Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 6226-6241
https://doi.org/10.28919/jmcs/6156
ISSN: 1927-5307

CLASSIFICATION OF ZERO DIVISOR GRAPHS OF COMMUTATIVE RING WITH ORDER LESS THAN OR EQUAL TO 22

HUSAM Q. MOHAMMAD*, NAZAR H. SHUKER

Department of Mathematics/College of Computers Sciences and Mathematics Mosul, Iraq

Abstract. Some authors investigated the classification of the zero divisor graphs of commutative ring of degree less than or equal 14. In this paper, we extend this classification to the zero divisor graph of commutative ring of degree less than or equal to 22.

Keywords: zero divisor graph; commutative rings; local rings.

2010 AMS Subject Classification: 13A99, 20M14, 05C25.

1. Introduction

For a commutative ring with identity, we mean (R,M) is a local ring R with maximal ideal M, F_{q_i} denoted by a field with degree q_i. A ring R is called reduced if R has no non zero nilpotent elements in R. Z_n denote the ring of integers modulo n. Clearly for every prime number p, Z_p is a field and $Z_p \cong F_p$. For every set S we denote $\mid S \mid$ the cardinality of S and $S^* = S - \{0\}$. Let $Z(R)$ be the set of all zero divisor elements in R and let $\Gamma(R)$ be a simple graph with $V(\Gamma(R)) = Z(R)^*$ and $ab \in E(\Gamma(R))$ if and only if $a.b = 0$. This concept was introduced by I. Beck in 1988 [5] and modified by D.F. Anderson and P.S. Livingston in 1999 [1] they proved this graph is connected with diameter less than or equal 3 and $\Gamma(R)$ finite graph if and only if

*Corresponding author
E-mail address: husamqm@uomosul.edu.iq
Received May 28, 2021

R is finite ring or integral domain. Many authors studied this concept see for example [3], [7], [10] and [11]. It is well known that every ring can be realizable as graph but the converse is not true in general, In [1] D.F. Anderson and P.S. Livingston classified the graphs with order less than or equal four. Later some authors classified the graphs with order less than or equal 14 see [3],[13] and [15]. In this paper we classify graphs to be realizable ring with order between 16 and 22 vertices. In section two we investigate special cases when R is local or reduced ring. In section three we extend this results for all ring with zero divisor lies between 16 and 22. In section four we use this results to find all possible cases to be matched by a ring.

2. Special Cases

Let R be a finite commutative ring with identity. Then R is local ring or $R \cong R_1 \times R_2 \times ... \times R_n$, where every R_i is local ring $i = 1, 2, \dots n$ *and* $n \geq 2$ [4, Lemmae 2]. Let (R, M) be a local ring with maximal ideal $M \neq \{0\}$, then $|R| = p^m$, where p is a prime number and integer $m \geq 2$ also $|Z(R)| = p^t$, where $t < n$. Hence there are three cases for local ring when $16 \leq |Z(R)^*| \leq 22$.

Proposition 2.1. *Let (R, M) be a local ring with $|Z(R)^*| = 16$, then $R \cong Z_{289}$ or $Z_{17}[A]/(A^2)$*

Proof. Since $|Z(R)^*| = 16$, then $|M| = |Z(R)| = 17$ and $|R| = 17^2 = 289$. So by [4, Theorem 2] $R \cong Z_{289}$ or $Z_{17}[A]/(A^2)$ □

By similar way we can prove the following result.

Proposition 2.2. *Let (R, M) be a local ring, if*

i. $|Z(R)^*| = 18$, *then* $R \cong Z_{361}$ *or* $Z_{19}[A]/(A^2)$

ii. $|Z(R)^*| = 22$, *then* $R \cong Z_{529}$ *or* $Z_{23}[A]/(A^2)$

Clearly if R is a finite reduced ring, then $R \cong F_{q_1} \times \ Fq_2 \ \times ... \ \times \ Fq_n$, where each F_{q_i} is a field, and n is a positive integer such that $n \geq 2$.

N. Ganesan in [8, Theorem 1] proved that for every commutative ring with identity if $|Z(R)| = t$, then R has at most $(t+1)^2$. Also if (R_1, M_1) and (R_2, M_2) are finite local rings, then $|Z(R_1 \times R_1)^*| = |R_1||M_2| + |R_2||M_1| - |M_1||M_2| - 1$. Consequently if R_i is a field for $i = 1, 2$, then $|M_i| = 1$ and $|Z(R_1 \times R_2)^*| = |R_1| + |R_2| - 2$.

