(1-z), & \text{m} = \text{-1}. \end{cases} \tag{5}$$

The single moment generating functions (mgf) of gOSs can be obtained, for $a \geq 1$, from (4) as

$$M^{(a)}_{r;\,n;\,m;\,k}(t) = E[e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}] = \frac{C_{r-1}}{(r-1)!}\int_0^{\infty} e^{t\,x^a}\,[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))\,dx. \tag{6}$$

Theorem 2.1. *Let X be a random variable. Then for integers a such that $a \geq 1$, the following recurrence relation is satisfied iff X has the cdf (1).*

$$\begin{aligned} M^{(a)}_{r;\,n;\,m;\,k}(t) - M^{(a)}_{r-1;\,n;\,m;\,k}(t) = \frac{a\,t}{\theta\,\gamma_r}\,E\Bigg[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}\; e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}}{\lambda'(X_{r;\,n;\,m;\,k})}\Bigg(1 - e^{\lambda(X_{r;\,n;\,m;\,k})} \\ + \; e^{\lambda(X_{r;\,n;\,m;\,k})}\,[1 - e^{-\lambda(X_{r;\,n;\,m;\,k})}]^{1-\theta}\Bigg)\Bigg], m \geq -1. \end{aligned} \tag{7}$$

Proof. If X has the cdf (1), then the mgf of the a^{th} power of the r^{th} gOSs, $X^a_{r;\,n;\,m;\,k}$, is given, from (6), as follows

$$M^{(a)}_{r;\,n;\,m;\,k}(t) = \frac{C_{r-1}}{\gamma_r (r-1)!}\int_0^{\infty} e^{t\,x^a}\; g_m^{r-1}(F(x))\,d\Big[-[\bar{F}(x)]^{\gamma_r}\Big]. \tag{8}$$

Integrating by parts, we obtain

$$\begin{aligned} M^{(a)}_{r;\,n;\,m;\,k}(t) = \frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!} \int_0^\infty x^{a-1}\,e^{t\,x^a}\,[\bar{F}(x)]^{\gamma_r}\,g_m^{r-1}(F(x)) \\ + \frac{(r-1)\,C_{r-1}}{\gamma_r\,(r-1)!} \int_0^\infty e^{t\,x^a}\,[\bar{F}(x)]^{\gamma_{r-1}-1}\,f(x)\,g_m^{r-2}(F(x))\,dx. \end{aligned} \tag{9}$$

The second term in the right hand side is $M^{(a)}_{r-1;\,n;\,m;\,k}(t)$, so we obtain

$$M^{(a)}_{r;\,n;\,m;\,k}(t) - M^{(a)}_{r-1;\,n;\,m;\,k}(t) = \frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!} \int_0^\infty x^{a-1}\,e^{t\,x^a}\,[\bar{F}(x)]^{\gamma_r}\,g_m^{r-1}(F(x))dx. \tag{10}$$

By rewriting $[\bar{F}(x)]^{\gamma_r} = [\bar{F}(x)]^{\gamma_r-1}\,[\bar{F}(x)]$, in (10), then making use of (3), we get

$$\begin{aligned} &M^{(a)}_{r;\,n;\,m;\,k}(t) - M^{(a)}_{r-1;\,n;\,m;\,k}(t) \\ &= \frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!} \int_0^\infty x^{a-1}\,e^{t\,x^a}[\bar{F}(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\Big[1 + \frac{(1-e^{\lambda(x)})}{\theta\,\lambda'(x)}\,f(x)\Big]\,dx \\ &= \frac{a\,t\,C_{r-1}}{\theta\,\gamma_r\,(r-1)!} \int_0^\infty \frac{x^{a-1}\,e^{t\,x^a}\,(1-e^{\lambda(x)})}{\lambda'(x)}[\bar{F}(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\,f(x)\,dx \\ &\quad + \frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!} \int_0^\infty x^{a-1}\,e^{t\,x^a}[\bar{F}(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\,dx. \end{aligned} \tag{11}$$

Since, $\left[\frac{e^{\lambda(x)}\,f(x)\,[1-e^{-\lambda(x)}]^{1-\theta}}{\theta\,\lambda'(x)} = 1\right]$. So, we can rewrite (11) in the form

$$\begin{aligned} &M^{(a)}_{r;\,n;\,m;\,k}(t) - M^{(a)}_{r-1;\,n;\,m;\,k}(t) \\ &= \frac{a\,t}{\theta\,\gamma_r}\,E\left[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}\;e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}\,(1-e^{\lambda(X_{r;\,n;\,m;\,k})})}{\lambda'(x_{r;\,n;\,m;\,k})}\right] \\ &\quad + \frac{a\,t\,C_{r-1}}{\gamma_r(r-1)!} \int_0^\infty x^{a-1}\,e^{t\,x^a}[\bar{F}(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\left[\frac{e^{\lambda(x)}\,f(x)\,[1-e^{-\lambda(x)}]^{1-\theta}}{\theta\,\lambda'(x)}\right]\,dx, \end{aligned}$$

