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TRIANGLE PRINCIPLE OF INERTIAL TRANSPORT IN INCOMPRESSIBLE TURBULENCE

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Abstract: It is proved that the inertial energy fluxes among any three Fourier components, where one wave vector is the sum or difference of the other two wave vectors, must be closed. This result is called *triangle principle* because such three vectors can form a triangle. Although the global energy conservation of inertial transport, which states that inertial transport does not change the total energy but only redistributes the energy among different Fourier components, is well known, the triangle principle gives the detailed mechanism for the global energy conservation and can therefore be called *detailed energy conservation* in inertial transport. A mechanism, similar to *Darwin's natural selection*, for the establishment of the quasi-equilibrium energy spectrum of dissipation range is discussed. Thus, the present analysis gives one example to show how detailed dynamical analysis may help to understand the establishment of certain statistical regularity.

Keywords: turbulence; transport.

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1. Introduction

There have been two major purposes for the research of turbulence: one for engineering applications, and the other one for better understanding of turbulence. Before the high-speed computers became available, there had been two major approaches for the

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two purposes respectively: the semi-empirical approach, represented by Prandtl's mixing length theory, and the statistical theory of homogeneous isotropic turbulence. (For an introductory review of the early development of the researches of turbulence, see e.g. Yih [1]. For more comprehensive reviews, see e.g. Hinze [2], Batchelor [3], and Lin [4]). Just as the Reynolds equations for the averaged quantities in the semi-empirical approach had the well-known non-closure problem represented by the unknown Reynolds stress tensor, the statistical theory of homogeneous isotropic turbulence also had non-closure problem. All the efforts trying to get exact close equations to govern the temporal evolution of the correlation function and the energy spectrum failed. Some artificial assumptions had to be proposed by various researchers in order to close the equations approximately.

The invention and development of high-speed computers made it possible to do direct numerical simulation (DNS) for the details of turbulent flow fields. Again, one type of DNS was to simulate specific turbulent flows under different boundary conditions for various engineering problems, while the other type of DNS, represented by box turbulence, was trying to get better understanding of turbulence. (For reviews of DNS, see e.g. Moin & Mahesh [5], and Ishihara, Gotoh & Kaneda [6].) The study of box turbulence started by the pioneering work of Orszag and Patterson (1972, [7]) with a modest resolution 32^3. Afterwards, improvements have been made by many researchers with better resolutions and higher Reynolds numbers by using more advanced computing facilities. For example, Chen et al. (1993, [8]) and She et al. (1993, [9]) carried out simulations at resolution 512^3. Ishihara et al. (2007, [10]) performed simulation with the resolution up to 4096^3.

Inertial energy transport and viscous energy dissipation are the two major aspects for the temporal change of energy spectrum of turbulence. Since the viscous dissipation is relatively simple, the present study will focus on the mechanism of inertial energy transport in terms of an analytical approach. For this purpose, we consider an incompressible turbulent flow in 3D space without any boundary, where the wave vectors of all Fourier components are linear combinations with integer coefficients of a limited number of starting wave vectors. All the Fourier components must be transverse waves because of the incompressibility of the fluid. Namely, for each wave, the amplitude vectors of velocity must be always perpendicular to the wave vector. This model can be called *multi-wave turbulence*. The physical relevance of this model is the following. The transition from laminar to turbulent flows starts from the loss of stability of the laminar flow. A few leading unstable modes with highest growth rates would dominate the transition process and create more and more waves through the nonlinear inertial term of Navier-Stokes equations and finally lead to turbulent flow.

The present study differs from the studies of *box turbulence* in two aspects. First, multi-wave turbulence does not have to be periodic in any direction. Actually, multi-wave turbulence is a generalization of box turbulence. In the special case that there are only three starting wave vectors of equal lengths which are perpendicular to each other, multi-wave turbulence would be reduced to box turbulence. Second, all the studies of box turbulence focused on the small-scale statistics, particularly on the comparison with Kolmogorov's similarity hypotheses ([11], [12]), while the present study focuses on the wave interactions, particularly the detailed mechanism of inertial energy transport, through an analytical approach.

2. Wave interactions

2.1. Single and multi- transverse waves as exact solutions of Navier-Stokes equations

We start from Navier-Stokes equations for viscous incompressible flow:

$$\operatorname{div} \vec{v} = 0 \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \Delta \vec{v} \tag{2}$$

where $\vec{v}$ is the velocity, p and ν are the pressure and the kinematic viscosity respectively, and the density ρ is assumed to be a constant. It is easy to see that the following single transverse wave is an exact solution of Navier-Stokes equations (1) and (2):

$$\vec{v} = \vec{v}_s + \vec{v}_c = [\vec{A}_s \sin(\vec{k}\cdot\vec{r}) + \vec{A}_c \cos(\vec{k}\cdot\vec{r})]e^{-\nu k^2 t} \tag{3}$$

$$p = const. \tag{4}$$

where the constant amplitude vectors $\vec{A}_s$ and $\vec{A}_c$ are both perpendicular to the wave vector $\vec{k}$:

$$\vec{A}_s \cdot \vec{k} = 0 \qquad \vec{A}_c \cdot \vec{k} = 0 \tag{5}$$

A noticeable property of such a single transverse wave is that the inertial term is zero:

$$(\vec{v}\cdot\nabla)\vec{v} = 0 \tag{6}$$

Therefore, the wave only decays by viscosity. Now we consider the combination of transverse waves

$$\vec{v} = \sum_{i=1}^{n} \vec{v}_i = \sum_{i=1}^{n} (\vec{v}_{si} + \vec{v}_{ci}) = \sum_{i=1}^{n} [\vec{A}_{si} \sin(\vec{k}_i \cdot \vec{r}) + \vec{A}_{ci} \cos(\vec{k}_i \cdot \vec{r})]e^{-\nu k_i^2 t} \tag{7}$$

where the constant amplitude vectors $\vec{A}_{si}$ and $\vec{A}_{ci}$ are both perpendicular to the wave vector $\vec{k}_i$ and independent of *t*. We will show that in the following two cases, (7) and (4) will still be the exact solution of Navier-Stokes equations:

Case (i): All wave vectors are in the same direction. Namely,

$$\vec{k}_i = k_i \vec{\alpha} \quad (i = 1, \ldots, n) \tag{8}$$

and

$$\vec{A}_{si} \cdot \vec{\alpha} = 0 \qquad \vec{A}_{ci} \cdot \vec{\alpha} = 0 \tag{9}$$

where $\vec{\alpha}$ is a given unit vector.

Case (ii): All wave vectors are in the same plane, while all amplitude vectors are perpendicular to this plane. If we denote this plane by *x*-*y* plane in a Cartesian system, we have

$$\vec{k}_i = k_{ix}\vec{e}_x + k_{iy}\vec{e}_y \quad (i = 1, \ldots, n) \tag{10}$$

and

$$\vec{A}_{si} = A_{si}\vec{e}_z \qquad \vec{A}_{ci} = A_{ci}\vec{e}_z \quad (i = 1, \ldots, n) \tag{11}$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are the unit vectors in *x*, *y* and *z* directions of the Cartesian system respectively. It is easy to see that in Case (i) and Case (ii), every amplitude vector is perpendicular to all wave vectors:

$$\vec{A}_{si} \cdot \vec{k}_j = 0 \qquad \vec{A}_{ci} \cdot \vec{k}_j = 0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n) \tag{12}$$

Thus, for these two cases, equation (6) is still valid. Namely, all the single waves included in equation (7) do not interact with each other. Each wave decays individually and independently from others. The damping speed of each wave depends on its own wave number. Except the above two cases, the condition (6) is generally not valid, and the interaction among individual waves would be inevitable. Thus, Equation (7) would not be the solution of Navier-Stokes equations any more.

2.2. Wave interactions

Now let us consider the combination of two transverse waves at the initial moment t=0:

$$\vec{v}(\vec{r},0)=\vec{v}_1+\vec{v}_2 \tag{13}$$

where

$$\vec{v}_i=\vec{A}_{si}\sin(\vec{k}_i\cdot\vec{r})+\vec{A}_{ci}\cos(\vec{k}_i\cdot\vec{r}) \qquad (i=1,2) \tag{14}$$

$$\vec{A}_{si}\cdot\vec{k}_i=0 \qquad \vec{A}_{ci}\cdot\vec{k}_i=0 \qquad (i=1,2) \tag{15}$$

Taking equation (15) into account, we can easily derive

$$(\vec{v}\cdot\nabla)\vec{v}=(\vec{v}_1\cdot\nabla)\vec{v}_2+(\vec{v}_2\cdot\nabla)\vec{v}_1 \tag{16}$$

where

$$\begin{aligned}
&(\vec{v}_1\cdot\nabla)\vec{v}_2=\\
&[(\vec{A}_{s1}\cdot\vec{k}_2)\sin(\vec{k}_1\cdot\vec{r})+(\vec{A}_{c1}\cdot\vec{k}_2)\cos(\vec{k}_1\cdot\vec{r})]\cdot\\
&[\vec{A}_{s2}\cos(\vec{k}_2\cdot\vec{r})-\vec{A}_{c2}\sin(\vec{k}_2\cdot\vec{r})]\\
&=\frac{1}{2}[(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{s2}-(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{c2}]\sin[(\vec{k}_1+\vec{k}_2)\cdot\vec{r}]\\
&+\frac{1}{2}[(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{s2}+(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{c2}]\cos[(\vec{k}_1+\vec{k}_2)\cdot\vec{r}]\\
&+\frac{1}{2}[(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{s2}+(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{c2}]\sin[(\vec{k}_1-\vec{k}_2)\cdot\vec{r}]\\
&+\frac{1}{2}[(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{s2}-(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{c2}]\cos[(\vec{k}_1-\vec{k}_2)\cdot\vec{r}]
\end{aligned} \tag{17}$$

By exchanging the subscripts 1 and 2, we have

$$
\begin{aligned}
&(\vec{v}_2 \cdot \nabla)\vec{v}_1 = \\
&\frac{1}{2}[(\vec{A}_{s2} \cdot \vec{k}_1)\vec{A}_{s1} - (\vec{A}_{c2} \cdot \vec{k}_1)\vec{A}_{c1}]\sin[(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}] \\
&+\frac{1}{2}[(\vec{A}_{c2} \cdot \vec{k}_1)\vec{A}_{s1} + (\vec{A}_{s2} \cdot \vec{k}_1)\vec{A}_{c1}]\cos[(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}] \\
&-\frac{1}{2}[(\vec{A}_{s2} \cdot \vec{k}_1)\vec{A}_{s1} + (\vec{A}_{c2} \cdot \vec{k}_1)\vec{A}_{c1}]\sin[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r}] \\
&+\frac{1}{2}[(\vec{A}_{c2} \cdot \vec{k}_1)\vec{A}_{s1} - (\vec{A}_{s2} \cdot \vec{k}_1)\vec{A}_{c1}]\cos[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r}]
\end{aligned}
\tag{18}
$$