Lemma 2.3. [14] *If* (R_1,M_1), (R_2,M_2) *and* (R_3,M_3) *are finite local rings, then* $|Z(R_1 \times R_2 \times R_3)^*| = |R_1||R_2||M_3| + |Z(R_1 \times R_2)|(|R_3| - |M_3|) - 1$, *where* $|Z(R_1 \times R_2)| = |R_1||M_2| + |R_2||M_1| - |M_1||M_2|$.

Lemma 2.4. [14] *If* F_{q_1}, F_{q_2} *and* F_{q_3} *are fields, then* $|Z(F_{q_1} \times F_{q_2} \times F_{q_3})^*| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}|$.

Lemma 2.5. [4] *Suppose R be a ring has t nonzero zero-divisors and Let k be the smallest positive integer such that,* $t<2^k - 2$. *Then R is a product of* k–1 *or fewer fields.*

Firs, we shall give the following result.

Lemma 2.6. *Let R be a ring and* $R \cong R_1 \times R_2 \times R_3 \times R_4$ *with* R_i *local for every* $1 \leq i \leq 4$, *then*

(1) If $|R_i| \geq 3$ *for some* $1 \leq i \leq 4$, *then* $|Z(R)^*| \geq 21$
(2) If $|R_i| \geq 4$ *for some* $1 \leq i \leq 4$, *then* $|Z(R)^*| \geq 30$
(3) If $|R_1|$ *and* $|R_2| \geq 3$, *then* $|Z(R)^*| \geq 34$
(4) $|Z(R)^*| = 14$ *if and only if* $R \cong F_2 \times F_2 \times F_2 \times F_2$

Proof. (1) Without loss of generality assume that $|R_1| \geq 3$. Then there exists $a \in R - \{0,1\}$. So, (1,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,1), (1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (0,1,1,1), (a,0,0,0), (a,1,0,0), (a,0,1,0), (a,0,0,1), (a,1,1,0), (a,1,0,1), (a,0,1,1) are non-zero zero-divisors of $Z(R_1 \times R_2 \times R_3 \times R_4)$. Therefore $|Z(R)| \geq 21$

By a similar way we can prove 2, 3, and 4. □

Remark 2.7. From Lemma 2.5, R is a product of at most four fields, when $16 \leq |Z(R)^*| \leq 22$. Also Lemma 2.6, R is not product of at four fields, when $16 \leq |Z(R)^*| \leq 22$ except the case $|Z(R)^*| = 21$ and hence $R \cong F_2 \times F_2 \times F_2 \times F_3$.

Now, we shall investigate all reduced rings with $16 \leq |Z(R)^*| \leq 22$.

Proposition 2.8. *Let R be a reduced ring with* $|Z(R)^*| = 16$, *then* $R \cong F_2 \times F_{16}$, $F_5 \times F_{13}$, $F_7 \times F_{11}$ *or* $F_9 \times F_9$

Proof. If R is a product of three fields and $|F_{q_i}| \geq 3$ for all $1 \leq i \leq 3$, then by Lemma 2.4 $|Z(R)^*| \geq 18$ which is a contradiction, so $|F_{q_i}| = 2$ for some $1 \leq i \leq 3$, without loss generality let $|F_{q_1}| = 2$, then by Lemma 2.4 we have $|F_{q_3}| = \dfrac{18 - |F_{q_2}|}{1 + |F_{q_2}|}$. Since $|F_{q_2}|$ is integer, then we have a contradiction. So R is a product of two fields and hence $|F_{q_1}| + |F_{q_2}| = 18$. Therefore $|F_{q_1}| = 2$, $|F_{q_2}| = 16$, $|F_{q_1}| = 5$, $|F_{q_2}| = 13$, $|F_{q_1}| = 7$, $|F_{q_2}| = 11$ or $|F_{q_1}| = 9$, $|F_{q_2}| = 9$. This gives $R \cong F_2 \times F_{16}$, $F_5 \times F_{13}$, $F_7 \times F_{11}$ or $F_9 \times F_9$. □

Proposition 2.9. *Let R be a reduced ring with $|Z(R)^*| = 17$, then $R \cong F_2 \times F_3 \times F_4$, $F_2 \times F_{17}$, $F_3 \times F_{16}$ or $F_8 \times F_{11}$*

