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UNIQUENESS THEOREMS CONCERNING L-FUNCTIONS AND WEAKLY WEIGHTED SHARING

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Abstract. In this paper, we establish some uniqueness theorems when $p(L)$ and $P(L)$ share "$(R(z),l)$" where p, P and R denote polynomial function, homogeneous differential polynomial function and rational function respectively and L denotes an L-function in the extended Selberg class. Our results improve and generalize some recent results due to Mandal, Datta [11].

Keywords: L-function; meromorphic function; uniqueness; weakly weighted sharing; homogeneous differential polynomial.

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1. INTRODUCTION

A model for L-functions is formulated by Selberg in 1992. The study of value distributions of L-functions is mainly concerned with the set $\{z \in \mathbb{C} : L(z) = a\}$ where $a \in \mathbb{C}$.

An L-function L in the Selberg class is a meromorphic function satisfying the following properties.

(i) $L(z)$ can be expressed as a Dirchlet series $L(z) = \sum_{n=1}^{\infty} a(n)/n^z$.

(ii) $\mid a(n) \mid = O(n^{\varepsilon})$, for any $\varepsilon > 0$.

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(iii) There exists a nonnegative integer k such that $(z-1)^k L(z)$ becomes an entire function of finite order.

(iv) Every L-function satisfies the functional equation

$$\lambda_L(z) = \omega\overline{\lambda_L(1-\bar{z})},$$

where

$$\lambda_L(z) = L(z)A^z \prod_{i=1}^{k} \Gamma(\eta_i z + \nu_i)$$

with positive real numbers A, η_i and complex numbers ν_i, ω with $Re(\nu_i) \geq 0$ and $|\ \omega\ | = 1$.

(v) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = exp(\sum_{k=1}^{\infty} b(p^k)/p^{kz})$ with $b(p^k) = O(p^{k\theta})$ for some $\theta < 1/2$ and p denotes prime number.

Clearly the Riemann zeta function is an L-function in the Selberg class. If L satisfies (i) - (iv) then we say that L is an L-function in the extended Selberg class. In this paper by an L-function we mean an L-function in the extended Selberg class with $a(1) = 1$.

In this paper, we study uniqueness problems with the help of Nevanlinna value distribution theory using the standard notations and definitions of the value distribution theory [4].

2. PRELIMINARIES

Let ξ and ψ be a meromorphic functions defined in the complex plane $\mathbb{C}$. We say that ξ and ψ share a value σ IM (CM) if $\xi - \sigma$ and $\psi - \sigma$ have same zeros ignoring(counting) multiplicities. If $\frac{1}{\xi}$ and $\frac{1}{\psi}$ share 0 CM (IM), we say that ξ and ψ share ∞ CM (IM). We denote by $S(r,\xi)$ any function satisfying $S(r,\xi) = o(T(r,\xi))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure. A meromorphic function ρ is said to be a small function of ξ if $T(r,\rho) = S(r,\xi)$. We denote by $S(\xi)$ the set of all the small functions of ξ.

In 2007 Steuding [13] proved the following uniqueness theorem.

Theorem 2.1. [13] *Let L_1 and L_2 be two L-functions with $a(1) = 1$ and $z_0 \neq \infty$ be a complex number. If L_1 and L_2 share z_0 CM, then $L_1 \equiv L_2$.*

In 2010 Li [8] proved the following theorem.

Theorem 2.2. [8] *If a meromorphic function F having finitely many poles and a nonconstant L-function L share α CM and β IM then $L \equiv F$, where α and β are two distinct finite values.*

In 2017, considering uniqueness problem of L-functions, Liu, Li and Yi [10] proved the following theorem.

Theorem 2.3. [10] *Let $k \geq 1$ and $j \geq 1$ be integers such that $k > 3j+6$. Also let L be an L-function and F be a nonconstant meromorphic function. If $\{F^k\}^{(j)}$ and $\{L^k\}^{(j)}$ share 1 CM then $F \equiv dL$ for some constant d satisfying $d^k = 1$.*

Definition 2.1. [5, 6] *Let ξ be a meromorphic function defined in the complex plane and m be an integer (≥ 0) or infinity . For $c \in \mathbb{C} \cup \{\infty\}$ we denote by $E_{m)}(c;\xi)$ the set of all zeros of $\xi - c$ with multiplicities not exceeding m, where a zero is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(c;\xi)$ the set of all zeros of $\xi - c$ with multiplicities not exceeding m, where a zero is counted ignoring multiplicity.*