Thus, we have

$$
\begin{aligned}
&(\vec{v} \cdot \nabla)\vec{v} = \\
&\vec{F}_s(\vec{k}_1, \vec{A}_{s1}, \vec{A}_{c1}, \vec{k}_2, \vec{A}_{s2}, \vec{A}_{c2})\sin[(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}] + \\
&\vec{F}_c(\vec{k}_1, \vec{A}_{s1}, \vec{A}_{c1}, \vec{k}_2, \vec{A}_{s2}, \vec{A}_{c2})\cos[(\vec{k}_1 + \vec{k}_2) \cdot \vec{r}] + \\
&\vec{G}_s(\vec{k}_1, \vec{A}_{s1}, \vec{A}_{c1}, \vec{k}_2, \vec{A}_{s2}, \vec{A}_{c2})\sin[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r}] + \\
&\vec{G}_c(\vec{k}_1, \vec{A}_{s1}, \vec{A}_{c1}, \vec{k}_2, \vec{A}_{s2}, \vec{A}_{c2})\cos[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r}]
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
&\vec{F}_s(\vec{k}_1, \vec{A}_{s1}, \vec{A}_{c1}, \vec{k}_2, \vec{A}_{s2}, \vec{A}_{c2}) = \\
&\frac{1}{2}[(\vec{A}_{s1} \cdot \vec{k}_2)\vec{A}_{s2} - (\vec{A}_{c1} \cdot \vec{k}_2)\vec{A}_{c2} + (\vec{A}_{s2} \cdot \vec{k}_1)\vec{A}_{s1} - (\vec{A}_{c2} \cdot \vec{k}_1)\vec{A}_{c1}]
\end{aligned}
\tag{20}
$$

$$\vec{F}_c(\vec{k}_1,\vec{A}_{s1},\vec{A}_{c1},\vec{k}_2,\vec{A}_{s2},\vec{A}_{c2}) =$$
$$\frac{1}{2}[(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{s2}+(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{c2}+(\vec{A}_{c2}\cdot\vec{k}_1)\vec{A}_{s1}+(\vec{A}_{s2}\cdot\vec{k}_1)\vec{A}_{c1}]$$

(21)

$$\vec{G}_s(\vec{k}_1,\vec{A}_{s1},\vec{A}_{c1},\vec{k}_2,\vec{A}_{s2},\vec{A}_{c2}) =$$
$$\frac{1}{2}[(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{s2}+(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{c2}-(\vec{A}_{s2}\cdot\vec{k}_1)\vec{A}_{s1}-(\vec{A}_{c2}\cdot\vec{k}_1)\vec{A}_{c1}]$$

(22)

$$\vec{G}_c(\vec{k}_1,\vec{A}_{s1},\vec{A}_{c1},\vec{k}_2,\vec{A}_{s2},\vec{A}_{c2}) =$$
$$\frac{1}{2}[(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{s2}-(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{c2}+(\vec{A}_{c2}\cdot\vec{k}_1)\vec{A}_{s1}-(\vec{A}_{s2}\cdot\vec{k}_1)\vec{A}_{c1}]$$

(23)

The right hand side of equation (19) represents two new waves with wave vectors $\vec{k}_1+\vec{k}_2$ and $\vec{k}_1-\vec{k}_2$ respectively. Incompressibility requires that the two new waves have to be also transverse. However, the amplitude vectors in equation (19) are generally not perpendicular to the wave vectors. Thus, only the transverse components of these amplitude vectors contribute to the new waves velocity components, while the longitudinal components, i.e. the components of the amplitude vectors parallel to the new wave vectors, will be cancelled out by the induced pressure field.

3. General solution of multi-wave incompressible turbulence starting from a finite number of transverse waves

Now we assume that the initial velocity field has only finite number of transverse waves

$$\vec{\mathrm{v}}(\vec{r},0)=\sum_{i=1}^{n}[\vec{A}_{si,0}\sin(\vec{k}_i\cdot\vec{r})+\vec{A}_{ci,0}\cos(\vec{k}_i\cdot\vec{r})] \tag{24}$$

where

$$\vec{A}_{si,0} \cdot \vec{k}_i = 0 \qquad \vec{A}_{ci,0} \cdot \vec{k}_i = 0 \qquad (i = 1, \ldots\ldots, n) \tag{25}$$

The interactions of all pairs of waves would generate more and more transverse waves as time increases by the mechanism discussed in the last section. Every generated wave vector must be a linear combination of the initial wave vectors with integer coefficients, i.e. in the form

$$\vec{k} = m_1 \vec{k}_1 + m_2 \vec{k}_2 + \ldots\ldots + m_n \vec{k}_n \tag{26}$$

Physically, the generated waves with extremely high wave numbers will be suppressed by viscosity. Thus, the final set of wave vectors developed by the initial condition (24) will be finite, denoted by κ in the (k_x, k_y, k_z)-space, shortly *k*-space. The corresponding set in (m_1, m_2, ……,m_n)-space, shortly *m*-space, is denoted by μ. Namely,

$$\vec{k} \in \kappa \quad \leftrightarrow \quad (m_1, \ldots, m_n) \in \mu \tag{27}$$

