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HYPONORMAL OPERATORS IN SOFT SETS

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Abstract. In this paper, we introduce the notion of hyponormal operators in soft Hilbert spaces, some properties of these operators are studied, and in a similar way to [6], a hyponormal soft operator is built from a family of operators. In addition, some results are obtained for self-adjoint, invertible, unitary, unit equivalent and normal soft linear operators that relate the properties of these soft operators with a family of operators in certain Hilbert spaces.

Keywords: soft sets; hyponormal soft operators; soft Hilbert spaces.

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1. Introduction

In what follows X will denote any nonempty set (possibly without algebraic structure) $\mathscr{P}(X)$ the set of parts of X and A a nonempty set of parameters.

Definition 1. [2] A soft set over X is a pair (F,A) where F is a mapping given by $F : A \to \mathscr{P}(X)$.

Definition 2. [4] *(Soft linear operator)* Let $T : SE(\check{X}) \to SE(\check{Y})$ be a operator. Then T is said to be soft linear if

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(L_1) $T(x_1+x_2) = T(x_1)+T(x_2)$ for all $x_1, x_2 \widetilde{\in} \check{X}$,

(L_2) $T(cx) = cT(x)$, for all soft scalar c and all $x \widetilde{\in} \check{X}$.

Theorem 3. [4] *Every soft linear operator can be descomposed into a family of crisp linear operators. This is, if $T : SE(\check{X}) \to SE(\check{Y})$ is a soft linear operator, then the family $\{T_\lambda : \lambda \in A\}$ where $T_\lambda : X \to Y$ is defined by $T_\lambda(\xi) = T(x)(\lambda)$ for all $\xi \in X$ and $x \widetilde{\in} \check{X}$ with $x(\lambda) = \xi$, is a family of linear operators.*

Theorem 4. [4] *Let $\{T_\lambda : \lambda \in A\}$ be a family of crisp linear operators of X to Y. Then the operador $T : SE(\check{X}) \to SE(\check{Y})$ defined for $T(x)(\lambda) = T_\lambda(\xi)$ with $x(\lambda) = \xi$, $\lambda \in A$, is soft linear.*

Definition 5. [4] Let $T : SE(\check{X}) \to SE(\check{Y})$ a soft linear operator, where $\check{X}, \check{Y}$ are soft and absolute normed vector spaces. The operator T is said to be bounded if exists $M \widetilde{\geq} \overline{0}$ such that $\|T(x)\| \widetilde{\leq} M\|x\|$, $\forall x \widetilde{\in} \check{X}$.

Proposition 6. [3] *Let $\check{H}$ be a soft Hilbert space and let $T : SE(\check{H}) \to SE(\check{H})$ be a bounded soft linear operator and T^* the adjoint operator for T. Then T^*_λ defined for $T^*_\lambda(x(\lambda)) = (T^*(x))(\lambda)$ is the adjoint operator of $T_\lambda, \forall \lambda \in A$.*

Proposition 7. [3] *Let $\check{H}$ be a soft Hilbert space and let $T : SE(\check{H}) \to SE(\check{H})$ be a continuous soft linear operator. Let $\{T^*_\lambda : \lambda \in A\}$ be a family of adjoint linear operators of T_λ. Then the soft linear operator T^* defined by $T^*(x)(\lambda) = T^*_\lambda(x(\lambda)), \forall \lambda \in A, \forall x \widetilde{\in} \check{H}$ is the adjoint operator of T.*

Definition 8. [3] A continuos soft linear operator $T : SE(\check{H}) \to SE(\check{H})$ is called self-adjoint soft linear operator if $T^* = T$.

Definition 9. [1] Let $\check{X}$, $\check{Y}$ soft normed and $T : SE(\check{\check{X}}) \to SE(\check{Y})$ a soft operator. T is said to be invertible if exists a bounded soft operator $S : SE(\check{H}) \to SE(\check{H})$ such that $TS(\widetilde{y}) = I_{\check{Y}}$ for all $\widetilde{y} \in \check{Y}$ and $ST(\widetilde{x}) = I_{\check{X}}$ for all $\widetilde{x} \in \check{X}$. We write $S = T^{-1}$

Definition 10. [5] Let $(\check{H}, A)$ be a soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ a bounded soft operator. If $T^*T = TT^*$ is said to be that T es normal soft.

Theorem 11. [3] *Suppose that $T \in B(\check{H}, \check{H})$. Then $\langle T(\widetilde{x}), \widetilde{y} \rangle = \overline{0}$ for all $\widetilde{x}, \widetilde{y} \widetilde{\in} \check{H}$ If and only if $T = O$, the soft zero linear operator.*

2. HYPONORMAL OPERATORS IN SOFT SETS

Next we show a result that allows us to introduce hyponormal operators in soft sets.