or equivalently,

$$\begin{aligned} &M^{(a)}_{r;\,n;\,m;\,k}(t) - M^{(a)}_{r-1;\,n;\,m;\,k}(t) \\ &= \frac{a\,t}{\theta\,\gamma_r}\,E\left[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}\;e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}\,(1-e^{\lambda(X_{r;\,n;\,m;\,k})})}{\lambda'(X_{r;\,n;\,m;\,k})}\right] \\ &\quad + \frac{a\,t\,C_{r-1}}{\gamma_r\,\theta\,(r-1)!} \int_0^\infty \left[\frac{x^{a-1}\,e^{t\,x^a}\,e^{\lambda(x)}\,[1-e^{-\lambda(x)}]^{1-\theta}}{\lambda'(x)}\right][\bar{F}(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\,f(x)\,dx, \end{aligned}$$

which can be written as

$$
\begin{aligned}
&M^{(a)}_{r;\,n;\,m;\,k}(t) - M^{(a)}_{r-1;\,n;\,m;\,k}(t)\\
&= \frac{a\,t}{\theta\,\gamma_r}\,E\left[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}\;e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}\,(1-e^{\lambda(X_{r;\,n;\,m;\,k})})}{\lambda'(X_{r;\,n;\,m;\,k})}\right]\\
&\quad+\frac{a\,t}{\theta\,\gamma_r}\,E\left[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}\,e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}\,e^{\lambda(X_{r;\,n;\,m;\,k})}\,[1-e^{-\lambda(X_{r;\,n;\,m;\,k})}]^{1-\theta}}{\lambda'(X_{r;\,n;\,m;\,k})}\right].
\end{aligned}
$$

So, we have the result.

Conversely, if the characterizing Condition (7) holds, then from (10) and (11), we have

$$
\begin{aligned}
&\frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!}\int_0^\infty x^{a-1}\,e^{t\,x^a}[\bar F(x)]^{\gamma_r}\,g_m^{r-1}(F(x))\,dx\\
&=\frac{a\,t\,C_{r-1}}{\theta\,\gamma_r\,(r-1)!}\int_0^\infty \frac{x^{a-1}\,e^{t\,x^a}\,(1-e^{\lambda(x)})}{\lambda'(x)}[\bar F_d(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\,f(x)\,dx\\
&\quad+\frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!}\int_0^\infty x^{a-1}\,e^{t\,x^a}[\bar F(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\,dx,
\end{aligned}
\tag{12}
$$

which can be written as

$$
\frac{a\,t\,C_{r-1}}{\gamma_r\,(r-1)!}\int_0^\infty x^{a-1}\,e^{t\,x^a}[\bar F(x)]^{\gamma_r-1}\,g_m^{r-1}(F(x))\left[\bar F(x)-1-\frac{f(x)\,(1-e^{\lambda(x)})}{\theta\,\lambda'(x)}\right]dx=0. \tag{13}
$$

The extension of Müntz-Szàsz theorem,[Hwang and Lin [9]] can be applied to obtain

$$
\bar F(x)=1+\frac{(1-e^{\lambda(x)})}{\theta\,\lambda'(x)}\,f(x).
$$

Remark 2.1. By differentiating both sides of Condition (7) with respect to t and then setting $t=0$, we obtain the following recurrence relation for single moment of gOSs

$$
\begin{aligned}
\mu^{(a)}_{r;\,n;\,m;\,k}-\mu^{(a)}_{r-1;\,n;\,m;\,k}=\frac{a}{\theta\,\gamma_r}\,E\Bigg[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}}{\lambda'(X_{r;\,n;\,m;\,k})}\Bigg(&1-e^{\lambda(X_{r;\,n;\,m;\,k})}\\
&+e^{\lambda(X_{r;\,n;\,m;\,k})}\,[1-e^{-\lambda(X_{r;\,n;\,m;\,k})}]^{1-\theta}\Bigg)\Bigg],\ m\ge -1,
\end{aligned}
\tag{14}
$$

where $\mu^{(a)}_{r;\,n;\,m;\,k}=E[X^a_{r;\,n;\,m;\,k}]$.

If we set $\theta=1$ in (7) and (14) we get

$$
M^{(a)}_{r;\,n;\,m;\,k}(t)-M^{(a)}_{r-1;\,n;\,m;\,k}(t)=\frac{a\,t}{\gamma_r}\,E\left[\frac{X^{(a-1)}_{r;\,n;\,m;\,k}\;e^{t\,X^{(a)}_{r;\,n;\,m;\,k}}}{\lambda'(X_{r;\,n;\,m;\,k})}\right],\qquad m\ge -1, \tag{15}
$$

$$\mu_{r;n;m;k}^{(a)} - \mu_{r-1;n;m;k}^{(a)} = \frac{a}{\gamma_r} E\left[\frac{X_{r;n;m;k}^{(a-1)}}{\lambda'(X_{r;n;m;k})}\right], \qquad m \geq -1. \tag{16}$$

Eqs. (15) and (16) agree with the results given by AL-Hussaini et al [5].