Definition 2.2. [5, 6] *Let ξ and χ be two meromorphic functions defined in the complex plane and m be an integer (≥ 0) or infinity . For $c \in \mathbb{C} \cup \{\infty\}$ we denote by $E_m(c;\xi)$ the set of all zeros of $f - c$ where an zero of multiplicity k is counted k times if $k \leq m$ and $m+1$ times if $k > m$. If $E_m(c;\xi) = E_m(c;\chi)$, we say that ξ, χ share the value c with weight m. We write ξ, χ share (c,m) to mean that ξ, χ share the value c with weight m.*

Definition 2.3. [11] *Let ξ be a meromorphic function defined in the complex plane and ρ be a small function of ξ. Then we denote by $E_{m)}(\rho;\xi)$, $\overline{E}_{m)}(\rho;\xi)$ and $E_m(\rho;\xi)$ the sets $E_{m)}(0;\xi - \rho)$, $\overline{E}_{m)}(0;\xi-\rho)$ and $E_m(0;\xi-\rho)$ respectively.*

Using weighted sharing in 2015, Wu and Hu [14] proved the following result.

Theorem 2.4. [14] *Let L and H be two L-functions, and let $\alpha,\beta \in \mathbb{C}$ be two distinct values. Take two positive integers m_1, m_2 with $m_1 m_2 > 1$. If $E_{m_1}(\alpha,L) = E_{m_1}(\alpha,H)$, and $E_{m_2}(\alpha,L) = E_{m_2}(\alpha,H)$, then $L \equiv H$.*

Considering weighted sharing in 2018 Hao and Chen [3] proved the following theorem.

Theorem 2.5. [3] *Let L be an L-function and F be a meromorphic function defined in the complex plane $\mathbb{C}$ with finitely many poles. Let $\alpha_1,\alpha_2 \in \mathbb{C}$ be distinct and m_1, m_2 be positive integers such that $m_1 m_2 > 1$. If $E_{m_j}(\alpha_j,F) = E_{m_j}(\alpha_j,L)$, $j = 1,2$, then $L \equiv F$.*

In 2020 using weighted sharing Datta and Mandal [2] proved the following theorem.

Theorem 2.6. [2] *Let ξ be a nonconstant meromorphic function and L be a nonconstant L-function. If $E_0(0;\xi) = E_0(0;L)$, $E_1(1;\xi) = E_1(1;L)$ and $N(r;0;\xi)+N(r;1;\xi) = S(r;\xi)$ then either $L \equiv \xi$ or $T(r;L) = N(r;0;L| \leq 2)+S(r;L)$ and $T(r;\xi) = N(r;0;L'| \leq 1)+S(r;L)$.*

Considering sharing of small function in 2020 Mandal and Datta [11] proved the following uniqueness theorem.

Theorem 2.7. [11] *Let L be a nonconstant L-function and ρ be a small function of L such that $\rho \not\equiv 0,\infty$. If $\overline{E}_{4)}(\rho;L) = \overline{E}_{4)}(\rho;(L^m)^{(k)})$, $E_{2)}(\rho;L) = E_{2)}(\rho;(L^m)^{(k)})$ and*

$$2N_{2+k}(r,0;L^m) \leq (\sigma + o(1))T(r,L), \tag{2.1}$$

where $m \geq 1$, $k \geq 1$ are integers and $0 < \sigma < 1$, then $L \equiv (L^m)^{(k)}$.

Definition 2.4. *Let ξ, ψ and χ be nonconstant meromorphic functions. We denote by $N_E(r,\chi;\xi,\psi)$ the counting function of all common zeros of $\xi-\chi$ and $\psi-\chi$ with same multiplicities. We denote by $\overline{N}_E(r,\chi;\xi,\psi)$ the corresponding reduced counting function.*

Definition 2.5. *Let ξ, ψ be nonconstant meromorphic functions and χ be a meromorphic function. We denote by $N_0(r,\chi;\xi,\psi)$ the counting function of all common zeros of $\xi-\chi$ and $\psi-\chi$. We denote by $\overline{N}_0(r,\chi;\xi,\psi)$ the corresponding reduced counting function.*

Definition 2.6. [9] *Let ξ, ψ be nonconstant meromorphic functions and $\rho \in S(\xi)\cap S(\psi)$. If*

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\psi)-2\overline{N}_E(r,\rho;\xi,\psi) = S(r,\xi)+S(r,\psi),$$

we say that ξ and ψ share ρ "CM". If

$$\overline{N}(r,\rho;\xi)+\overline{N}(r,\rho;\psi)-2\overline{N}_0(r,\rho;\xi,\psi) = S(r,\xi)+S(r,\psi),$$

we say that ξ and ψ share ρ "IM".