Proof. If R is a product of three fields, then by Lemma 2.4 $17 = |Z(F_{q_1} \times F_{q_2} \times F_{q_3})^*| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}|$. So that $|F_{q_1}| = 2$, $|F_{q_2}| = 3$ and $|F_{q_3}| = 4$, then we have $R \cong F_2 \times F_3 \times F_4$. Finally if R is a product of two fields, then $|F_{q_1}| + |F_{q_2}| = 19$. So $|F_{q_1}| = 2$, $|F_{q_2}| = 17$, $|F_{q_1}| = 3$, $|F_{q_2}| = 16$ or $|F_{q_1}| = 8$, $|F_{q_2}| = 11$. Thus $R \cong F_2 \times F_{17}$, $F_3 \times F_{16}$ or $F_8 \times F_{11}$. □

Proposition 2.10. *Let R be a reduced ring with $|Z(R)^*| = 18$, then $R \cong F_3 \times F_3 \times F_3$, $F_3 \times F_{17}$, $F_4 \times F_{16}$, $F_7 \times F_{13}$ or $F_9 \times F_{11}$.*

Proof. If R is a product of three fields, then by Lemma 2.4 $18 = |Z(F_{q_1} \times F_{q_2} \times F_{q_3})^*| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}|$. So $|F_{q_1}| = |F_{q_2}| = |F_{q_3}| = 3$, and we have $R \cong F_3 \times F_3 \times F_3$. Finally if R is a product of two fields, then $|F_{q_1}| + |F_{q_2}| = 20$. So $|F_{q_1}| = 3$, $|F_{q_2}| = 17$, $|F_{q_1}| = 4$, $|F_{q_2}| = 16$, $|F_{q_1}| = 7$, $|F_{q_2}| = 13$ or $|F_{q_1}| = 9$, $|F_{q_2}| = 11$. Hence $R \cong F_3 \times F_{17}$, $F_4 \times F_{16}$, $F_7 \times F_{13}$ or $F_9 \times F_{11}$ □

Proposition 2.11. *Let R be a reduced ring with $|Z(R)^*| = 19$, then $R \cong F_2 \times F_{19}$, $F_4 \times F_{17}$, $F_5 \times F_{16}$ or $F_8 \times F_{13}$*

Proof. If R is a product of three fields, then $|Z(R)^*| = |Z(F_{q_1} \times F_{q_2} \times F_{q_3})^*| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}| = 19$, this leads to contradiction. Therefore R is a product of two fields and we have $|F_{q_1}| + |F_{q_2}| = 21$. So $|F_{q_1}| = 2$, $|F_{q_2}| = 19$, $|F_{q_1}| = 4$, $|F_{q_2}| = 17$, $|F_{q_1}| = 5$, $|F_{q_2}| = 16$ or $|F_{q_1}| = 8$, $|F_{q_2}| = 13$. Therefore $R \cong F_2 \times F_{19}$, $F_4 \times F_{17}$, $F_5 \times F_{16}$ or $F_8 \times F_{13}$. □

Proposition 2.12. *Let R be a reduced ring with* $|Z(R)^*| = 20$*, then* $R \cong F_3 \times F_{19}$*,* $F_5 \times F_{17}$*,* $F_9 \times F_{13}$ *or* $F_{11} \times F_{11}$.

Proof. If R is a product of three fields, then $|Z(R)^*| = |Z(F_{q_1} \times F_{q_2} \times F_{q_3})^*| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}| = 20$, which is a contradiction. So that R is a product of two fields, and we have $|F_{q_1}| + |F_{q_2}| = 22$. Therefore $R \cong F_3 \times F_{19}$, $F_5 \times F_{17}$, $F_9 \times F_{13}$ or $F_{11} \times F_{11}$. □

Proposition 2.13. *Let R be a reduced ring with* $|Z(R)^*| = 21$*, then* $R \cong F_2 \times F_2 \times F_2 \times F_3$*,* $F_2 \times F_2 \times F_7$*,* $F_2 \times F_3 \times F_5$*,* $F_4 \times F_{19}$ *or* $F_7 \times F_{16}$.