Proposition 12. *Let $(\check{H},A)$ be a soft Hilbert space. If $T : SE(\check{H}) \to SE(\check{H})$ is a bounded operator, then $T^*T - TT^*$ is self-adjoint soft.*

Proof. Follow from [3, Theorem 11]. □

Theorem 13. *Let $T : SE(\check{H}) \to SE(\check{H})$ be a soft bounded operator with $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and for all $\widetilde{x} \in SE(\check{H})$. If T is invertible, then T_λ is invertible for all $\lambda \in A$.*

Proof. Let $T : SE(\check{H}) \to SE(\check{H})$ be a invertible soft operator, then exists $W : SE(\check{H}) \to SE(\check{H})$ such that $TW = WT = I_{\check{H}}$. By Theorem 3 exists a family $\{W_\lambda : \lambda \in A\}$ of operators such that $(W\widetilde{x})(\lambda) = W_\lambda(\widetilde{x}(\lambda))$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$.
We make $W\widetilde{x} = \widetilde{y}$, then

$$\begin{aligned}(T_\lambda W_\lambda)(\widetilde{x}(\lambda)) &= T_\lambda(W_\lambda(\widetilde{x}(\lambda))) = T_\lambda((W\widetilde{x})(\lambda)) = T_\lambda(\widetilde{y}(\lambda)) \\ &= (T\widetilde{y})(\lambda) = (T(W\widetilde{x}))(\lambda) = (TW\widetilde{x})(\lambda) = (I_{\check{H}}\widetilde{x})(\lambda) = \widetilde{x}(\lambda)\end{aligned}$$

Furthermore, if $T\widetilde{x} = \widetilde{z}$, then

$$\begin{aligned}(W_\lambda T_\lambda)(\widetilde{x}(\lambda)) &= W_\lambda(T_\lambda(\widetilde{x}(\lambda))) = W_\lambda((T\widetilde{x})(\lambda)) = W_\lambda(\widetilde{z}(\lambda)) = (W\widetilde{z})(\lambda) \\ &= (W(T\widetilde{x}))(\lambda) = (WT\widetilde{x})(\lambda) = (I_{\check{H}}\widetilde{x})(\lambda) = \widetilde{x}(\lambda)\end{aligned}$$

So T_λ is invertible for all $\lambda \in A$. □

Theorem 14. *Let $\{T_\lambda : \lambda \in A\}$ be a family of invertible linear operators, then a soft invertible and bounded operator can be determined $T : SE(\check{H}) \to SE(\check{H})$ such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and for all $\widetilde{x} \in SE(\check{H})$.*

Proof. Suppose that T_λ is invertible for all $\lambda \in A$, then exists a W_λ such that $T_\lambda W_\lambda = I_\lambda$ and $W_\lambda T_\lambda = I_\lambda$. By Theorem 4 exists $W : SE(\check{H}) \to SE(\check{H})$ such that $(W(\widetilde{x}))(\lambda) = W_\lambda(\widetilde{x}(\lambda))$.

We make $W\widetilde{x} = \widetilde{y}$, then

$$(TW\widetilde{x})(\lambda) = T(W(\widetilde{x}))(\lambda) = (T\widetilde{y})(\lambda) = T_\lambda(\widetilde{y}(\lambda)) = T_\lambda((W\widetilde{x})(\lambda))$$

$$= T_\lambda(W_\lambda(\widetilde{x}(\lambda))) = (T_\lambda W_\lambda)(\widetilde{x}(\lambda)) = (I_\lambda \widetilde{x})(\lambda) = (I\widetilde{x})(\lambda)$$

Therefore, if $T\widetilde{x} = \widetilde{z}$

$$(WT\widetilde{x})(\lambda) = W(T\widetilde{x})(\lambda) = (W\widetilde{z})(\lambda) = W_\lambda(\widetilde{z}(\lambda)) = W_\lambda((T\widetilde{x})(\lambda))$$

$$= W_\lambda(T_\lambda(\widetilde{x}(\lambda)) = (W_\lambda T_\lambda)(\widetilde{x}(\lambda)) = I_\lambda(\widetilde{x}(\lambda)) = (I\widetilde{x})(\lambda)$$

So T is invertible. □

Following Das and Samanta [5], a soft bounded linear operator $T : SE(\check{H}) \to SE(\check{H})$ is said unitary if satisfies the condition $TT^* = T^*T = I_{\check{H}}$. In the following result we give a characterization of soft unitary operators in terms of a family of operators in the classical sense.

Proposition 15. *Let $T : SE(\check{H}) \to SE(\check{H})$ be a soft operator such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$. If T is unitary then T_λ is unitary for all $\lambda \in A$.*

Proof. Let $T : SE(\check{H}) \to SE(\check{H})$ be a soft unitary operator, then $TT^* = T^*T = I$ and by Theorem 3 exists a family $\{T_\lambda : \lambda \in A\}$ of operators such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$. We make $T^*\widetilde{x} = \widetilde{y}$, then

$$(T_\lambda T_\lambda^*)(\widetilde{x}(\lambda)) = T_\lambda(T_\lambda^*(\widetilde{x}(\lambda)) = T_\lambda((T^*\widetilde{x})(\lambda)) = T_\lambda(\widetilde{y}(\lambda)) = (T\widetilde{y})(\lambda)$$

$$= (TT^*\widetilde{x})(\lambda) = (I\widetilde{x})(\lambda) = I_\lambda(\widetilde{x}(\lambda))$$