Remark 2.2. If we put $m = 0$ and $k = 1$ in (7) and (14), we obtain the recurrence relations of oOSs, $[\gamma_r = n - r + 1$ and $X_{r;n;m;k} \equiv X_{r;n}]$ in the form

$$\begin{aligned} M_{r;n}^{(a)}(t) - M_{r-1;n}^{(a)}(t) = \frac{a\,t}{\theta\,(n-r+1)} E\Bigg[& \frac{X_{r;n}^{(a-1)}\; e^{t X_{r;n}^{(a)}}}{\lambda'(X_{r;n})} \\ & \times \left(1 - e^{\lambda(X_{r;n})} + e^{\lambda(X_{r;n})}\,[1 - e^{-\lambda(X_{r;n})}]^{1-\theta}\right)\Bigg], \end{aligned} \tag{17}$$

$$\mu_{r;n}^{(a)} - \mu_{r-1;n}^{(a)} = \frac{a}{\theta\,(n-r+1)} E\left[\frac{X_{r;n}^{(a-1)}}{\lambda'(X_{r;n})}\left(1 - e^{\lambda(X_{r;n})} + e^{\lambda(X_{r;n})}\,[1 - e^{-\lambda(X_{r;n})}]^{1-\theta}\right)\right]. \tag{18}$$

Remark 2.3. If we put $m = -1$ and $k = 1$ in (7) and (14), oRVs, $[\gamma_r = k$ and $X_{r;n;m;k} \equiv X_{U(r)}]$ we have

$$\begin{aligned} M_{U(r)}^{(a)}(t) - M_{U(r-1)}^{(a)}(t) = \frac{a\,t}{\theta} E\Bigg[& \frac{X_{U(r)}^{(a-1)}\; e^{t X_{U(r)}^{(a)}}}{\lambda'(X_{U(r)})} \\ & \times \left(1 - e^{\lambda(X_{U(r)})} + e^{\lambda(X_{U(r)})}\left(1 - e^{-\lambda(X_{U(r)})}\right)^{1-\theta}\right)\Bigg], \end{aligned} \tag{19}$$

$$\mu_{r}^{(a)} - \mu_{r-1}^{(a)} = \frac{a}{\theta} E\left[\frac{X_{U(s)}^{(a-1)}}{\lambda'(X_{U(r)})}\left(1 - e^{\lambda(X_{U(r)})} + e^{\lambda(X_{U(r)})}\left(1 - e^{-\lambda(X_{U(r)})}\right)^{1-\theta}\right)\right]. \tag{20}$$

3. Characterizations Based on Recurrence Relations for Conditional Moment Generating Functions of gOSs

The joint density function of the gOSs $X_{s;n;m;k}$ and $X_{r;n;m;k}$, $s > r$ is given by Kamps [10], as follows

$$\begin{aligned} f_{X(s;n;m;k),\,X(r;n;m;k)}(x,y) = {} & \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} \\ & \times\; f(y)\,[\bar{F}(y)]^m\,[\bar{F}(x)]^{\gamma_s - 1}\,f(x)\,g_m^{r-1}(F(y)) \\ & \times\; [h_m(F(x)) - h_m(F(y))]^{s-r-1}, \qquad x > y, \end{aligned} \tag{21}$$

$$h_m(z) = \begin{cases} \frac{-(1-z)^{m+1}}{m+1}, & \text{m} \neq \text{-1}, \\ \\ -\ln(1-z), & \text{m = -1}. \end{cases} \tag{22}$$

Using (4) and (21), the conditional distribution function of $X(s;\, n;\, m;\, k)$ given $X(r;\, n;\, m;\, k)$ is given by

$$f\Big(X(s;\, n;\, m;\, k)|X(r;\, n;\, m;\, k) = y\Big) = B\, f(x)\, [\bar{F}(x)]^{\gamma_s - 1}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1}, \tag{23}$$

where

$$B = \frac{C_{s-1}\, [\bar{F}(y)]^{1+m-\gamma_r}}{C_{r-1}\, (s-r-1)!}. \tag{24}$$

Theorem 3.1. *Let X be a random variable, r, s be two integers such that $1 \le r \le s \le n$, m and k be real numbers such that $m \ge -1$, $k \ge 1$. Then for integers a such that $a \ge 1$, the following recurrence relation is satisfied iff X has the cdf (1).*

$$\begin{aligned} & M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) - M_{X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) \\ &= \frac{a\, t}{\theta\, \gamma_s}\, E\Bigg[\frac{X^{(a-1)}_{s;\,n;\,m;\,k}\; e^{t\, X^{(a)}_{s;\,n;\,m;\,k}}}{\lambda'(X_{s;\,n;\,m;\,k})} \Bigg(1 - e^{\lambda(X_{s;\,n;\,m;\,k})} \\ &\quad + e^{\lambda(X_{s;\,n;\,m;\,k})}\, [1 - e^{-\lambda(X_{s;\,n;\,m;\,k})}]^{1-\theta}\Bigg) |\, X_{r;\,n;\,m;\,k} = y\Bigg]. \end{aligned} \tag{25}$$

Proof. From (23) we get

$$\begin{aligned} & M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) \\ &= E[e^{t\, X^{(a)}_{s;\,n;\,m;\,k}}|X_{r;\,n;\,m;\,k} = y] \\ &= B \int_y^\infty e^{t\, X^a}\, [\bar{F}(x)]^{\gamma_s - 1}\, f(x)\, [h_m(F(x)) - h_m(F(y))]^{s-r-1}\, dx \\ &= \frac{B}{\gamma_s} \int_y^\infty e^{t\, X^a}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1}\, d\Big[-[\bar{F}(x)]^{\gamma_s}\Big], \end{aligned} \tag{26}$$

where B is given by (24).