Proof. If R is a product of four fields, then by Remark 2.7 $R \cong F_2 \times F_2 \times F_2 \times F_3$. If R is a product of three fields, then $|Z(R)| = |Z(F_{q_1} \times F_{q_2} \times F_{q_3})| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}| = 21$. Then the solutions of this equation are $|F_{q_1}| = |F_{q_2}| = 2$ and $|F_{q_3}| = 7$ or $|F_{q_1}| = 2$, $|F_{q_2}| = 3$ and $|F_{q_3}| = 5$. So $R \cong F_2 \times F_2 \times F_7$ or $F_2 \times F_3 \times F_5$. If R is a product of two fields, then $|F_{q_1}| = |F_{q_2}| = 23$, therefore $R \cong F_4 \times F_{19}$ or $F_7 \times F_{16}$. □

Proposition 2.14. *Let R be a reduced ring with* $|Z(R)^*| = 22$*, then* $R \cong F_2 \times F_4 \times F_4$*,* $F_5 \times F_{19}$*,* $F_7 \times F_{17}$*,* $F_8 \times F_{16}$ *or* $F_{11} \times F_{13}$.

Proof. If R is a product of three fields, then $|Z(R)^*| = |Z(F_{q_1} \times F_{q_2} \times F_{q_3})^*| = |F_{q_1}||F_{q_2}| + |F_{q_1}||F_{q_3}| + |F_{q_2}||F_{q_3}| - |F_{q_1}| - |F_{q_2}| - |F_{q_3}| = 22$. This implies that $|F_{q_3}| = \dfrac{22 + |F_{q_1}| + |F_{q_2}| - |F_{q_1}||Fq_2|}{|F_{q_1}| + |F_{q_2}| - 1}$, the only solution of this equation is $|F_{q_1}| = 2$ and $|F_{q_2}| = |F_{q_3}| = 4$. So $R \cong F_2 \times F_4 \times F_4$. Finally if R is a product of two fields, then $|F_{q_1}| + |F_{q_2}| = 24$. Therefore $R \cong F_5 \times F_{19}$, $F_7 \times F_{17}$, $F_8 \times F_{16}$ or $F_{11} \times F_{13}$. □

3. General Cases

In this section we investigate the classification of every commutative ring R with zero divisors of degree less than or equal to 22 .

Lemma 3.1. *Let* $R \cong R_1 \times R_2 \times R_3$*, where* R_i *local rings for every* $1 \leq i \leq 3$*, then*

i. If R_i *not field for some* $1 \leq i_1, i_2 \leq 3$*, then* $|Z(R)^*| \geq 27$.

ii. If $|R_1|$*,* $|R_2| \geq 3$ *and* R_3 *not field, then* $|Z(R)^*| \geq 27$.

iii. If $|R_2| \geq 3$ and R_3 not field, then $|Z(R)^*| \geq 19$

iv. If R_3 not field and $|R_3| \neq 4$, then $|Z(R)^*| \geq 26$

Proof. Directed by Lemma 2.3 □

By [4, Corollary 1], if $|Z(R)| = p$, then R is a local or reduced. If $|Z(R)^*| = 16$, 18 or 22 are done. So we need to investigate the cases when $|Z(R)^*| = 17$, 19, 20 or 21.

Next, we shall give the following main results.

Theorem 3.2. *Let R be a ring with $|Z(R)^*| = 17$, then $R \cong F_2 \times F_3 \times F_4$, $F_2 \times F_{17}$, $F_3 \times F_{16}$, $F_8 \times F_{11}$, $F_4 \times Z_9$ or $F_4 \times Z_3[A]/(A^2)$.*

Proof. By Proposition 2.9, if R is a reduced ring, then $R \cong F_2 \times F_3 \times F_4$, $F_2 \times F_{17}$, $F_3 \times F_{16}$ or $F_8 \times F_{11}$. If R is not reduced and product of three local rings we get a contradiction. Also If R_1, R_2 are not fields and $R_i \neq 4$ for some $i = 1,2$, then $17 = |Z(R)^*| \geq 21$ which is a contradiction. Also if $|R_1| = |R_2| = 4$, then $|Z(R)^*| = 11$. If R_1 is a field and R_2 not field, then $|R_2| + (|R_1| - 1)|M_2| = 18$ which implies $|R_1| = 4$, $|R_2| = 9$ and $|M_2| = 3$. Hence $R \cong F_4 \times Z_9$ or $F_4 \times Z_3[A]/(A^2)$. □