Also, if $T\widetilde{x} = \widetilde{z}$, then

$$(T_\lambda^* T_\lambda)(\widetilde{x}(\lambda)) = T_\lambda^*(T_\lambda(\widetilde{x}(\lambda))) = T_\lambda^*((T\widetilde{x})(\lambda)) = (T_\lambda^*(\widetilde{z}(\lambda)) = (T^*\widetilde{z})(\lambda)$$

$$= (TT^*\widetilde{x})(\lambda) = (I\widetilde{x})(\lambda) = I_\lambda(\widetilde{x}(\lambda))$$

Thus T_λ is unitary for all $\lambda \in A$. □

Proposition 16. *Let $\{T_\lambda : \lambda \in A\}$ be a family of unitary operators, then a soft unitary operator can be determined $T : SE(\check{H}) \to SE(\check{H})$ such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and for all $\widetilde{x} \in SE(\check{H})$.*

Proof. Suppose that T_λ is unitary for all $\lambda \in A$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$. If $T^*\widetilde{x} = \widetilde{y}$ we have

$$(TT^*\widetilde{x})(\lambda) = (T(T^*\widetilde{x}))(\lambda) = (T\widetilde{y})(\lambda) = T_\lambda(\widetilde{y}(\lambda)) = T_\lambda((T^*\widetilde{x})(\lambda))$$

$$= T_\lambda(T_\lambda^*(\widetilde{x}(\lambda))) = (T_\lambda T_\lambda^*)(\widetilde{x}(\lambda)) = (I_\lambda(\widetilde{x}(\lambda)) = (I\widetilde{x})(\lambda)$$

Also, if $T\widetilde{x} = \widetilde{z}$

$$(T^*T\widetilde{x})(\lambda) = (T^*(T\widetilde{x}))(\lambda) = (T^*\widetilde{z})(\lambda) = T_\lambda^*(\widetilde{z}(\lambda)) = T_\lambda^*((T\widetilde{x})(\lambda))$$

$$= T_\lambda^*(T_\lambda(\widetilde{x}(\lambda))) = (T_\lambda^* T_\lambda)(\widetilde{x}(\lambda)) = (I_\lambda(\widetilde{x}(\lambda)) = (I\widetilde{x})(\lambda)$$

Thus T is soft unitary. □

Next, we introduce a new class of soft linear operators.

Definition 17. Let $(\check{H}, A)$ be a complex soft Hilbert space and S, $T : SE(\check{H}) \to SE(\check{H})$ bounded soft linear operator. If there is a unitary soft operator $U : SE(\check{H}) \to SE(\check{H})$ such that $S = UTU^*$, is said to be that S is unitarily equivalent with T.

Proposition 18. *Let $(\check{H}, A)$ be a soft Hilbert space and $S, T : SE(\check{H}) \to SE(\check{H})$ bounded soft operators such that $(S\widetilde{x})(\lambda) = S_\lambda(\widetilde{x}(\lambda))$, $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$. If S, T are unitarily equivalent, then S_λ, T_λ are unitarily equivalent for all $\lambda \in A$.*

Proof. Let S, T unitarily equivalent soft operators, then exists a operator $U : SE(\check{H}) \to SE(\check{H})$ soft unitary such that $S = UTU^*$. Then by Proposition 15 there is a family of unitary operators $\{U_\lambda : \lambda \in A\}$ such that $(U\widetilde{x})(\lambda) = U_\lambda(\widetilde{x}(\lambda))$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$. If $U^*\widetilde{x} = \widetilde{y}$ and $T\widetilde{y} = \widetilde{z}$, we have

$$S_\lambda(\widetilde{x}(\lambda)) = (S\widetilde{x})(\lambda) = (UTU^*\widetilde{x})(\lambda) = (U(T\widetilde{y}))(\lambda) = U_\lambda(\widetilde{z}(\lambda))$$

$$= U_\lambda((T\widetilde{y})(\lambda)) = U_\lambda(T_\lambda(\widetilde{y}(\lambda))) = U_\lambda(T_\lambda(U_\lambda^*(\widetilde{x}(\lambda))))$$

$$= (U_\lambda T_\lambda U_\lambda^*)(\widetilde{x}(\lambda))$$

Thus S_λ is unitarily equivalent with T_λ for all $\lambda \in A$. □

Proposition 19. *Let $(\check{H}, A)$ a soft Hilbert space and $\{T_\lambda : \lambda \in A\}$, $\{S_\lambda : \lambda \in A\}$ two families of unitarily equivalent linear operators, then two unitarily equivalent soft linear operators S, T :*

$SE(\check{H}) \to SE(\check{H})$ can be determined such that $(S\widetilde{x})(\lambda) = S_\lambda(\widetilde{x}(\lambda))$, $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and for all $\widetilde{x} \in SE(\check{H})$.