The solution can be expressed as

$$\vec{\mathrm{v}}(\vec{r}, t) = \sum_j [\vec{A}_{sj}(t) \sin(\vec{k}_j \cdot \vec{r}) + \vec{A}_{cj}(t) \cos(\vec{k}_j \cdot \vec{r})] \tag{28}$$

where

$$\vec{A}_{sj}(t) \cdot \vec{k}_j = 0 \qquad \vec{A}_{cj}(t) \cdot \vec{k}_j = 0 \tag{29}$$

and the summation on the right hand side of equation (28) is over all

$$\vec{k}_j = m_{1j}\vec{k}_1 + m_{2j}\vec{k}_2 + ... + m_{nj}\vec{k}_n \tag{30}$$

in κ, or equivalently over all (m_{1j}, …, m_{nj}) in μ.

For the general expression of velocity (28), we denote

$$(\vec{v}\cdot\nabla)\vec{v} = \sum_j [\vec{B}_{sj}(t)\sin(\vec{k}_j\cdot\vec{r}) + \vec{B}_{cj}(t)\cos(\vec{k}_j\cdot\vec{r})] \tag{31}$$

The amplitude vectors $\vec{B}_{sj}(t)$ and $\vec{B}_{cj}(t)$ can be decomposed into components perpendicular and parallel to the wave vector $\vec{k}_j$ respectively:

$$\vec{B}_{sj}(t) = \vec{B}_{sj,\perp}(t) + \vec{B}_{sj,//}(t) \qquad \vec{B}_{cj}(t) = \vec{B}_{cj,\perp}(t) + \vec{B}_{cj,//}(t) \tag{32}$$

Thus, for the longitudinal and the transverse components of the inertial term, we have

$$[(\vec{v}\cdot\nabla)\vec{v}]_{//} = \sum_j [\vec{B}_{sj,//}(t)\sin(\vec{k}_j\cdot\vec{r}) + \vec{B}_{cj,//}(t)\cos(\vec{k}_j\cdot\vec{r})] \tag{33}$$

$$[(\vec{v}\cdot\nabla)\vec{v}]_{\perp} = \sum_j [\vec{B}_{sj,\perp}(t)\sin(\vec{k}_j\cdot\vec{r}) + \vec{B}_{cj,\perp}(t)\cos(\vec{k}_j\cdot\vec{r})] \tag{34}$$

By applying equation (19) to each pair of ($\vec{k}_i, \vec{k}_j$) in equation (28), we have

$$\begin{aligned}\vec{B}_{sj}(t) = &\sum_{i_1,i_2} \vec{F}_s(\vec{k}_{i_1}, \vec{A}_{si_1}, \vec{A}_{ci_1}, \vec{k}_{i_2}, \vec{A}_{si_2}, \vec{A}_{ci_2}) + \\ &\sum_{j_1,j_2} \vec{G}_s(\vec{k}_{j_1}, \vec{A}_{sj_1}, \vec{A}_{cj_1}, \vec{k}_{j_2}, \vec{A}_{sj_2}, \vec{A}_{cj_2})\end{aligned} \tag{35}$$

$$\vec{B}_{cj}(t) = \sum_{i_1, i_2} \vec{F}_c(\vec{k}_{i_1}, \vec{A}_{si_1}, \vec{A}_{ci_1}, \vec{k}_{i_2}, \vec{A}_{si_2}, \vec{A}_{ci_2}) + \sum_{j_1, j_2} \vec{G}_c(\vec{k}_{j_1}, \vec{A}_{sj_1}, \vec{A}_{cj_1}, \vec{k}_{j_2}, \vec{A}_{sj_2}, \vec{A}_{cj_2}) \quad (36)$$

The two double summations on the right hand side of each of Equations (35) and (36) should be over all the pairs ($\vec{k}_{i_1}, \vec{k}_{i_2}$) and all the pairs ($\vec{k}_{j_1}, \vec{k}_{j_2}$) respectively such that

$$\vec{k}_{i_1} \neq \vec{k}_{i_2}, \quad \vec{k}_{i_1} \in \kappa, \quad \vec{k}_{i_2} \in \kappa, \quad \vec{k}_{i_1} + \vec{k}_{i_2} = \vec{k}_j \quad (37)$$

$$\vec{k}_{j_1} \neq \vec{k}_{j_2}, \quad \vec{k}_{j_1} \in \kappa, \quad \vec{k}_{j_2} \in \kappa, \quad \vec{k}_{j_1} - \vec{k}_{j_2} = \vec{k}_j \quad (38)$$

For the term of viscous force, we have

$$\nu \Delta \vec{v} = -\nu \sum_j k_j^2 [\vec{A}_{sj}(t) \sin(\vec{k}_j \cdot \vec{r}) + \vec{A}_{cj}(t) \cos(\vec{k}_j \cdot \vec{r})] \quad (39)$$

The longitudinal and the transverse components of the inertial term (equations (33) and (34)) contribute to the induced pressure field and the temporal change of the velocity components respectively as follows:

$$\nabla p = -\rho \sum_j [\vec{B}_{sj,//}(t) \sin(\vec{k}_j \cdot \vec{r}) + \vec{B}_{cj,//}(t) \cos(\vec{k}_j \cdot \vec{r})] \quad (40)$$

$$\frac{d\vec{A}_{sj}}{dt} = -\vec{B}_{sj,\perp}(t) - \nu k_j^2 \vec{A}_{sj} \quad (41)$$

$$\frac{d\vec{A}_{cj}}{dt} = -\vec{B}_{cj,\perp}(t) - \nu k_j^2 \vec{A}_{cj} \tag{42}$$