Theorem 3.3. *Let R be a ring with $|Z(R)^*| = 19$, then $R \cong F_2 \times F_{19}$, $F_4 \times F_{17}$, $F_5 \times F_{16}$, $F_8 \times F_{13}$, $F_2 \times F_3 \times Z_4$, $F_2 \times F_3 \times Z_2[A]/(A^2)$, $F_2 \times F_4[A]/(A^2)$, $F_2 \times F_8[A]/(A^2, 2A)$, $F_4 \times Z_8$, $F_4 \times F_2[A]/(A^3)$, $F_4 \times Z_4[A]/(2A, A^2 - 2)$, $F_4 \times Z_4[A]/(2A, A^2)$, $F_4 \times F_2[A_1, A_2]/(A_1, A_2)^2$, $F_9 \times Z_4$ or $F_9 \times F_2[A]/(A^2)$*

Proof. If R is a reduced, then by Proposition, 2.11 $R \cong F_2 \times F_{19}$, $F_4 \times F_{17}$, $F_5 \times F_{16}$ or $F_8 \times F_{13}$. Let R is not reduced and $R \cong R_1 \times R_2 \times R_3$, where R_i local rings for all $1 \leq i \leq 3$. If R_i not field for some $1 \leq i_1, i_2 \leq 3$ or $|R_1|$, $|R_2| \geq 3$, then $19 = |Z(R)^*| \geq 27$ which is a contradiction. Also if $|R_1| \neq 4$ or $|R_1| \geq 4$, then $|Z(R)^*| \geq 25$ which is a contradiction. So $|R_1| = 2$ and $|R_2| = 2$ or 3 and $|R_3| = 4$. If $|R_2| = 2$ leads to contradiction. If $|R_2| = 3$, then we have $R \cong F_2 \times F_3 \times Z_4$ or $F_2 \times F_3 \times Z_2[A]/(A^2)$. If $R \cong R_1 \times R_2$, since R not reduced, then R_1 and R_2 not fields or R_1 is a field and R_2 not field, If R_1 and R_2 not fields, then we get a contradiction. If R_1 is a field and R_2 not field with maximal ideal M_2, then we have six cases:

Case1: If $|R_1| = 2$, then $|R_2| = 16$ and $|M_2| = 4$. By [6] there are 21 rings of order $16 = p^4$, the only two rings are maximal ideal of order 4. Hence $R_2 \cong F_4[A]/(A^2)$ or $F_8[A]/(A^2,2A)$ and $R \cong F_2 \times F_4[A]/(A^2)$ or $F_2 \times F_8[A]/(A^2,2A)$.

Case2: If $|R_1| = 3$, then $|R_2| = 20 - 2|M_2|$, this is leads to contradiction.

Case3: If $|R_1| = 4$, then $|R_2| = 20–3|M_2|$, implies that $|R_2| = 8$ and $|M_2| = 4$. Therefore by [6] $R_2 \cong Z_8$, $F_2[A]/(A^3)$, $Z_4[A]/(2A,A^2-2)$, $Z_4[A]/(2A,A^2)$ or $F_2[A_1,A_2]/(A_1,A_2)^2$. Thus $R \cong F_4 \times Z_8$, $F_4 \times F_2[A]/(A^3)$, $F_4 \times Z_4[A]/(2A,A^2-2)$, $F_4 \times Z_4[A]/(2A,A^2)$ or $F_4 \times F_2[A_1,A_2]/(A_1,A_2)^2$.

Case4: If $|R_1| = 5$, 7 or 8, then we get a contradiction.

Case5: If $|R_1| = 9$, then $|R_2| = 20 - 8|M_2|$, implies that $|R_2| = 4$ and $|M_2| = 2$. Therefore $R_2 \cong Z_4$ or $F_2[A]/(A^2)$ and $R \cong F_9 \times Z_4$ or $F_9 \times F_2[A]/(A^2)$.