Proof. Suppose that S_λ, T_λ are unitarily equivalent, then exists a operator $U_\lambda : H \to H$ unitary such that $S_\lambda = U_\lambda T_\lambda U_\lambda^*$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$.
Let's make $U^*\widetilde{x} = \widetilde{y}$ and $T\widetilde{y} = \widetilde{z}$.

$$\begin{aligned}(S\widetilde{x})(\lambda) &= (S_\lambda(\widetilde{x}(\lambda)) = U_\lambda(T_\lambda(U_\lambda^*(\widetilde{x}(\lambda)))) = U_\lambda(T_\lambda(\widetilde{y}(\lambda))) \\ &= U_\lambda((T\widetilde{y})(\lambda)) = (U\widetilde{z})(\lambda) = (U(T\widetilde{y}))(\lambda) = (UTU^*\widetilde{x})(\lambda)\end{aligned}$$

Thus S is unitarily equivalent soft with T. □

Theorem 20. *Let $(\check{H},A)$ be a complex soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ bounded soft such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\widetilde{x} \in SE(\check{H})$ and all $\lambda \in A$. If T is soft normal then T_λ is normal for all $\lambda \in A$.*

Proof. Let T be a normal soft operator, then $TT^* = T^*T$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$. Let's make $T^*\widetilde{x} = \widetilde{y}$ and $T\widetilde{x} = \widetilde{z}$, then

$$\begin{aligned}(T_\lambda T_\lambda^*)(\widetilde{x}(\lambda)) &= T_\lambda((T^*\widetilde{x})(\lambda)) = T_\lambda(\widetilde{y}(\lambda)) = (T\widetilde{y})(\lambda) = (TT^*\widetilde{x})(\lambda) \\ &= (T^*T\widetilde{x})(\lambda) = (T^*\widetilde{z})(\lambda) = T_\lambda^*(\widetilde{z}(\lambda)) = T_\lambda^*((T\widetilde{x})(\lambda)) \\ &= T_\lambda^*(T_\lambda(\widetilde{x}(\lambda))) = (T_\lambda^* T_\lambda)(\widetilde{x}(\lambda))\end{aligned}$$

So $T_\lambda T_\lambda^* = T_\lambda^* T_\lambda$ for all $\lambda \in A$. Thus T_λ is normal for all $\lambda \in A$. □

Theorem 21. *Let A be a set of parameters and $(\check{H},A)$ be a soft Hilbert space. If $\{T_\lambda : \lambda \in A\}$ is a family of normal linear operators, then the linear operator $T : SE(\check{H}) \to SE(\check{H})$ defined by $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and for all $\widetilde{x} \in SE(\check{H})$ is a normal soft operator.*

Proof. Suppose that $T_\lambda T_\lambda^* = T_\lambda^* T_\lambda$ for all $\lambda \in A$. By Theorem 3 exists a family $\{T_\lambda : \lambda \in A\}$ of operators such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$. Let's take $T^*\widetilde{x} = \widetilde{y}$ and $T\widetilde{x} = \widetilde{z}$

$$\begin{aligned}(TT^*\widetilde{x})(\lambda) &= (T(T^*\widetilde{x}))(\lambda) = (T\widetilde{y})(\lambda) = T_\lambda(\widetilde{y}(\lambda)) = T_\lambda((T^*\widetilde{x})(\lambda)) \\ &= T_\lambda^*(T_\lambda(\widetilde{x}(\lambda))) = T_\lambda^*(T\widetilde{x})(\lambda)) = T_\lambda^*(\widetilde{z}(\lambda)) = (T^*T\widetilde{x})(\lambda)\end{aligned}$$

Thus $TT^* = T^*T$. ☐

Proposition 22. *Let $(\check{H},A)$ be a soft Hilbert space $S,T \in B(\check{H})$ unitarily equivalent soft. If T is a normal soft operator, then so is S.*

Proof. Let S, $T : SE(\check{H}) \to SE(\check{H})$ unitarily equivalent soft, then exists $U : SE(\check{H}) \to SE(\check{H})$ unitary soft such that $S = UTU^*$, from where

$$S^* = (UTU^*)^* = ((UT)U^*)^* = (U^*)^*(UT)^* = U(T^*U^*) = UT^*U^*.$$

then for $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$ we have

$$\begin{aligned} SS^* =& (UTU^*)(UT^*U^*) = UTT^*U^* = UT^*TU^* \\ =& UT^*ITU^* = (UT^*U^*)(UTU^*) = S^*S \end{aligned}$$

Thus S is normal soft. ☐

Proposition 23. *Let $(\check{H},A)$ be a complex soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ self-adjoint soft, then $\langle T\widetilde{x},\widetilde{x}\rangle(\lambda) \in \mathbb{R}$ for all $\widetilde{x} \in SE(\check{H})$ and for all $\lambda \in A$.*