Integrating by parts, yields

$$M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y)$$
$$= \frac{a\,t\,B}{\gamma_s} \int_y^\infty x^{a-1}\, e^{t\,x^a}\, [\bar{F}(x)]^{\gamma_s}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1}\, dx$$
$$+ \frac{C_{s-2}\,[\bar{F}(y)]^{1+m-\gamma_r}}{C_{r-1}\,(s-r-2)!} \int_y^\infty e^{t\,x^a}\, [\bar{F}(x)]^{\gamma_{s-1}-1}\, f(x)\, [h_m(F(x)) - h_m(F(y))]^{s-r-2}\, dx.$$

The second term in the right hand side is $M_{X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y)$, so we obtain

$$\begin{aligned} &M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) - M_{X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) \\ &= \frac{a\,t\,B}{\gamma_s} \int_y^\infty x^{a-1}\, e^{t\,x^a}\, [\bar{F}(x)]^{\gamma_s}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1} dx. \end{aligned} \tag{27}$$

By rewriting $[\bar{F}(x)]^{\gamma_s} = [\bar{F}(x)]^{\gamma_s - 1}\, [\bar{F}(x)]$, in (27), then making use of (3), gives

$$\begin{aligned} &M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) - M_{X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) \\ &= \frac{a\,t\,B}{\gamma_s} \int_y^\infty x^{a-1}\, e^{t\,x^a}\, [\bar{F}(x)]^{\gamma_s-1}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1} \Big[1 + \frac{(1-e^{\lambda(x)})}{\theta\,\lambda'(x)}\, f(x)\Big] dx \\ &= \frac{a\,t\,B}{\theta\,\gamma_s} \int_y^\infty \frac{x^{a-1}\, e^{t\,x^a}\, (1-e^{\lambda(x)})}{\lambda'(x)}\, f(x)\, [\bar{F}(x)]^{\gamma_s-1}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1}\, dx \\ &\quad + \frac{a\,t\,B}{\gamma_s} \int_y^\infty x^{a-1}\, e^{t\,x^a}\, [\bar{F}(x)]^{\gamma_s-1}\, [h_m(F(x)) - h_m(F(y))]^{s-r-1}\, dx. \end{aligned} \tag{28}$$

Since, $\left[\frac{e^{\lambda(x)}\, f(x)\, [1-e^{-\lambda(x)}]^{1-\theta}}{\theta\,\lambda'(x)} = 1\right]$. So, we can rewrite (28) in the form

$$\begin{aligned} &M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) - M_{X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) \\ &= \frac{a\,t}{\theta\,\gamma_s}\, E\left[\frac{X^{(a-1)}_{s;\,n;\,m;\,k}\; e^{t\,X^{(a)}_{s;\,n;\,m;\,k}} \left(1 - e^{\lambda(X_{s;\,n;\,m;\,k})}\right)}{\lambda'(X_{s;\,n;\,m;\,k})} |X_{r;\,n;\,m;\,k}\right] \\ &\quad + \frac{a\,t\,B}{\gamma_s} \int_y^\infty x^{a-1}\, e^{t\,x^a} [\bar{F}(x)]^{\gamma_s-1}\, [h_m(F(x)) - h_m(F(x))]^{s-r-1} \\ &\quad \times \left[\frac{e^{\lambda(x)}\, f(x)\, [1-e^{-\lambda(x)}]^{1-\theta}}{\theta\,\lambda'(x)}\right] dx. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&M_{X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) - M_{X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}}(t|y) \\
&= \frac{a\,t}{\theta\,\gamma_s}\,E\left[\frac{X^{(a-1)}_{s;\,n;\,m;\,k}\;e^{t\,X^{(a)}_{s;\,n;\,m;\,k}}\left(1-e^{\lambda(X_{s;\,n;\,m;\,k})}\right)}{\lambda'(X_{s;\,n;\,m;\,k})}|X_{r;\,n;\,m;\,k}\right] \\
&\quad + \frac{a\,t}{\theta\,\gamma_s}\,E\left[\frac{X^{(a-1)}_{s;\,n;\,m;\,k}\,e^{t\,X^{(a)}_{s;\,n;\,m;\,k}}\,e^{\lambda(X_{s;\,n;\,m;\,k})}\,[1-e^{-\lambda(X_{s;\,n;\,m;\,k})}]^{1-\theta}}{\lambda'(X_{s;\,n;\,m;\,k})}|\,X_{r;\,n;\,m;\,k}=y\right].
\end{aligned}
\tag{29}
$$

So, we have the result.