Definition 2.7. *Let ξ, ψ be nonconstant meromorphic functions and χ be a meromorphic function. We say that ξ and ψ share χ "CM"("IM") if $\xi-\chi$ and $\psi-\chi$ share* 0 *"CM"("IM").*

Definition 2.8. [7]. *Let ξ be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha\in\mathbb{C}\cup\{\infty\}$. By $N(r,\alpha;\xi\mid\leq n)$ we denote the counting function of the α points of ξ with multiplicity $\leq n$ and by $\overline{N}(r,\alpha;\xi\mid\leq n)$ the reduced counting function. Also by $N(r,\alpha;\xi\mid\geq n)$ we denote the counting function of the α points of ξ with multiplicity $\geq n$ and by $\overline{N}(r,\alpha;\xi\mid\geq n)$ the reduced counting function. We define*

$$N_n(r,\alpha;\xi)=\overline{N}(r,\alpha;\xi)+\overline{N}(r,\alpha;\xi\mid\geq 2)+\cdots+\overline{N}(r,\alpha;\xi\mid\geq n).$$

Definition 2.9. [7]. *Let ξ and ψ be two meromorphic functions defined in the complex plane. Then we denote by $N(r,\psi;\xi\mid\leq m)$, $\overline{N}(r,\psi;\xi\mid\leq m)$, $N(r,\psi;\xi\mid\geq m)$, $\overline{N}(r,\psi;\xi\mid\geq m)$, $N_m(r,\psi;\xi)$ etc. the counting functions $N(r,0;\xi-\psi\mid\leq m)$, $\overline{N}(r,0;\xi-\psi\mid\leq m)$, $N(r,0;\xi-\psi\mid\geq m)$, $\overline{N}(r,0;\xi-\psi\mid\geq m)$, $N_m(r,0;\xi-\psi)$ etc. respectively.*

Definition 2.10. *Let two nonconstant meromorphic functions ξ and ψ share a value α "IM" and m be a positive integer or ∞. We denote by $\overline{N}_E(r,\alpha;\xi,\psi|\leq m)$ the counting function of the α-points of ξ and ψ with multiplicities not greater than m and the multiplicities with respect to ξ is equal to the multiplicities with respect to ψ, where each α-point is counted once only.*

Definition 2.11. *Let two nonconstant meromorphic functions ξ and ψ share a value α "IM" and m be a positive integer or ∞. We denote by $\overline{N}_0(r,\alpha;\xi,\psi|\geq m)$ the counting function of the common α-points of ξ and ψ with multiplicities not less than m, where each α-point is counted once only.*

Definition 2.12. *Let two nonconstant meromorphic functions ξ, ψ share a meromorphic functions χ "IM". By $\overline{N}_E(r,\chi;\xi,\psi|\leq m)$, $\overline{N}_0(r,\chi;\xi,\psi|\geq m)$ we denote the counting functions $\overline{N}_E(r,0;\xi-\chi,\psi-\chi|\leq m)$ and $\overline{N}_0(r,0;\xi-\chi,\psi-\chi|\geq m)$ respectively.*

Definition 2.13. *Let $\rho\in S(\xi)\cap S(\psi)$ and two nonconstant meromorphic functions ξ, ψ share ρ "IM". If m is a positive integer or ∞ and*

$$\overline{N}(r,\rho;\xi|\leq m)-\overline{N}_E(r,\rho;\xi,\psi|\leq m)=S(r,\xi)$$

$$\overline{N}(r,\rho;\psi|\leq m)-\overline{N}_E(r,\rho;\xi,\psi|\leq m)=S(r,\psi)$$

$$\overline{N}(r,\rho;\xi|\geq m+1)-\overline{N}_0(r,\rho;\xi,\psi|\geq m+1)=S(r,\xi)$$