4. Energy equation, inertial transport rate and dissipation rate

The kinetic energy per unit mass is

$$e(\vec{r},t) = \frac{1}{2}\mathrm{v}^2(\vec{r},t) \tag{43}$$

From equation (28), the spatial average of e over the entire space, denoted by $E(t)$, is given by

$$E(t) = \frac{1}{4}\sum_j (|\vec{A}_{sj}(t)|^2 + |\vec{A}_{cj}(t)|^2) \tag{44}$$

We define

$$E_j(t) = \frac{1}{4}(|\vec{A}_{sj}(t)|^2 + |\vec{A}_{cj}(t)|^2) \tag{45}$$

We then have

$$E(t) = \sum_j E_j(t) \tag{46}$$

From equation (2), we get

$$\frac{\partial e}{\partial t} + \vec{\mathrm{v}}\cdot[(\vec{\mathrm{v}}\cdot\nabla)\vec{\mathrm{v}}] = -\frac{1}{\rho}\vec{\mathrm{v}}\cdot\nabla p + \nu\vec{\mathrm{v}}\cdot\Delta\vec{\mathrm{v}} \tag{47}$$

where

$$\vec{v}\cdot[(\vec{v}\cdot\nabla)\vec{v}]=\vec{v}\cdot[\nabla e+rot\vec{v}\times\vec{v}]=\vec{v}\cdot\nabla e=div(e\vec{v}) \tag{48}$$

and

$$-\frac{1}{\rho}\vec{v}\cdot\nabla p=-div(\frac{p}{\rho}\vec{v}) \tag{49}$$

In the derivations of equations (48) and (49), equation (1) and constant ρ have been taken into account. Now, we consider the turbulence in 3D space without any boundary and take the average of equation (47) over the entire space. For the average of the inertial term, we have

$$\begin{aligned}&\overline{\vec{v}\cdot[(\vec{v}\cdot\nabla)\vec{v}]}=\overline{div(e\vec{v})}\\&=\lim_{R\to\infty}\frac{3}{4\pi R^3}\iiint_{|\vec{r}|\le R}div(e\vec{v})\mathrm{d}\tau=\lim_{R\to\infty}\frac{3}{4\pi R^3}\oint\!\!\!\oint_{|\vec{r}|=R}e\vec{v}\cdot\overrightarrow{\mathrm{ds}}=0\end{aligned} \tag{50}$$

because the surface integral over the sphere $|\vec{r}|=R$ is only of order R^2 as long as e and $|\vec{v}|$ are bounded in the entire 3D space which is indeed the case. Similarly, for the pressure term, we have

$$\begin{aligned}&\overline{-\frac{1}{\rho}\vec{v}\cdot\nabla p}=\overline{-div(\frac{p}{\rho}\vec{v})}\\&=-\lim_{R\to\infty}\frac{3}{4\pi R^3}\iiint_{|\vec{r}|\le R}div(\frac{p}{\rho}\vec{v})\mathrm{d}\tau=-\lim_{R\to\infty}\frac{3}{4\pi R^3}\oint\!\!\!\oint_{|\vec{r}|=R}\frac{p}{\rho}\vec{v}\cdot\overrightarrow{\mathrm{ds}}=0\end{aligned} \tag{51}$$

as long as p/ρ is also bounded in the entire 3D space which is also indeed the case. Thus, the average of equation (47) over the entire space yields

$$\varepsilon = -\frac{\partial E}{\partial t} = -\nu \overline{\vec{v} \cdot \Delta \vec{v}} \tag{52}$$

From equations (28) and (31), we obtain

$$\overline{\vec{v} \cdot [(\vec{v} \cdot \nabla)\vec{v}]} = \frac{1}{2} \sum_j (\vec{A}_{sj} \cdot \vec{B}_{sj} + \vec{A}_{cj} \cdot \vec{B}_{cj}) \tag{53}$$

We define

$$T_j = -\frac{1}{2} (\vec{A}_{sj} \cdot \vec{B}_{sj} + \vec{A}_{cj} \cdot \vec{B}_{cj}) \qquad T = \sum_j T_j \tag{54}$$

where $T_j(t)$ is the power exerted on, or equivalently the energy flux given to, the wave of the wave vector $\vec{k}_j$ by the inertial transport, and T is the rate of change of the total energy due to the inertial transport. From equations (50), (53) and (54), we have

$$T = \sum_j T_j = -\frac{1}{2} \sum_j (\vec{A}_{sj} \cdot \vec{B}_{sj} + \vec{A}_{cj} \cdot \vec{B}_{cj}) = 0 \tag{55}$$

This is the well-known global energy conservation of inertial transport, which states that inertial transport only redistributes the energy among different Fourier components without changing the total energy. By substituting equation (28) into equation (52), we obtain

$$\varepsilon = \frac{1}{2} \nu \sum_j k_j^2 (|\vec{A}_{sj}|^2 + |\vec{A}_{cj}|^2) = 2\nu \sum_j k_j^2 E_j \tag{56}$$

From equation (45), we have

$$\frac{dE_j(t)}{dt} = \frac{1}{2}[\vec{A}_{sj}(t)\frac{d\vec{A}_{sj}(t)}{dt} + \vec{A}_{cj}(t)\frac{d\vec{A}_{cj}(t)}{dt}] \tag{57}$$