Conversely, if the characterizing Condition (25), is satisfied, then from (27) and (28), we have

$$
\begin{aligned}
&\frac{a\,t\,B}{\gamma_s}\int_y^\infty x^{a-1}\,e^{t\,x^a}\,[\bar F(x)]^{\gamma_s}\,[h_m(F(x))-h_m(F(x))]^{s-r-1}\,dx \\
&= \frac{a\,t\,B}{\theta\,\gamma_s}\int_y^\infty \frac{x^{a-1}\,e^{t\,x^a}\,(1-e^{\lambda(x)})}{\lambda'(x)}\,f(x)\,[\bar F(x)]^{\gamma_s-1}\,[h_m(F(x))-h_m(F(x))]^{s-r-1}\,dx \\
&\quad + \frac{a\,t\,B}{\gamma_s}\int_y^\infty x^{a-1}\,e^{t\,x^a}\,[\bar F(x)]^{\gamma_s-1}\,[h_m(F(x))-h_m(F(x))]^{s-r-1}\,dx,
\end{aligned}
$$

which can be written as

$$
\int_y^\infty x^{a-1}\,e^{t\,x^a}[\bar F(x)]^{\gamma_s-1}\,[h_m(F(x))-h_m(F(x))]^{s-r-1}\left[\bar F(x)-1-\frac{f(x)\,(1-e^{\lambda(x)})}{\theta\,\lambda'(x)}\right]dx=0.
$$

Applying the extension of Müntz-Szàsz theorem, we get

$$
\bar F(x) = 1 + \frac{(1-e^{\lambda(x)})}{\theta\,\lambda'(x)}\,f(x).
$$

Remark 3.1. By differentiating both sides of Condition (25) with respect to t and then setting $t=0$, we obtain the following recurrence relation for moments of gOSs

$$
\begin{aligned}
&E[X^a_{s;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}=y] - E[X^a_{s-1;\,n;\,m;\,k}|X_{r;\,n;\,m;\,k}=y] \\
&= \frac{a}{\gamma_s\,\theta}\,E\Bigg[\frac{X^{(a-1)}_{s;\,n;\,m;\,k}}{\lambda'(X_{s;\,n;\,m;\,k})}\Bigg(1-e^{\lambda(X_{s;\,n;\,m;\,k})}+e^{\lambda(X_{s;\,n;\,m;\,k})} \\
&\quad\times\left(1-e^{-\lambda(X_{s;\,n;\,m;\,k})}\right)^{1-\theta}\Bigg)|X_{r;\,n;\,m;\,k}=y\Bigg],\ m\ge -1.
\end{aligned}
\tag{30}
$$

If we set $\theta = 1$ in (25) and (30) we get

$$\begin{aligned} & M_{X^a_{s;n;m;k}|X_{r;n;m;k}}(t|y) - M_{X^a_{s-1;n;m;k}|X_{r;n;m;k}}(t|y) \\ & = \frac{a\,t}{\gamma_s}\,E\left[\frac{X^{(a-1)}_{s;n;m;k}\ e^{t\,X^{(a)}_{s;n;m;k}}}{\lambda'(X_{s;n;m;k})}\,|\,X_{r;n;m;k}=y\right],\ m\geq -1. \end{aligned} \tag{31}$$

$$\begin{aligned} & E[X^a_{s;n;m;k}|X_{r;n;m;k}=y] - E[X^a_{s-1;n;m;k}|X_{r;n;m;k}=y] \\ & = \frac{a}{\gamma_s}\,E\left[\frac{X^{(a-1)}_{s;n;m;k}}{\lambda'(X_{s;n;m;k})}|X_{r;n;m;k}=y\right],\ m\geq -1. \end{aligned} \tag{32}$$

Eqs. (31) and (32) coincide with the result, given by AL-Hussaini et al [5].

Remark 3.2. If we put $m = 0$ and $k = 1$ in (25) and (30), we obtain the recurrence relations of oOSs, $[\gamma_s = n - s + 1$ and $X_{s;n;m;k} \equiv X_{s;n}]$ as follows

$$\begin{aligned} & M_{X^a_{s;n}|X_{r;n}}(t|y) - M_{X^a_{s-1;n}|X_{r;n}}(t|y) \\ & = \frac{a\,t}{\theta\,(n-s+1)}\,E\left[\frac{X^{(a-1)}_{s;n}\ e^{t\,X^{(a)}_{s;n}}}{\lambda'(X_{s;n})}\left(1-e^{\lambda(X_{s;n})}+e^{\lambda(X_{s;n})}\,[1-e^{-\lambda(X_{s;n})}]^{1-\theta}\right)|\,X_{r;n}=y\right]. \end{aligned} \tag{33}$$

$$\begin{aligned} & E[X^a_{s;n}|X_{r;n}=y] - E[X^a_{s-1;n}|X_{r;n}=y] \\ & = \frac{a}{\theta\,(n-s+1)}\,\lambda'(X_{s;n})\left(1-e^{\lambda(X_{s;n})}+e^{\lambda(X_{s;n})}\left(1-e^{-\lambda(X_{s;n})}\right)^{1-\theta}\right)|X_{r;n}=y\Bigg]. \end{aligned} \tag{34}$$