Case6: If $|R_1| \geq 11$, then $|Z(R)^*| \geq 23$ which is a contradiction. □

Theorem 3.4. *Let R be a ring with $|Z(R)^*| = 20$, then $R \cong F_3 \times F_{19}$, $F_5 \times F_{17}$, $F_9 \times F_{13}$, $F_{11} \times F_{11}$, $F_5 \times Z_9$ or $F_5 \times Z_3[A]/(A^2)$,*

Proof. If R is a reduced ring, then by Proposition 2.12, $R \cong F_3 \times F_{19}$, $F_5 \times F_{17}$, $F_9 \times F_{13}$ or $F_{11} \times F_{11}$. If R is not reduced, since R not local and from Lemmas 2.5 and 2.6 we have R is a product three or two local rings. If R is a product three local rings, $R_1 \times R_2 \times R_3$ and R_i not field for some $1 \leq i_1, i_2 \leq 3$, then $20 = |Z(R)^*| \geq 27$ which is a contradiction. So it's enough to discuss the case when R_1 and R_2 are fields and R_3 not field. If $|R_3| \neq 4$, then Lemma 3.1 leads a contradiction. Also if $|R_3| = 4$, then R_3 has a maximal ideal M_3 such that $|M_3| = 2$. Hence $20 = |Z(R)^*| = |R_1||R_2| \cdot 2 + (|R_1| + |R_2| - 1) \cdot (4-2) - 1$. Which implies that $|R_1| = \dfrac{23 - 2|R_2|}{2(|R_2|+1)}$, which is a contradiction. If R is a product two local rings, $R_1 \times R_2$, then we have two cases:

Case1: R_1 and R_2 are not fields, then by [14], $20 = |Z(R)^*| = |R_1||M_2| + |R_2||M_1| - |M_1||M_2| - 1$. This lead to a contradiction.

Case2: R_1 is a field and R_2 is not a field, then $|M_1| = 1$ and so $20 = |Z(R)^*| = |R_1||M_2| + |R_2| - |M_2| - 1$. Therefore $R_1 \cong F_5$ and $R_2 \cong Z_9$ or $Z_3[A]/(A^2)$ and we have $R \cong F_5 \times Z_9$ or $F_5 \times Z_3[A]/(A^2)$. □

Theorem 3.5. *Let R be a ring with $|Z(R)^*| = 21$, then R is reduced. Consequently $R \cong F_2 \times F_2 \times F_3$, $F_2 \times F_2 \times F_7$, $F_2 \times F_3 \times F_5$, $F_4 \times F_{19}$ or $F_7 \times F_{16}$.*

Proof. If R is a product of three local rings, $R_1 \times R_2 \times R_3$ and R_i not field for some $1 \leq i_1, i_2 \leq 3$, then by Lemma 3.1 $|Z(R)^*| \geq 27$ that is a contradicts. If R_1 and R_2 are fields and R_3 not field and $|R_3| \neq 4$, then $|R_3| \geq 8$ and $|M_3| \geq 3$ applying Lemma 2.3 we get $|Z(R)^*| = |R_1||R_2||M_3| + (|R_1|+|R_2|-1)(|R_3|-|M_3|)-1 \geq 26$ which is a contradiction. If $|R_3| = 4$, then by [6] $|M_3| = 2$ and we have $21 = |Z(R)^*| = 2 \cdot |R_1||R_2| + 2(|R_1|+|R_2|-1)-1$, this lead a contradicts. Finally, if R is a product of two local rings. By a same way of prove Theorem 3.4, we have a contradiction. Therefore R is reduced and by Proposition 2.13 $R \cong F_2 \times F_2 \times F_2 \times F_3$, $F_2 \times F_2 \times F_7$, $F_2 \times F_3 \times F_5$, $F_4 \times F_{19}$ or $F_7 \times F_{16}$. □

4. CLASSIFICATION OF ZERO DIVISOR GRAPH WITH ORDER BETWEEN 16 AND 22

In this section we classify of all zero divisor graphs with vertices between 16 and 21. Recall that K_n ($K_{n,m}$ res.) denoted a complete graph of order n (complete bipartite graph res.).

vertices	Ring type	Graph
16	Z_{289} or $Z_{17}[A]/(A^2)$	K_{16}
	$F_2 \times F_{16}$	$K_{1,15}$
	$F_5 \times F_{13}$	$K_{4,12}$
	$F_7 \times F_{11}$	$K_{6,10}$
	$F_9 \times F_9$	$K_{8,8}$
17	$F_2 \times F_3 \times F_4$	Fig. 1
	$F_2 \times F_{17}$	$K_{1,16}$
	$F_3 \times F_{16}$	$K_{2,15}$
	$F_8 \times F_{11}$	$K_{7,1o}$
	$F_4 \times Z_9$ or $F_4 \times Z_3[A]/(A^2)$	Fig. 2
18	Z_{361} or $Z_{19}[A]/(A^2)$	K_{18}
	$F_3 \times F_{17}$	$K_{2,16}$
	$F_4 \times F_{16}$	$K_{3,15}$
	$F_7 \times F_{13}$	$K_{6,12}$
	$F_9 \times F_{11}$	$K_{8,10}$
	$F_3 \times F_3 \times F_3$	Fig. 3