By substituting equations (41) and (42) into equation (57), we obtain

$$\frac{dE_j(t)}{dt} = T_j(t) - 2\nu k_j^2 E_j(t) \tag{58}$$

5. “Triangle principle”

Now we consider any three Fourier components where one wave vector is the sum of the other two. Without losing generality, the three wave vectors are denoted by $\vec{k}_1,\ \vec{k}_2,\ \vec{k}_3$ and are assumed to satisfy

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 \tag{59a}$$

$$\vec{k}_3 - \vec{k}_1 = \vec{k}_2 \tag{59b}$$

$$\vec{k}_3 - \vec{k}_2 = \vec{k}_1 \tag{59c}$$

Note, if one of the above three relations is true, the other two must also be true. Now we assume that the amplitude vectors for the wave vectors $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ are{ $\vec{A}_{s1}, \vec{A}_{c1}$ }, { $\vec{A}_{s2}, \vec{A}_{c2}$ }, and { $\vec{A}_{s3}, \vec{A}_{c3}$ } respectively, where

$$\vec{A}_{si} \cdot \vec{k}_i = 0 \qquad \vec{A}_{ci} \cdot \vec{k}_i = 0 \qquad (i = 1,\ 2,\ 3) \tag{60}$$

For simplicity, we denote the left hand sides of equations (20) to (23) by $\vec{F}_{s,12}$, $\vec{F}_{c,12}$,

$\vec{G}_{s,12}$, $\vec{G}_{c,12}$ respectively. For the interaction of the above three waves alone without including the contributions of other Fourier components, equations (35) and (36) are reduced to the following:

$$\left.\begin{aligned}\vec{B}_{s1}(t)&=\vec{G}_{s,32} & \vec{B}_{c1}(t)&=\vec{G}_{c,32}\\ \vec{B}_{s2}(t)&=\vec{G}_{s,31} & \vec{B}_{c2}(t)&=\vec{G}_{c,31}\\ \vec{B}_{s3}(t)&=\vec{F}_{s,12} & \vec{B}_{c3}(t)&=\vec{F}_{c,12}\end{aligned}\right\} \tag{61}$$

Thus, equation (54) is reduced to

$$T_{32,1}=-\frac{1}{2}(\vec{G}_{s,32}\cdot\vec{A}_{s1}+\vec{G}_{c,32}\cdot\vec{A}_{c1}) \tag{62}$$

$$T_{31,2}=-\frac{1}{2}(\vec{G}_{s,31}\cdot\vec{A}_{s2}+\vec{G}_{c,31}\cdot\vec{A}_{c2}) \tag{63}$$

$$T_{12,3}=-\frac{1}{2}(\vec{F}_{s,12}\cdot\vec{A}_{s3}+\vec{F}_{c,12}\cdot\vec{A}_{c3}) \tag{64}$$

where $T_{32,1}$ stands for the energy flux to the wave $\vec{k}_1$ due to the interaction of the waves $\vec{k}_3$ and $\vec{k}_2$, $T_{31,2}$ for the energy flux to the wave $\vec{k}_2$ due to the interaction of the waves $\vec{k}_3$ and $\vec{k}_1$, while $T_{12,3}$ for the energy flux to the wave $\vec{k}_3$ due to the interaction of the waves $\vec{k}_1$ and $\vec{k}_2$. By substituting the right hand sides of equations (20) to (23), where (1, 2) should be replaced by (3, 1) or (3, 2) if necessary, into equations (62) to (64), we obtain

$$\begin{aligned}T_{32,1}=&-\frac{1}{4}\{[(\vec{A}_{s3}\cdot\vec{k}_2)\vec{A}_{s2}+(\vec{A}_{c3}\cdot\vec{k}_2)\vec{A}_{c2}-(\vec{A}_{s2}\cdot\vec{k}_3)\vec{A}_{s3}-(\vec{A}_{c2}\cdot\vec{k}_3)\vec{A}_{c3}]\cdot\vec{A}_{s1}\\ &+[(\vec{A}_{c3}\cdot\vec{k}_2)\vec{A}_{s2}-(\vec{A}_{s3}\cdot\vec{k}_2)\vec{A}_{c2}+(\vec{A}_{c2}\cdot\vec{k}_3)\vec{A}_{s3}-(\vec{A}_{s2}\cdot\vec{k}_3)\vec{A}_{c3}]\cdot\vec{A}_{c1}\}\end{aligned} \tag{65}$$

$$T_{31,2} = -\frac{1}{4}\{[(\vec{A}_{s3}\cdot\vec{k}_1)\vec{A}_{s1} + (\vec{A}_{c3}\cdot\vec{k}_1)\vec{A}_{c1} - (\vec{A}_{s1}\cdot\vec{k}_3)\vec{A}_{s3} - (\vec{A}_{c1}\cdot\vec{k}_3)\vec{A}_{c3}]\cdot\vec{A}_{s2}$$
$$+[(\vec{A}_{c3}\cdot\vec{k}_1)\vec{A}_{s1} - (\vec{A}_{s3}\cdot\vec{k}_1)\vec{A}_{c1} + (\vec{A}_{c1}\cdot\vec{k}_3)\vec{A}_{s3} - (\vec{A}_{s1}\cdot\vec{k}_3)\vec{A}_{c3}]\cdot\vec{A}_{c2}\} \tag{66}$$