Remark 3.3. If we put $m = -1$ and $k = 1$ in (25) and (30), the following relations of oRVs, $[\gamma_s = k$ and $X_{s;n;m;k} \equiv X_{U(s)}]$ can be deduced

$$\begin{aligned} & M_{X^a_{U(s)}|X_{U(r)}}(t|y) - M_{X^a_{U(s-1)}|X_{U(r)}}(t|y) \\ & = \frac{a\,t}{\theta}\,E\left[\frac{X^{(a-1)}_{U(s)}\ e^{t\,X^{(a)}_{U(s)}}}{\lambda'(X_{U(s)})}\left(1-e^{\lambda(X_{U(s)})}\ +\ e^{\lambda(X_{U(s)})}\,[1-e^{-\lambda(X_{U(s)})}]^{1-\theta}|\,X_{U(r)}=y\right)\right]. \end{aligned} \tag{35}$$

$$\begin{aligned} & E[X^a_{U(s)}|X_{U(r)}=y] - E[X^a_{s-1}|X_{U(r)}=y] \\ & = \frac{a}{\theta}\,E\left[\frac{X^{(a-1)}_{U(s)}}{\lambda'(X_{U(s)})}\left(1-e^{\lambda(X_{U(s)})}+e^{\lambda(X_{U(s)})}\left(1-e^{-\lambda(X_{U(s)})}\right)^{1-\theta}\right)|X_{U(r)}=y\right]. \end{aligned} \tag{36}$$

4. Characterizations Based on Recurrence Relations for Product Moments of gOSs

Lemma 4.1. *Mahmoud and Al-Nagar [13]. For every absolutely continuous function $\phi(x,y)$, integers $0<r<s\le n$ and real m,k ,$k\ge 1$*

$$\begin{aligned} &E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s;\,n;\,m;\,k}\Big)\Big] - E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s-1;\,n;\,m;\,k}\Big)\Big] \\ &= \frac{C_{s-2}}{(r-1)!\,(s-r-1)!} \int_0^\infty \int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\, f(x)\,[\bar F(x)]^m\, g_m^{r-1}(F(x)) \\ &\quad \times\, [h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s} dy\,dx, \qquad x\ge 0. \end{aligned} \tag{37}$$

Theorem 4.1. *Let X be a random variable with distribution $F(x)$, Then for every continuous function $\phi(x,y)$, some integers $0<r<s<n$ and real m, k with $k\ge 1, k+m>0$, the following recurrence relation is satisfied iff X has the cdf (1)*

$$\begin{aligned} &E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s;\,n;\,m;\,k}\Big)\Big] - E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s-1;\,n;\,m;\,k}\Big)\Big] \\ &= \frac{1}{\theta\,\gamma_s}\, E\Big[\xi\Big(X_{r;\,n;\,m;\,k}, X_{s;\,n;\,m;\,k}\Big)\Big], \end{aligned} \tag{38}$$

where

$$\xi(x,y) = \left[\frac{\phi'(x,y)}{\lambda'(y)}\Big(1 - e^{\lambda(y)} +\, e^{\lambda(y)}\,[1-e^{-\lambda(y)}]^{1-\theta}\Big)\right]. \tag{39}$$

Proof. From (37) and (3), we have

$$\begin{aligned} &E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s;\,n;\,m;\,k}\Big)\Big] - E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s-1;\,n;\,m;\,k}\Big)\Big] \\ &= \frac{C_{s-2}}{(r-1)!\,(s-r-1)!} \int_0^\infty \int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\, f(x)\,[\bar F(x)]^m\, g_m^{r-1}(F(x)) \\ &\quad \times\, [h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\Big[1 + \frac{(1-e^{\lambda(y)})}{\theta\,\lambda'(y)}\, f(y)\Big] dy\,dx, \end{aligned}$$

or equivalently,

$$\begin{aligned} &E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s;\,n;\,m;\,k}\Big)\Big] - E\Big[\phi\Big(X_{r;\,n;\,m;\,k}, X_{s-1;\,n;\,m;\,k}\Big)\Big] \\ &= \frac{C_{s-2}}{\theta\,(r-1)!\,(s-r-1)!} \int_0^\infty \int_x^\infty \Big(\frac{\partial\,\phi(x,y)}{\partial\,y}\,\frac{(1-e^{\lambda(y)})}{\lambda'(y)}\Big) f(x)\,[\bar F(x)]^m\, g_m^{r-1}(F(x)) \\ &\times\, [h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\, f(y)\,dy\,dx \\ &+\, \frac{C_{s-2}}{(r-1)!\,(s-r-1)!} \int_0^\infty \int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\, f(x)\,[\bar F(x)]^m\, g_m^{r-1}(F(x)) \\ &\times\, [h_m(F(y)) - h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\,dy\,dx. \end{aligned} \tag{40}$$