vertices	Ring type	Graph
18	Z_{361} or $Z_{19}[A]/(A^2)$	K_{18}
	$F_3 \times F_{17}$	$K_{2,16}$
	$F_4 \times F_{16}$	$K_{3,15}$
	$F_7 \times F_{13}$	$K_{6,12}$
	$F_9 \times F_{11}$	$K_{8,10}$
	$F_3 \times F_3 \times F_3$	Fig. 3
19	$F_2 \times F_{19}$	$K_{1,18}$
	$F_4 \times F_{17}$	$K_{3,16}$
	$F_5 \times F_{16}$	$K_{4,15}$
	$F_8 \times F_{13}$	$K_{7,12}$
	$F_2 \times F_3 \times Z_4$ or $F_2 \times F_3 \times F_2[A]/(A^2)$	Fig. 4
	$F_2 \times F_4[A]/(A^2)$	Fig. 5
	$F_2 \times F_8[A]/(A^2, 2A)$	Fig. 6
	$F_4 \times Z_8$, $F_4 \times F_2[A]/(A^3)$ or $F_4 \times Z_4[A]/(2A, A^2-2)$	Fig. 7
	$F_4 \times Z_4[A]/(2A, A^2)$ or $F_4 \times F_2[A_1, A_2]/(A_1, A_2)^2$	Fig. 8
	$F_9 \times Z_4$ or $F_9 \times F_2[A]/(A^2)$	Fig. 9
20	$F_3 \times F_{19}$	$K_{2,18}$
	$F_5 \times F_{17}$	$K_{4,16}$
	$F_9 \times F_{13}$	$K_{8,12}$
	$F_{11} \times F_{11}$	$K_{10,10}$
	$F_5 \times Z_9$ or $F_5 \times F_3[A]/(A)^2$	Fig. 10
21	$F_2 \times F_2 \times F_2 \times F_3$	Fig. 11
	$F_2 \times F_2 \times F_7$	Fig. 12
	$F_2 \times F_3 \times F_5$	Fig. 13
	$F_4 \times F_{19}$	$K_{3,18}$
	$F_7 \times F_{16}$	$K_{6,15}$

vertices	Ring type	Graph
22	Z_{529} or $F_{23}[A]/(A^2)$	K_{22}
	$F_2 \times F_4 \times F_4$	Fig. 14
	$F_5 \times F_{19}$	$K_{4,18}$
	$F_7 \times F_{17}$	$K_{6,16}$
	$F_8 \times F_{16}$	$K_{7,15}$
	$F_{11} \times F_{13}$	$K_{10,12}$