$$\overline{N}(r,\rho;\psi|\geq m+1)-\overline{N}_0(r,\rho;\xi,\psi|\geq m+1)=S(r,\psi)$$

or $m=0$ *and*

$$\overline{N}(r,\rho;\xi)-\overline{N}_0(r,\rho;\xi,\psi)=S(r,\xi)$$

$$\overline{N}(r,\rho;\psi)-\overline{N}_0(r,\rho;\xi,\psi)=S(r,\psi),$$

then we say ξ *and* ψ *weakly share* ρ *with weight m. We write* ξ *and* ψ *share* "(ρ,m)" *to mean that* ξ *and* ψ *weakly share* ρ *with weight m.*

Definition 2.14. *Let two nonconstant meromorphic functions* ξ, ψ *share a meromorphic functions* χ "*IM*". *Also let m be a positive integer or* ∞. *We say that* ξ, ψ *share* "(χ,m)" *if* $\xi-\chi$, $\psi-\chi$ *share* "$(0,m)$".

Definition 2.15. *Let* ξ *be a meromorphic function,* t_{ij} $(i=0,1,2,.....n,\ j=1,2,....,m)$ *be non-negative integers and* $\rho_j\in S(\xi)$ *such that* $\rho_j\not\equiv 0$ *for* $j=1,2,....,m$. *We define the differential polynomial* $P(\xi)$ *of* ξ *by* $P(\xi)=\sum_{j=1}^{m}M_j(\xi)$, *where* $M_j(\xi)=\rho_j\prod_{i=0}^{n}(\xi^{(i)})^{t_{ij}}$. *The numbers* $\overline{d}(P)=\max_{1\leq j\leq m}\sum_{i=0}^{n}t_{ij}$ *and* $\underline{d}(P)=\min_{1\leq j\leq m}\sum_{i=0}^{n}t_{ij}$ *are called degree and lower degree of* $P(\xi)$ *respectively. If* $\overline{d}(P)=\underline{d}(P)=d$ *(say), then we say that* $P(\xi)$ *is a homogeneous differential polynomial of degree d generated by* ξ. *We define Q by* $Q=\max_{1\leq j\leq m}\sum_{i=0}^{n}it_{ij}$.

Now the following question comes naturally.

Question 2.1. *In place of small function sharing if we consider rational function sharing in theorem 2.6 then what happens?*

Question 2.2. *Can we take polynomial of L and differential polynomial generated by L in place of L and* $(L^m)^{(k)}$ *in theorem 2.6?*

3. Main Results

Let L be a nonconstant L-function and $a_i, b_j \in S(L)$, $i = 0,1,2,...t,\ j = 0,1,2,...,s$. Henceforth we denote by $R(z)$ the function $R(z) = \frac{\sum_{i=0}^{t} a_i z^i}{\sum_{j=0}^{s} b_j z^j}$, where $a_t \not\equiv 0$ and $b_s \not\equiv 0$. Also we denote by $P(L)$ a homogeneous differential polynomial of degree d generated by L as defined in definition 2.15.

Using the concept of weakly weighted sharing we try to solve Questions 2.1, 2.2 and prove the following theorem.

Theorem 3.1. *Let L be a nonconstant L-function and $p(z)$ be a polynomial of degree $\lambda \geq 1$ with $p(0) = 0$. Let $P(L)$ be a homogeneous differential polynomial of degree d generated by L. If $p(L)$ and $P(L)$ share "$(R(z),l)$" where $2 \leq l \leq \infty$ and*

$$N_2(r,0;p(L)) + N_{2+n}(r,0;L) < (d+o(1))T(r,L), \tag{3.1}$$

then $p(L) \equiv P(L)$.

4. Lemmas

In this section we present some necessary lemmas.