$$T_{12,3} = -\frac{1}{4}\{[(\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{s2} - (\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{c2} + (\vec{A}_{s2}\cdot\vec{k}_1)\vec{A}_{s1} - (\vec{A}_{c2}\cdot\vec{k}_1)\vec{A}_{c1}]\cdot\vec{A}_{s3}$$
$$+[(\vec{A}_{c1}\cdot\vec{k}_2)\vec{A}_{s2} + (\vec{A}_{s1}\cdot\vec{k}_2)\vec{A}_{c2} + (\vec{A}_{c2}\cdot\vec{k}_1)\vec{A}_{s1} + (\vec{A}_{s2}\cdot\vec{k}_1)\vec{A}_{c1}]\cdot\vec{A}_{c3}\} \tag{67}$$

Now we make the summation of equations (65) to (67). The 24 terms on the right hand sides of these equations can be reorganized as follows:

$$\begin{aligned}
T_{32,1} + T_{31,2} + T_{12,3} = -\frac{1}{4}\{&[(\vec{A}_{s3}\cdot\vec{k}_2) + (\vec{A}_{s3}\cdot\vec{k}_1)](\vec{A}_{s1}\cdot\vec{A}_{s2} - \vec{A}_{c1}\cdot\vec{A}_{c2}) + \\
&[(\vec{A}_{c3}\cdot\vec{k}_2) + (\vec{A}_{c3}\cdot\vec{k}_1)](\vec{A}_{s1}\cdot\vec{A}_{c2} + \vec{A}_{s2}\cdot\vec{A}_{c1}) - \\
&[(\vec{A}_{s2}\cdot\vec{k}_3) - (\vec{A}_{s2}\cdot\vec{k}_1)](\vec{A}_{s1}\cdot\vec{A}_{s3} + \vec{A}_{c1}\cdot\vec{A}_{c3}) - \\
&[(\vec{A}_{c2}\cdot\vec{k}_3) - (\vec{A}_{c2}\cdot\vec{k}_1)](\vec{A}_{s1}\cdot\vec{A}_{c3} - \vec{A}_{s3}\cdot\vec{A}_{c1}) - \\
&[(\vec{A}_{s1}\cdot\vec{k}_3) - (\vec{A}_{s1}\cdot\vec{k}_2)](\vec{A}_{s2}\cdot\vec{A}_{s3} + \vec{A}_{c2}\cdot\vec{A}_{c3}) - \\
&[(\vec{A}_{c1}\cdot\vec{k}_3) - (\vec{A}_{c1}\cdot\vec{k}_2)](\vec{A}_{s2}\cdot\vec{A}_{c3} - \vec{A}_{s3}\cdot\vec{A}_{c2})\}
\end{aligned} \tag{68}$$

It is easy to see that all the square brackets in equation (68) are zero:

$$\left.\begin{aligned}
(\vec{A}_{s3}\cdot\vec{k}_2) + (\vec{A}_{s3}\cdot\vec{k}_1) &= \vec{A}_{s3}\cdot(\vec{k}_1 + \vec{k}_2) = \vec{A}_{s3}\cdot\vec{k}_3 = 0 \\
(\vec{A}_{c3}\cdot\vec{k}_2) + (\vec{A}_{c3}\cdot\vec{k}_1) &= \vec{A}_{c3}\cdot(\vec{k}_1 + \vec{k}_2) = \vec{A}_{c3}\cdot\vec{k}_3 = 0 \\
(\vec{A}_{s2}\cdot\vec{k}_3) - (\vec{A}_{s2}\cdot\vec{k}_1) &= \vec{A}_{s2}\cdot(\vec{k}_3 - \vec{k}_1) = \vec{A}_{s2}\cdot\vec{k}_2 = 0 \\
(\vec{A}_{c2}\cdot\vec{k}_3) - (\vec{A}_{c2}\cdot\vec{k}_1) &= \vec{A}_{c2}\cdot(\vec{k}_3 - \vec{k}_1) = \vec{A}_{c2}\cdot\vec{k}_2 = 0 \\
(\vec{A}_{s1}\cdot\vec{k}_3) - (\vec{A}_{s1}\cdot\vec{k}_2) &= \vec{A}_{s1}\cdot(\vec{k}_3 - \vec{k}_2) = \vec{A}_{s1}\cdot\vec{k}_1 = 0 \\
(\vec{A}_{c1}\cdot\vec{k}_3) - (\vec{A}_{c1}\cdot\vec{k}_2) &= \vec{A}_{c1}\cdot(\vec{k}_3 - \vec{k}_2) = \vec{A}_{c1}\cdot\vec{k}_1 = 0
\end{aligned}\right\} \tag{69}$$

Therefore, we have

$$T_{32,1} + T_{31,2} + T_{12,3} = 0 \tag{70}$$

This means that *the inertial energy fluxes among any three wave vectors with sum and* *difference relations must be closed* in the sense of equation (70). We call the inertial energy transport among three wave vectors with sum and difference relations (59a) to (59c) a *triangle action* because such three vectors can form a triangle, and call the above finding shown in equation (70), i.e. the closed energy fluxes during any triangle action, the *triangle principle*, which shows the *detailed energy conservation* during inertial transport. Since the only way for a Fourier component of wave vector $\vec{k}_1$ to influence the energy of another Fourier component of wave vector $\vec{k}_2$ is through a third Fourier component with wave vector either $\vec{k}_3 = \vec{k}_1 + \vec{k}_2$ or $\vec{k}_3 = \vec{k}_1 - \vec{k}_2$, a triangle action would be involved in any interaction in an incompressible turbulence. The *inertial energy transport internet* among all the wave vectors in Fourier space consists of millions of triangle actions. Since the inertial energy fluxes in each triangle action must be closed according to (70), the total energy is conserved during inertial transport. Thus, the triangle principle gives the detailed mechanism for the global energy conservation during inertial transport.