Proof. Let $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$. Since T is self-adjoint soft then $T = T^*$ and by theorem 3 exists a family $\{T_\lambda : \lambda \in A\}$ of operators that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$. Then

$$\begin{aligned} \overline{\langle T\widetilde{x},\widetilde{x}\rangle(\lambda)} &= \overline{\langle (T\widetilde{x})(\lambda),\widetilde{x}(\lambda)\rangle_\lambda} = \overline{\langle T_\lambda(\widetilde{x}(\lambda)),\widetilde{x}(\lambda)\rangle_\lambda} \\ &= \overline{\langle \widetilde{x}(\lambda), T_\lambda(\widetilde{x}(\lambda))\rangle_\lambda} \quad \text{since } T_\lambda \text{ is self-adjoint } \forall \lambda \in A \\ &= \langle T_\lambda(\widetilde{x}(\lambda)),\widetilde{x}(\lambda)\rangle_\lambda = \langle (T\widetilde{x})(\lambda),\widetilde{x}(\lambda)\rangle_\lambda = \langle T\widetilde{x},\widetilde{x}\rangle(\lambda) \end{aligned}$$

Thus $\langle T\widetilde{x},\widetilde{x}\rangle(\lambda) \in \mathbb{R}$ for all $\widetilde{x} \in SE(\check{H})$ and for all $\lambda \in A$. ☐

Definition 24. Let $(\check{H},A)$ be a complex soft Hilbert space and S, T self-adjoint soft. If $\langle S\widetilde{x},\widetilde{x}\rangle(\lambda) \leq \langle T\widetilde{x},\widetilde{x}\rangle(\lambda)$ for all $\widetilde{x} \in SE(\check{H})$, we write $S \,\widetilde{\leq}\, T$.

Definition 25. Let $(\check{H},A)$ a complex soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ self-adjoint soft. If $\langle T\widetilde{x},\widetilde{x}\rangle(\lambda) \geq 0$ for all $\widetilde{x} \in SE(\check{H})$ and for all $\lambda \in A$, we write $T \,\widetilde{\geq}\, O$, where O is the null soft operator

According to [3, Theorem 11], if $T : SE(\check{H}) \to SE(\check{H})$ is a bounded soft operator, then T^*T, TT^* and $T+T^*$ and $T+T^*$ are self-adjoint soft, this implies that $T^*T - TT^*$ is self-adjoint soft. This fact, together with the Proposition 23 motivates the following definition.

Definition 26. Let $(\check{H},A)$ a complex soft Hilbert space and $T : SE(\check{H}) \to SE(\check{H})$ a bounded soft operator. If $TT^* \widetilde{\leq} T^*T$, T is called operador hyponormal soft.

Lema 27. *Let $T : SE(\check{H}) \to SE(\check{H})$ be a bounded soft operator. T is hyponormal soft If and only if $\|T^*\widetilde{x}\| \widetilde{\leq} \|T\widetilde{x}\|$ for all $\widetilde{x} \widetilde{\in} \check{H}$.*

Example 28. Let $H = \ell^2$ and $T : SE(\check{H}) \to SE(\check{H})$ a soft operator defined as follows:

$$\begin{aligned} T\{\widetilde{x}_n\} &= T(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \ldots) \\ &= (\widetilde{x}_2, \widetilde{x}_3 + 2\widetilde{x}_1, \widetilde{x}_4 + 2\widetilde{x}_2, \widetilde{x}_5 + 2\widetilde{x}_3, \ldots), \end{aligned}$$

For any $\{\widetilde{x}_n\} \in \ell^2$. clearly T is a bounded soft linear operator.
On the other hand, a calculation shows that T^* is given by:

$$\begin{aligned} T^*\{\widetilde{x}_n\} &= T^*(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \ldots) \\ &= (2\widetilde{x}_2, \widetilde{x}_1 + 2\widetilde{x}_3, \widetilde{x}_2 + 2\widetilde{x}_4, \widetilde{x}_3 + 2\widetilde{x}_5, \ldots) \end{aligned}$$

Finally,

$$\begin{aligned} \|T\{\widetilde{x}_n\}\|^2 - \|T^*\{\widetilde{x}_n\}\|^2 &= \|(\widetilde{x}_2, \widetilde{x}_3 + 2\widetilde{x}_1, \widetilde{x}_4 + 2\widetilde{x}_2, \widetilde{x}_5 + 2\widetilde{x}_3, \ldots)\|^2 \\ &\quad - \|(2\widetilde{x}_2, \widetilde{x}_1 + 2\widetilde{x}_3, \widetilde{x}_2 + 2\widetilde{x}_4, \widetilde{x}_3 + 2\widetilde{x}_5, \ldots)\|^2 \\ &= \widetilde{x}_2^2 + \sum_{n=2}^{\infty} (\widetilde{x}_{n+1} + 2\widetilde{x}_{n-1})^2 - (4\widetilde{x}_2^2 + \sum_{n=2}^{\infty} (\widetilde{x}_{n-1} + 2\widetilde{x}_{n+1})^2) \\ &= -3\widetilde{x}_2^2 + \sum_{n=2}^{\infty} [(\widetilde{x}_{n+1} + 2\widetilde{x}_{n-1})^2 - (\widetilde{x}_{n-1} + 2\widetilde{x}_{n+1})^2)] \\ &= -3\widetilde{x}_2^2 + \sum_{n=2}^{\infty} [-3\widetilde{x}_{n+1}^2 + 3\widetilde{x}_{n-1}^2] \\ &= 3\widetilde{x}_1^2. \end{aligned}$$

So, $[\|T\{\widetilde{x}_n\}\|(\lambda)]^2 - [\|T^*\{\widetilde{x}_n\}\|(\lambda)]^2 \geq 0$. Thus, by the previous lemma we have that T is hyponormal soft.