Henceforth we denote by Ω the function defined by

$$\Omega = \left(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1}\right)$$

Lemma 4.1. [13]. *Let L be an L-function with degree q. Then*

$$T(r,L) = \frac{q}{\pi} r \log r + O(r).$$

Lemma 4.2. [11]. *Let L be an L-function. Then $N(r,\infty;L) = S(r,L) = O(\log r)$.*

Lemma 4.3. [15]. *Let $\xi(z) = \frac{\alpha_0 + \alpha_1 z + ... + \alpha_t z^t}{\beta_0 + \beta_1 z + ... + \beta_s z^s}$ be a nonconstant rational function defined in the complex plane $\mathbb{C}$, where $\alpha_0, \alpha_1, ..., \alpha_t (\neq 0)$ and $\beta_0, \beta_1, ..., \beta_s (\neq 0)$ are complex constants. Then*

$$T(r,\xi) = max\{t,s\} \log r + O(1).$$

Lemma 4.4. *Let ξ be a nonconstant meromorphic function defined in the complex plane and $P(\xi)$ be a homogeneous differential polynomial of degree d generated by ξ defined as in definition 2.15. If $P(\xi) \not\equiv 0$ then for any positive integer k*

(i) $N_k(r,0;P(\xi)) \leq N_{k+n}(r,0;\xi) + T(r,P(\xi)) - dT(r,\xi) + S(r,\xi)$

(ii) $N_k(r,0;P(\xi)) \leq N_{k+n}(r,0;\xi) + Q\overline{N}(r,\infty;\xi) + S(r,\xi)$.

Proof. Using first fundamental theorem we have

$$\begin{aligned} N_k(r,0;P(\xi)) &\leq N(r,0;P(\xi)) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) \\ &= T(r,P(\xi)) - m(r,0;P(\xi)) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + O(1) \\ &\leq T(r,P(\xi)) + m(r,\infty;\frac{P(\xi)}{\xi^d}) - m(r,0;\xi^d) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + O(1) \\ &\leq T(r,P(\xi)) - dT(r,\xi) + N(r,0;\xi^d) - \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + S(r,\xi) \\ &\leq T(r,P(\xi)) - dT(r,\xi) + N_{(k+n)d}(r,0;\xi^d) + \sum_{l=(k+n+1)d}^{\infty} \overline{N}(r,0;\xi^d| \geq l) \\ &- \sum_{l=k}^{\infty} \overline{N}(r,0;P(\xi)| \geq l) + S(r,\xi) \\ &\leq T(r,P(\xi)) - dT(r,\xi) + N_{k+n}(r,0;\xi) + S(r,\xi)) \qquad (4.1) \end{aligned}$$

This proves (i).

Now

$$\begin{aligned} T(r,P(\xi)) &= N(r,\infty;P(\xi)) + m(r,\infty;P(\xi)) \\ &\leq N(r,\infty;P(\xi)) + m(r,\infty;\xi^d) + m(r,\infty;\frac{P(\xi)}{\xi^d}) \\ &\leq N(r,\infty;P(\xi)) + dm(r,\infty;\xi) + S(r,\xi) \\ &\leq dN(r,\infty;\xi) + Q\overline{N}(r,\infty;\xi) + dm(r,\infty;\xi) + S(r,\xi) \\ &\leq dT(r,\xi) + Q\overline{N}(r,\infty;\xi) + S(r,\xi) \qquad (4.2) \end{aligned}$$

From (4.1) and (4.2) we have

$$N_k(r,0;P(\xi)) \leq N_{k+n}(r,0;\xi) + Q\overline{N}(r,\infty;\xi) + S(r,\xi)$$

This proves (ii).

This completes the proof. □

Lemma 4.5. [12] *Let ξ be a nonconstant meromorphic function and let $\Phi(\xi) = \frac{\sum_{i=0}^{t} \alpha_i \xi^i}{\sum_{j=0}^{s} \beta_j \xi^j}$ be irreducible rational function in ξ with coefficients α_i and β_j, $i = 0,1,2,..,t$, $j = 0,1,2,...,s$ where $\alpha_t \neq 0$ and $\beta_s \neq 0$. Then $T(r,\Phi(\xi)) = \max\{t,s\}T(r,\xi) + S(r,\xi)$.*

Lemma 4.6. *Let ξ be a nonconstant meromorphic function and $\mu_i, \nu_j \in S(\xi)$, $i = 0,1,2,...t$, $j = 0,1,2,...,s$. Also let $H(\xi) = \frac{\sum_{i=0}^{t} \mu_i \xi^i}{\sum_{j=0}^{s} \nu_j \xi^j}$, where $\mu_t \not\equiv 0$ and $\nu_s \not\equiv 0$. Then $T(r,H(\xi)) = \max\{t,s\}T(r,\xi) + S(r,\xi)$.*