6. Discussions

6.1. On the non-closure problem in the classical theory of homogeneous isotropic turbulence

As we have mentioned in the introduction, all the efforts trying to find exact closed equations governing the averaged quantities, e.g. velocity correlations and energy spectrum, in a homogeneous and isotropic turbulence failed. For example, the 1D energy spectrum $E(k,t)$ is governed by the equation

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu^2 E(k,t) \tag{71}$$

where $T(k,t)$ is the 1D inertial transport rate. Equation (71) is actually the integration of equation (58) over the spherical surface of constant k in Fourier space. The analysis in section 4 and section 5 showed that the inertial transport of energy among all the

Fourier components is determined by the 3D velocity field, more specifically by all the amplitude vectors $\{\vec{A}_{sj}(t), \vec{A}_{cj}(t)\}$, which cannot be uniquely determined by *E(k,t)*. Thus, the non-closure problem in the classical theory of homogeneous and isotropic turbulence was simply caused by the fact that the averaged quantities alone do not give entire information of the 3D velocity field, while only the 3D velocity field can determine the 3D velocity field as well as all the average quantities at the next moment.

6.2. On the establishment of the quasi-equilibrium energy spectrum of dissipation range

Even the 3D energy spectrum $\{E_j(t)\}$ cannot uniquely determine the 3D velocity field. If the 3D velocity field with amplitude vectors $\{\vec{A}_{sj}(t), \vec{A}_{cj}(t)\}$ has a 3D energy spectrum $\{E_j(t)\}$, any rotations of $\{\vec{A}_{sj}(t), \vec{A}_{cj}(t)\}$ about the corresponding wave vectors $\{\vec{k}_j\}$, in particular $\{-\vec{A}_{sj}(t), -\vec{A}_{cj}(t)\}$, would generate the same 3D energy spectrum $\{E_j(t)\}$. However, if the 1D inertial transport rate for $\{\vec{A}_{sj}(t), \vec{A}_{cj}(t)\}$ is *T(k,t)*, it would be $-T(k,t)$ for $\{-\vec{A}_{sj}(t), -\vec{A}_{cj}(t)\}$.

For a given 1D energy spectrum $\{E(k,t)\}$, or even for a given 3D energy spectrum $\{E_j(t)\}$, there are infinite number of possible instantaneous 3D velocity fields. However, they are not equally realistic. In order to see this more intuitively, let us roughly divide the entire Fourier space into “energy containing range” and “dissipation range” by assuming a certain k^* such that all the Fourier components with $k<k^*$ are “large eddies” containing the large portion of the total energy, while all the Fourier components with $k>k^*$ are “small eddies” containing a small portion of the total energy but causing the major portion of energy dissipation. For a given Fourier component in the dissipation range with wave vector $\vec{k}_j$ ($|\vec{k}_j|=k_j>k^*$), the velocity amplitudes $\{\vec{A}_{sj}(t), \vec{A}_{cj}(t)\}$ may have any orientation as long as they are perpendicular to $\vec{k}_j$. Thus, all the triangle actions acting on this Fourier component may lead to a gain or a loss of energy for this Fourier component, depending on the specific orientation of $\{\vec{A}_{sj}(t), \vec{A}_{cj}(t)\}$ at that moment. If we consider all Fourier components in the dissipation range at a certain moment, some of them may gain, and some of them may lose, energy from the corresponding triangle actions. On the other hand, all of them contain relatively small amount energy and dissipate rapidly because of the high wave numbers. Thus, the Fourier components of dissipation range which lose energy due to inertial transport cannot survive a long time, while those gaining energy from inertial transport can survive because the viscous dissipation can be compensated by the energy gain from the inertial transport. If we regard all the Fourier components of the dissipation range as a “species”, with each individual Fourier component in this range as an “organism”, and the energy containing range as an environment, the above argument is similar to *Darwin's natural selection*, which helps to establish a quasi-equilibrium state of the energy spectrum of the dissipation range.

6.3. Statistical regularity and dynamic regularity

As a general principle, it is not possible to prove or to derive a statistical regularity without involving another statistical assumption. In other words, it is not possible to prove or to derive a statistical result by dynamics alone. Nevertheless, detailed dynamical analysis may help to understand the mechanism for the establishment of statistical regularity. The argument in Section 6.2 offered one example for this.

7. Summary

The main result of this paper is to prove the triangle principle, which states that the inertial energy fluxes among any three Fourier components, where one wave vector is the sum or difference of the other two wave vectors, must be closed. This gives the *detailed energy conservation* during inertial transport. The Fourier components in the energy containing range will influence the orientations of the velocity amplitudes of the Fourier components in the dissipation range, through a mechanism similar to *Darwin's natural selection*, so that a statistically quasi-equilibrium energy spectrum of the dissipation range could be reached. Thus, the present analysis gives one example to show how detailed dynamical analysis may help to understand the establishment of certain statistical regularity.

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