The right hand side of (40) can be written as

$$\begin{aligned} &\frac{C_{s-2}}{\theta\,(r-1)!\,(s-r-1)!}\int_0^\infty\int_x^\infty \Big(\frac{\partial\,\phi(x,y)}{\partial\,y}\,\frac{(1-e^{\lambda(y)})}{\lambda'(y)}\Big)\,f(x)\,[\bar F(x)]^m\,g_m^{r-1}(F(x)) \\ &\quad\times\,[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\,f(y)\,dy\,dx \\ &\quad+\,\frac{C_{s-2}}{(r-1)!\,(s-r-1)!}\int_0^\infty\int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\,f(x)\,[\bar F(x)]^m\,g_m^{r-1}(F(x)) \\ &\quad\times\,[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\left[\frac{e^{\lambda(y)}\,f(y)\,[1-e^{-\lambda(y)}]^{1-\theta}}{\theta\,\lambda'(y)}\right]dy\,dx. \end{aligned} \tag{41}$$

Eq. (41), can be written in the following form

$$\begin{aligned} &\frac{1}{\theta\,\gamma_s}\,\frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\int_0^\infty\int_x^\infty \xi(x,y)\,[\bar F(x)]^m\,g_m^{r-1}(F(x)) \\ &\quad\times\,[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}f(x)\,f(y)\,dy\,dx, \end{aligned} \tag{42}$$

where $\xi(x,y)$ is given in (39). We get

$$\frac{1}{\theta\,\gamma_s}\,E\Big[\xi\Big(X_{r;\,n;\,m;\,k},X_{s;\,n;\,m;\,k}\Big)\Big]. \tag{43}$$

Using (39) in (43), the result is proved. Conversely, if the characterizing Condition (38), $m\ge -1$ is satisfied, then from (37) and (40), we get

$$\begin{aligned} &\frac{C_{s-2}}{(r-1)!\,(s-r-1)!}\int_0^\infty\int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\,f(x)\,[\bar F(x)]^m\,g_m^{r-1}(F(x)) \\ &\quad\times\,[h_m(F(y))-h_m((x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s}dy\,dx \\ &=\frac{C_{s-2}}{\theta\,(r-1)!\,(s-r-1)!}\int_0^\infty\int_x^\infty \Big(\frac{\partial\,\phi(x,y)}{\partial\,y}\,\frac{(1-e^{\lambda(y)})}{\lambda'(y)}\Big)\,f(x)\,[\bar F(x)]^m\,g_m^{r-1}(F(x)) \\ &\quad\times\,[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\,f(y)\,dy\,dx \\ &\quad+\,\frac{C_{s-2}}{(r-1)!\,(s-r-1)!}\int_0^\infty\int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\,f(x)\,[\bar F(x)]^m\,g_m^{r-1}(F(x)) \\ &\quad\times\,[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1}\,dy\,dx, \end{aligned} \tag{44}$$

which can be written as

$$\begin{aligned} &\int_0^\infty\int_x^\infty \frac{\partial\,\phi(x,y)}{\partial\,y}\,f(x)\,[\bar F(x)]^m\,g_m^{r-1}(F(x))\,[h_m(F(y))-h_m(F(x))]^{s-r-1}\,[\bar F(y)]^{\gamma_s-1} \\ &\quad\times\,\left[\bar F(y)-1-\frac{f(y)\,(1-e^{\lambda(y)})}{\theta\,\lambda'(y)}\right]dy\,dx=0. \end{aligned}$$

After simplifying we obtain

$$\bar{F}(y) = 1 + \frac{(1 - e^{\lambda(y)})}{\theta\, \lambda'(y)}\, f(y).$$

Setting $\theta = 1$ in (38) gives

$$\begin{aligned} &E\Big[\phi\Big(X_{r,n,m,k}, X_{s,n,m,k}\Big)\Big] - E\Big[\phi\Big(X_{r,n,m,k}, X_{s-1,n,m,k}\Big)\Big] \\ &= \frac{1}{\theta\, \gamma_s}\, E\Big[\eta\Big(X_{r,n,m,k}, X_{s,n,m,k}\Big)\Big], \end{aligned} \tag{45}$$

where $\eta(x, y) = \frac{\phi'(x,y)}{\lambda'(y)}$.

Remark 4.1. If we put $m = 0$ and $k = 1$ in (38), we obtain the recurrence relation of oOSs, $[\gamma_s = n - s + 1$ and $X_{s;\,n;\,m;\,k} \equiv X_{s;\,n}]$ as follows

$$\begin{aligned} &E\Big[\phi\Big(X_{r,n}, X_{s,n}\Big)\Big] - E\Big[\phi\Big(X_{r,n}, X_{s-1,n}\Big)\Big] \\ &= \frac{1}{\theta\,(n - s + 1)}\, E\Big[\xi\Big(X_{r,n}, X_{s,n}\Big)\Big]. \end{aligned} \tag{46}$$