Proof. Since $\mu_i, \nu_j \in S(\xi)$, $i = 0,1,2,..,t$, $j = 0,1,2,...,s$, therefore $T(r,\mu_i) = S(r,\xi)$, $i = 0,1,.....,t$ and $T(r,\nu_j) = S(r,\xi)$, $j = 0,1,.....,s$. Hence the result follows by lemma 4.5 □

Lemma 4.7. *Let L be a nonconstant L-function. Then $T(r,R(z)) = S(r,L)$.*

Proof. By lemma 4.1, lemma 4.3 and lemma 4.6 we get the required result. □

Lemma 4.8. [1] *Let $P(\xi)$ be a homogeneous differential polynomial of degree d generated by a nonconstant meromorphic function ξ as defined in definition 2.15. Then*

$$N(r,0;\frac{P(\xi)}{f^d}) \leq Q(\overline{N}(r,0;\xi) + \overline{N}(r,\infty;\xi)) + S(r,\xi).$$

Lemma 4.9. [9] *Let l be a nonnegative integer and two nonconstant meromorphic functions Φ and Ψ share "$(1,l)$". If $\Omega \not\equiv 0$ and $2 \leq l \leq \infty$, then*

$$T(r,\Phi) \leq N_2(r,\infty;\Phi) + N_2(r,0;\Phi) + N_2(r,\infty;\Psi) + N_2(r,0;\Psi) + S(r,\Phi) + S(r,\Psi)$$

and

$$T(r,\Psi) \leq N_2(r,\infty;\Phi) + N_2(r,0;\Phi) + N_2(r,\infty;\Psi) + N_2(r,0;\Psi) + S(r,\Phi) + S(r,\Psi).$$

5. PROOF OF THE THEOREM 3.1

Proof. Let $\Phi(z) = \frac{(p(L(z))}{R(z)}$ and $\Psi(z) = \frac{P(L(z))}{R(z)}$. Then Φ, Ψ share "$(1,l)$", except for zeros and poles of $R(z)$.

By lemma 4.7 we have $T(r,R(z)) = S(r,L)$. Now we have to consider the following two cases

Case 1 Let $\Omega \not\equiv 0$.

Using lemma 4.2, lemma 4.4 and lemma 4.9 we have

$$\begin{aligned}
T(r,P(L)) &= T(r,\Psi)+S(r,L)\\
&\leq N_2(r,\infty;\Phi)+N_2(r,0;\Phi)+N_2(r,\infty;\Psi)+N_2(r,0;\Psi)+S(r,L)\\
&\leq N_2(r,\infty;p(L))+N_2(r,0;p(L))+N_2(r,\infty;P(L))+N_2(r,0;P(L))+S(r,L)\\
&\leq N_2(r,\infty;L)+N_2(r,0;p(L))+N_2(r,\infty;L)+N_2(r,0;P(L))+S(r,L)\\
&\leq N_2(r,0;p(L))+N_{2+n}(r,0;L)+T(r,P(L))-dT(r,L)+S(r,L). \qquad (5.1)
\end{aligned}$$

From (5.1) we have

$$dT(r,L) \leq N_2(r,0;p(L))+N_{2+n}(r,0;L)+S(r,L),$$

which contradicts (3.1).

Case 2 Let $\Omega \equiv 0$.

Hence

$$\left(\frac{\Phi''}{\Phi'}-\frac{2\Phi'}{\Phi-1}\right)-\left(\frac{\Psi''}{\Psi'}-\frac{2\Psi'}{\Psi-1}\right)=0. \qquad (5.2)$$

Integrating (5.2) we get

$$\Phi=\frac{(D+1)\Psi+(C-D-1)}{D\Psi+(C-D)}, \qquad (5.3)$$

where $C \neq 0$ and D are constants.

Now we have to consider the following three subcases.

Subcase 2.1 Let $D = 0$. Then from (5.3) we have

$$\Phi=\frac{\Psi+C-1}{C}. \qquad (5.4)$$

If $C \neq 1$, then from (5.4) we get

$$\overline{N}(r,1-C,\Psi)=\overline{N}(r,0;\Phi). \tag{5.5}$$

Using lemma 4.2, lemma 4.4, (5.5) we get by Nevanlinna second fundamental theorem

$$\begin{aligned}
T(r,P(L)) &= T(r,\Psi)+S(r,L)\\
&\leq \overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Psi)+\overline{N}(r,1-C;\Psi)+S(r,L)\\
&\leq \overline{N}(r,\infty;L)+\overline{N}(r,0;\Psi)+\overline{N}(r,0;\Phi)+S(r,L)\\
&\leq \overline{N}(r,0;P(L))+\overline{N}(r,0;p(L))+S(r,L)\\
&\leq T(r,P(L))-dT(r,L)+N_{1+n}(r,0;L)+\overline{N}(r,0;p(L))+S(r,L)\\
&\leq T(r,P(L))-dT(r,L)+N_{2+n}(r,0;L)+N_2(r,0;p(L)+S(r,L) \qquad (5.6)
\end{aligned}$$