Theorem 29. *Let $(\check{H},A)$ be a complex soft Hilbert space and $\{T_\lambda : \lambda \in A\}$ a family of operators in $(\check{H},A)$. If $T : SE(\check{H}) \to SE(\check{H})$ defined by $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and all $\widetilde{x}\widetilde{\in}\check{H}$, is hyponormal soft, then T_λ is hyponormal for all $\lambda \in A$.*

Proof. Let T hyponormal soft, then $TT^*\widetilde{\leq}T^*T$. If $T^*\widetilde{x} = \widetilde{y}$ and $T\widetilde{x} = \widetilde{z}$ we have

$$\begin{aligned}
\langle (T_\lambda T_\lambda^*)(\widetilde{x}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda &= \langle T_\lambda((T^*\widetilde{x})(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle T_\lambda(\widetilde{y}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda = \langle (TT^*\widetilde{x})(\lambda), \widetilde{x}(\lambda) \rangle_\lambda \\
&\leq \langle (T^*T\widetilde{x})(\lambda), \widetilde{x}(\lambda) \rangle_\lambda = \langle (T^*\widetilde{z})(\lambda), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle T_\lambda^*(\widetilde{z}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda = \langle T_\lambda^*((T\widetilde{x})(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle (T_\lambda^* T_\lambda)(\widetilde{x}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda.
\end{aligned}$$

Thus T_λ is hyponormal for all $\lambda \in A$. □

Theorem 30. *Let $(\check{H},A)$ be a complex soft Hilbert space and $\{T_\lambda : \lambda \in A\}$ a family of hyponormal operators, then we can determine an operator $T : SE(\check{H}) \to SE(\check{H})$ hyponormal bounded soft linear operator.*

Proof. Let T_λ hyponormal, then $T_\lambda T_\lambda^* \leq T_\lambda^* T_\lambda$ for all $\lambda \in A$. By Theorem 4 exists $T : SE(\check{H}) \to SE(\check{H})$ such that $(T(\widetilde{x}))(\lambda) = T_\lambda(\widetilde{x}(\lambda))$. Let $\lambda \in A$ and $\widetilde{x} \in SE(\check{H})$. If $T^*\widetilde{x} = \widetilde{y}$ and $T\widetilde{x} = \widetilde{z}$ then

$$\begin{aligned}
\langle (TT^*\widetilde{x}), \widetilde{x} \rangle(\lambda) &= \langle (T(T^*\widetilde{x}))(\lambda), \widetilde{x}(\lambda) \rangle_\lambda = \langle (T\widetilde{y})(\lambda), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle T_\lambda(\widetilde{y}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda = \langle T_\lambda((T^*\widetilde{x})(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle (T_\lambda T_\lambda^*)(\widetilde{x}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda \leq \langle (T_\lambda^* T_\lambda)(\widetilde{x}(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle T_\lambda^*((T\widetilde{x})(\lambda)), \widetilde{x}(\lambda) \rangle_\lambda = \langle (T^*\widetilde{z})(\lambda), \widetilde{x}(\lambda) \rangle_\lambda \\
&= \langle (T^*(T\widetilde{x}))(\lambda), \widetilde{x}(\lambda) \rangle_\lambda = \langle (T^*T\widetilde{x}), \widetilde{x} \rangle(\lambda)
\end{aligned}$$

So, $TT^* \widetilde{\leq} T^*T$. Thus T is hyponormal soft. □

Theorem 31. *Let $(\check{H},A)$ a complex soft Hilbert space and S, $T : SE(\check{H}) \to SE(\check{H})$ bounded soft operators. If T is hyponormal soft and S is unitarily equivalent soft with T, then S is hyponormal soft.*

Proof. Exists $U : SE(\check{H}) \to SE(\check{H})$ unitary soft such that $S = UTU^*$ and exist $\{T_\lambda : \lambda \in A\}$, $\{S_\lambda : \lambda \in A\}$ and $\{U_\lambda : \lambda \in A\}$ families of operators such that $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$, $(S\widetilde{x})(\lambda) = S_\lambda(\widetilde{x}(\lambda))$ and $(U\widetilde{x})(\lambda) = U_\lambda(\widetilde{x}(\lambda))$ for all $\lambda \in A$ and for all $\widetilde{x} \in SE(\check{H})$

Now if $\widetilde{x} \in SE(\check{H})$ and $\lambda \in A$ we have

$$\begin{aligned}
\langle (SS^*\widetilde{x}), \widetilde{x}\rangle(\lambda) &= \langle ((UTU^*)(UT^*U^*)\widetilde{x}), \widetilde{x}\rangle(\lambda) \\
&= \langle (UTT^*U^*\widetilde{x}), \widetilde{x}\rangle(\lambda) = \langle (TT^*)(U^*\widetilde{x}), U^*\widetilde{x}\rangle(\lambda) \\
&\leq \langle (T^*T)(U^*\widetilde{x}), U^*\widetilde{x}\rangle(\lambda) = \langle (UT^*TU^*\widetilde{x}), \widetilde{x}\rangle(\lambda) \\
&= \langle (UT^*U^*UTU^*\widetilde{x}), \widetilde{x}\rangle(\lambda) = \langle (S^*S\widetilde{x}), \widetilde{x}\rangle(\lambda)
\end{aligned}$$