Remark 4.2. If we put $m = -1$ and $k = 1$ in (38), we obtain the recurrence relation of oRVs, $[\gamma_s = k$ and $X_{s;\,n;\,m;\,k} \equiv X_{U(s)}]$ as follows

$$E\Big[\phi\Big(X_{U(r)}, X_{U(s)}\Big)\Big] - E\Big[\phi\Big(X_{U(r)}, X_{U(s-1)}\Big)\Big] = \frac{1}{\theta}\, E\Big[\xi\Big(X_{U(r)}, X_{U(s)}\Big)\Big] \tag{47}$$

Remark 4.3. If we choose $\phi(x, y) = x^a\, y^b\,, a,\, b \geq 0$, then the relations (38), (46) and (47) become

$$\begin{aligned} \mu^{(a,b)}_{r;s;\,n;\,m;\,k} - \mu^{(a,b)}_{r;s-1;\,n;\,m;\,k} = \frac{b}{\theta\, \gamma_s}\, E\Bigg[\frac{X^{(a)}_{r;\,n;\,m;\,k}\, X^{(b-1)}_{s;\,n;\,m;\,k}}{\lambda'(X_{s;\,n;\,m;\,k})}\, \Bigg(1 - e^{\lambda(X_{s;\,n;\,m;\,k})} \\ + e^{\lambda(X_{s;\,n;\,m;\,k})}\, [1 - e^{-\lambda(X_{s;\,n;\,m;\,k})}]^{1-\theta}\Bigg)\Bigg], m \geq -1. \end{aligned} \tag{48}$$

$$\mu^{(a,b)}_{r;s;\,n} - \mu^{(a,b)}_{r;s-1;\,n} = \frac{b}{\theta\,(n - s + 1)}\, E\Bigg[\frac{X^{(a)}_{r;\,n}\, X^{(b-1)}_{s;\,n}}{\lambda'(X_{s;\,n})}\, \Bigg(1 - e^{\lambda(X_{s;\,n})} + e^{\lambda(X_{s;\,n})}\, [1 - e^{-\lambda(X_{s;\,n})}]^{1-\theta}\Bigg)\Bigg]. \tag{49}$$

$$\mu^{(a,b)}_{r;s} - \mu^{(a,b)}_{r;s-1} = \frac{b}{\theta}\, E\Bigg[\frac{X^{(a)}_{U(r)}\, X^{(b-1)}_{U(s)}}{\lambda'(X_{U(s)})}\, \Bigg(1 - e^{\lambda(X_{U(s)})} + e^{\lambda(X_{U(s)})}\, [1 - e^{-\lambda(X_{U(s)})}]^{1-\theta}\Bigg)\Bigg]. \tag{50}$$

If $\theta = 1$ in Eq. (49) coincides with the result, given by AL-Hussaini et al [6].

The following table (1) gives some distributions with proper choice of $\lambda(x)$ as examples on Theorems (2.1), (3.1) and (4.1).

Table (1) examples of cdf (1) distributions

Distribution	cdf	$\lambda(x)$
exponentiated linear failure rate	$[1-e^{-(a\,x+\frac{b}{2}\,x^2)}]^\theta$	$(a\,x+\frac{b}{2}\,x^2),\ a,\,b\geq 0,\ x>0$
exponentiated Weibull	$[1-e^{-\alpha\,x^\beta}]^\theta$	$\alpha\,x^\beta,\ \alpha,\,x,\,\beta>0$
exponentiated Rayleigh	$[1-e^{-\alpha\,x^2}]^\theta$	$\alpha\,x^2,\ x,\,\alpha>0$
exponentiated exponential	$[1-e^{-\alpha\,x}]^\theta$	$\alpha\,x,\ x,\,\alpha>0$
exponentiated modified Weibull	$[1-e^{-\alpha\,x^\beta\,e^{\gamma\,x}}]^\theta$	$\alpha\,x^\beta\,e^{\gamma\,x},\ x,\,\alpha>0,\ \beta,\,\gamma\geq 0$
exponentiated Gompertz	$[1-e^{-\frac{\alpha}{c}\,(e^{c\,x}-1)}]^\theta$	$\frac{\alpha}{c}\,(e^{c\,x}-1),\ x,\,\alpha>0,\ c\geq 0$
exponentiated Burr Type XII	$[1-(1+x^\beta)^{-\alpha}]^\theta$	$\alpha\,\ln(1+x^\beta),\ x,\,\alpha,\,\beta>0$
exponentiated Lomax	$[1-(1+\beta\,x)^{-\alpha}]^\theta$	$\alpha\,\ln(1+\beta\,x)\,,x,\,\alpha,\,\beta>0$
exponentiated Pareto	$[1-(1+x)^{-\alpha}]^\theta$	$\alpha\,\ln(1+x),\ x,\,\alpha>0$
exponentiated Gamma	$[1-e^{-\alpha\,x}\,(1+\alpha\,x)]^\theta$	$\alpha\,x-\ln(1+\alpha\,x),\ x,\,\alpha>0$

5. Conclusions

This paper deals with the generalized order statistics based on exponentiated family of distributions. Recurrence relations for moment, conditional moment generating functions and product moments are derived. These recurrence relations are used to characterize this family.

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