From (5.6)we get $dT(r,L)\leq N_2(r,0;p(L))+N_{2+n}(r,0;L)+S(r,L)$, which contradicts (3.1). Hence $C=1$ and therefore $p(L)\equiv P(L)$.

Subcase 2.2 Let $D=-1$. Then from (5.3) we have

$$\Phi=\frac{C}{C+1-\Psi}. \tag{5.7}$$

If $C\neq -1$, then using lemma 4.2 we get from (5.7)

$$\overline{N}(r,1+C,\Psi)=\overline{N}(r,\infty;\Phi)=\overline{N}(r,\infty;L)+S(r,L)=S(r,L). \tag{5.8}$$

Using lemma 4.2, lemma 4.4, (5.8) we get by Nevanlinna second fundamental theorem

$$\begin{aligned}
T(r,P(L)) &= T(r,\Psi)+S(r,L)\\
&\leq \overline{N}(r,\infty;\Psi)+\overline{N}(r,0;\Psi)+\overline{N}(r,1+C;\Psi)+S(r,L)\\
&\leq \overline{N}(r,\infty;L)+\overline{N}(r,0;\Psi)+\overline{N}(r,\infty;\Phi)+S(r,L)\\
&\leq \overline{N}(r,0;P(L))+\overline{N}(r,\infty;L)+S(r,L)\\
&\leq T(r,P(L))-dT(r,L)+N_{1+n}(r,0;L)+S(r,L)\\
&\leq T(r,P(L))-dT(r,L)+N_{2+n}(r,0;L)+N_2(r,0;p(L)+S(r,L) \qquad (5.9)
\end{aligned}$$

From (5.9)we get $dT(r,L) \le N_2(r,0;p(L)) + N_{2+n}(r,0;L) + S(r,L)$, which contradicts (3.1).
If $C = -1$, then

$$\Phi\Psi = 1 \tag{5.10}$$

From (5.10) we have

$$p(L)P(L) = R^2(z) \tag{5.11}$$

From (5.11) we have

$$\overline{N}(r,\infty;L) + \overline{N}(r,0;L) = S(r,L) \tag{5.12}$$

Using lemma 4.8 and (5.12) we get $N(r,\infty;\frac{P(L)}{L^d}) = S(r,L)$ and hence

$$T(r,\frac{P(L)}{L^d}) = N(r,\infty;\frac{P(L)}{L^d}) + m(r,\infty;\frac{P(L)}{L^d}) = S(r,L). \tag{5.13}$$

Using lemma 4.6 and (5.13) we have

$$\begin{aligned}(d+\lambda)T(r,L) &\le T(r,\frac{R^2(z)}{d^{\lambda+d}}) + O(1)\\ &\le T(r,(1+\frac{\rho_{\lambda-1}}{L}+...+\frac{\rho_1}{L^{\lambda-1}})\frac{P(L)}{L^d}) + O(1)\\ &\le (\lambda-1)T(r,L) + T(r,\frac{P(L)}{L^d}) + S(r,L)\\ &\le (\lambda-1)T(r,L) + S(r,L) \qquad (5.14)\end{aligned}$$

From (5.14) we get $T(r,L) = S(r,L)$, which is a contradiction.

Subcase 2.3 Let $D \neq 0,-1$.
If $C-D \neq 1$, the from (5.3) we get $\overline{N}(r,\frac{-C+D+1}{D+1};\Psi) = \overline{N}(r,0;\Phi)$. Now proceeding as in Subcase 2.1 we arrive at a contradiction.
If $C-D = 1$, then by (5.3) we get $\overline{N}(r,\frac{-1}{D};\Psi) = \overline{N}(r,\infty;\Phi)$. Now proceeding as in Subcase 2.2 we arrive at a contradiction.
This completes the proof. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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