So, $SS^* \widetilde{\leq} S^*S$. Thus S is hyponormal soft. $\square$

Corollary 32. *Let $(\check{H}, A)$ a complex soft Hilbert space, $T \in B(\check{H})$. If T and T^* are hyponormal soft, then T is normal soft.*

Proof. Let T and T^* hyponormal soft, then $TT^* \widetilde{\leq} T^*T$ and $T^*(T^*)^* \widetilde{\leq} (T^*)^*T^*$, which implies that $T^*T \widetilde{\leq} TT^*$. Then to $\widetilde{x} \in SE(\check{H})$ it is true that $\langle TT^*\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq \langle T^*T\widetilde{x}, \widetilde{x}\rangle(\lambda)$ and $\langle T^*T\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq \langle TT^*\widetilde{x}, \widetilde{x}\rangle(\lambda)$, then $\langle TT^*\widetilde{x}, \widetilde{x}\rangle(\lambda) - \langle T^*T\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ and $\langle T^*T\widetilde{x}, \widetilde{x}\rangle(\lambda) - \langle TT^*\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ so $\langle (TT^* - T^*T)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ and $\langle (T^*T\widetilde{x} - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ here $\langle (-1)(T^*T - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ and $\langle (T^*T\widetilde{x} - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ so $(-1)\langle (T^*T - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ and $\langle (T^*T\widetilde{x} - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ then $\langle (T^*T - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \geq 0$ and $\langle (T^*T\widetilde{x} - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) \leq 0$ then $\langle (T^*T - TT^*)\widetilde{x}, \widetilde{x}\rangle(\lambda) = 0$, thus by Theorem 11 we have that $T^*T - TT^* = O$, which implies that $T^*T = TT^*$. Thus T is normal soft. $\square$

Proposition 33. *Let $(\check{H}, A)$ be a complex soft Hilbert space and S, $T : SE(\check{H}) \to SE(\check{H})$ bounded soft linear operators. If S and T commute, then S_λ, T_λ commute for each $\lambda \in A$.*

Proof. Suppose that $ST = TS$. Let $\widetilde{x} \in SE(\widetilde{H})$ and $\lambda \in A$. If $T\widetilde{x} = \widetilde{y}$ and $S\widetilde{x} = \widetilde{w}$, then

$$\begin{aligned}
(S_\lambda T_\lambda)(\widetilde{x}(\lambda)) &= S_\lambda((T\widetilde{x})(\lambda)) = (S\widetilde{y})(\lambda) = (ST\widetilde{x})(\lambda) = (TS\widetilde{x})(\lambda) \\
&= (T\widetilde{w})(\lambda) = T_\lambda(\widetilde{w}(\lambda)) = T_\lambda((S\widetilde{x})(\lambda)) = T_\lambda(S_\lambda(\widetilde{x}(\lambda))) \\
&= (T_\lambda S_\lambda)(\widetilde{x}(\lambda))
\end{aligned}$$

So $S_\lambda T_\lambda = T_\lambda S_\lambda$. Thus, S_λ and T_λ conmute. □

Proposition 34. *Let $(\check{H},A)$ be a complex soft Hilbert space, S, $T : SE(\check{H}) \to SE(\check{H})$ two bounded soft linear operators such that $(S\widetilde{x})(\lambda) = S_\lambda(\widetilde{x}(\lambda))$ and $(T\widetilde{x})(\lambda) = T_\lambda(\widetilde{x}(\lambda))$ for each $\lambda \in A$ and all $\widetilde{x} \in SE(\check{H})$. Si S_λ, T_λ conmute for each $\lambda \in A$, then S and T conmute.*

Proof. Suppose that S_λ and T_λ commute for all $\lambda \in A$. Let $\widetilde{x} \in SE(\widetilde{H})$ and $\lambda \in A$. we make $T\widetilde{x} = \widetilde{y}$ and $S\widetilde{x} = \widetilde{z}$ we have

$$\begin{aligned}(ST\widetilde{x})(\lambda) &= (S(T\widetilde{x}))(\lambda) = S_\lambda(\widetilde{y}(\lambda)) = S_\lambda((T\widetilde{x})(\lambda)) = S_\lambda(T_\lambda(\widetilde{x}(\lambda)))\\ &= (S_\lambda T_\lambda)(\widetilde{x}(\lambda)) = (T_\lambda S_\lambda)(\widetilde{x}(\lambda)) = T_\lambda((S\widetilde{x})(\lambda))\\ &= T_\lambda(\widetilde{z}(\lambda)) = (T\widetilde{z})(\lambda) = T((S\widetilde{x})(\lambda)) = (TS\widetilde{x})(\lambda)\end{aligned}$$

So, $ST = TS$. Thus S and T commute □

Theorem 35. *Let $(\check{H},A)$ be a complex soft Hilbert space and $S, T : SE(\check{H}) \to SE(\check{H})$ hyponormal soft operators. If S and T commute and $T^*S = ST^*$, then ST is hyponormal soft.*

Proof. Since S, $T \in B(\check{H})$ are commutative hyponormal soft operators then $SS^* \leq S^*S$, $TT^* \leq T^*T$ and $ST = TS$. Si $ST^* = T^*S$, then $(ST^*)^* = (T^*S)^*$, where $(T^*)^*S^* = (S^*)(T^*)^*$, So $TS^* = S^*T$.
Let $\widetilde{x} \in SE(\check{H})$, then

$$\begin{aligned}\langle (ST)(ST)^*\widetilde{x},\widetilde{x}\rangle(\lambda) &= \langle STT^*S^*\widetilde{x},\widetilde{x}\rangle(\lambda) = \langle TT^*(S^*\widetilde{x}),S^*\widetilde{x}\rangle(\lambda)\\ &\leq \langle T^*T(S^*\widetilde{x}),S^*\widetilde{x}\rangle(\lambda) = \langle S^*T\widetilde{x},TS^*\widetilde{x}\rangle(\lambda)\\ &= \langle S^*T\widetilde{x},S^*T\widetilde{x}\rangle(\lambda) = \langle SS^*(T\widetilde{x}),T\widetilde{x}\rangle(\lambda)\\ &\leq \langle S^*S(T\widetilde{x}),T\widetilde{x}\rangle(\lambda) = \langle (ST)^*(ST)\widetilde{x},\widetilde{x}\rangle(\lambda)\end{aligned}$$

So $(ST)(ST)^* \widetilde{\leq} (ST)^*(ST)$. Thus ST is hyponormal soft. □

Theorem 36. *Let $(\check{H},A)$ be a complex soft Hilbert space and S, $T : SE(\check{H}) \to SE(\check{H})$ hyponormal soft operators, such that $TS^* = S^*T$, then $S+T$ is hyponormal soft.*

Proof. Since $TS^* = S^*T$, then $(TS^*)^* = (S^*T)^*$, where $(S^*)^*T^* = T(S^*)^*$, So $ST^* = T^*S$. Also since T and S are hyponormal soft, then $\langle\, TT^*\widetilde{x},\widetilde{x}\,\rangle(\lambda)\widetilde{\leq}\langle\, T^*T\widetilde{x},\widetilde{x}\,\rangle(\lambda)$ and $\langle SS^*\widetilde{x},\widetilde{x}\rangle(\lambda)\widetilde{\leq}\langle S^*S\widetilde{x},\widetilde{x}\rangle(\lambda)$. Let $\widetilde{x}\in SE(\check{H})$ and $\lambda \in A$ we have

$$\begin{aligned}
\langle\,(S+T)(S+T)^*\widetilde{x},\widetilde{x}\rangle(\lambda) &= \langle\,(SS^*\widetilde{x}+ST^*\widetilde{x}+TS^*\widetilde{x}+TT^*\widetilde{x}),\widetilde{x}\rangle(\lambda)\\
&= \langle SS^*\widetilde{x},\widetilde{x}\rangle(\lambda)+\langle ST^*\widetilde{x},\widetilde{x}\rangle(\lambda)+\langle TS^*\widetilde{x},\widetilde{x}\rangle(\lambda)+\langle TT^*\widetilde{x},\widetilde{x})\,\rangle(\lambda)\\
&\leq \langle S^*S\widetilde{x},\widetilde{x}\rangle(\lambda)+\langle ST^*\widetilde{x},\widetilde{x}\rangle(\lambda)+\langle TS^*\widetilde{x},\widetilde{x}\rangle(\lambda)+\langle T^*T\widetilde{x},\widetilde{x})\,\rangle(\lambda)\\
&= \langle\,(S^*S+ST^*+TS^*+T^*T)\widetilde{x},\widetilde{x}\rangle(\lambda)\\
&= \langle\,((S^*S+S^*T)+(T^*S+T^*T))\widetilde{x},\widetilde{x}\rangle(\lambda)\\
&= \langle\,(S^*(S+T)+T^*(S+T))\widetilde{x},\widetilde{x}\rangle(\lambda)\\
&= \langle\,((S^*+T^*)(S+T))\widetilde{x},\widetilde{x}\rangle(\lambda)\\
&= \langle\,((S+T)^*(S+T))\widetilde{x},\widetilde{x}\rangle(\lambda)
\end{aligned}$$

So $(S+T)(S+T)^* \widetilde{\leq} (S+T)^*(S+T)$. Thus $S+T$ is hyponormal soft. □

3. Conclusion

Through a family of operators in Hilbert spaces, a soft linear operator can be constructed. Furthermore, if the family of linear operators in a set of Hilbert spaces has the property of being hyponormal, unitarily equivalent or normal then the soft operator associated with the soft set of said Hilbert spaces inherits these properties.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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