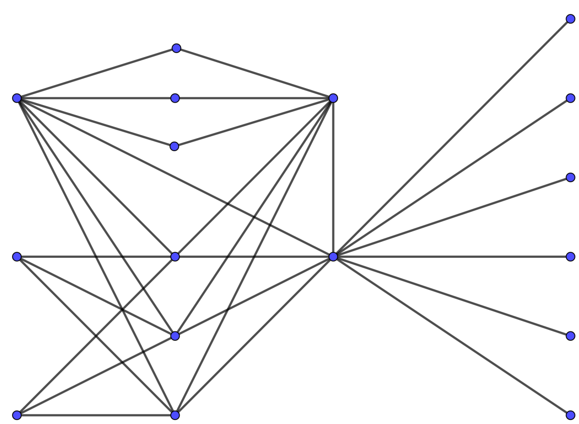

FIGURE 1. $\Gamma(F_2 \times F_3 \times F_4)$

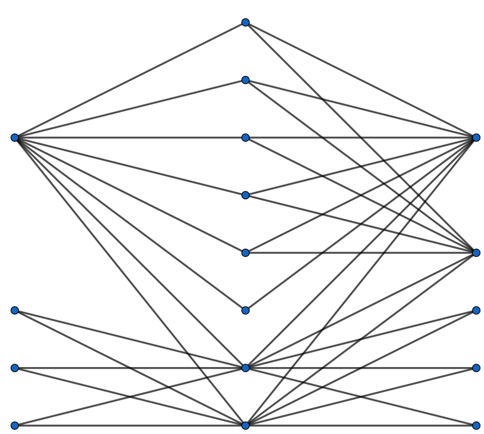

FIGURE 2. $\Gamma(F_4 \times Z_9)$ or $\Gamma(F_4 \times Z_3[A]/(A^2))$

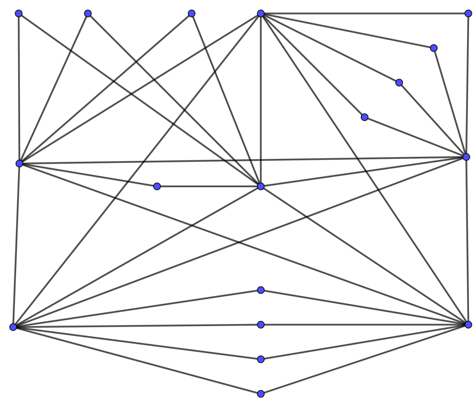

FIGURE 3. $\Gamma(F_3 \times F_3 \times F_3)$

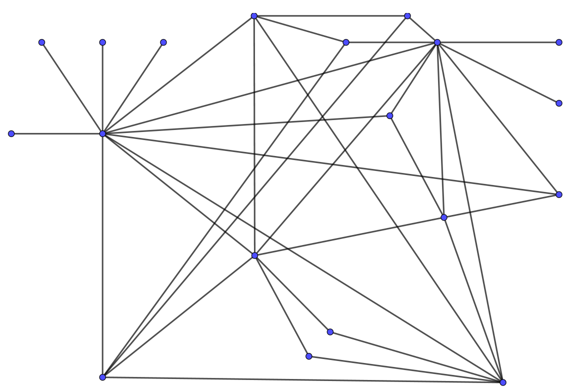

FIGURE 4. $\Gamma(F_2 \times F_3 \times Z_4)$ or $\Gamma(F_2 \times F_3 \times F_2[A]/(A^2))$

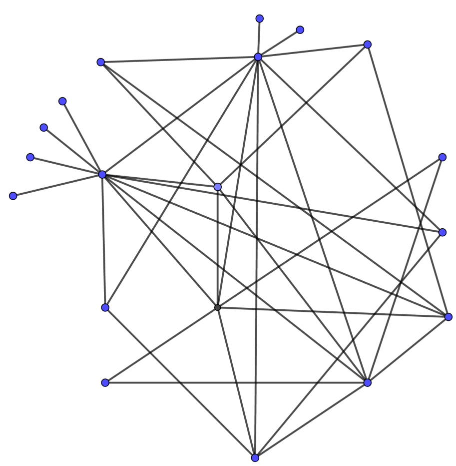

FIGURE 5. $\Gamma(F_2 \times F_4[A]/(A^2))$

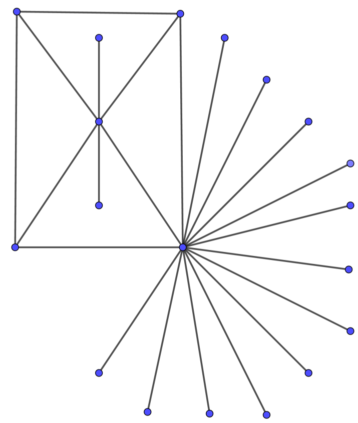

FIGURE 6. $\Gamma(F_2 \times F_8[A]/(A^2, 2A))$

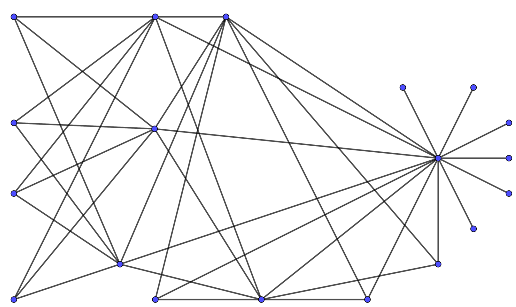

FIGURE 7. $\Gamma(F_4 \times Z_8)$, $\Gamma(F_4 \times Z_2[A]/(A^3))$ or $\Gamma(F_4 \times Z_4[A]/(2A, A^2 - 2))$

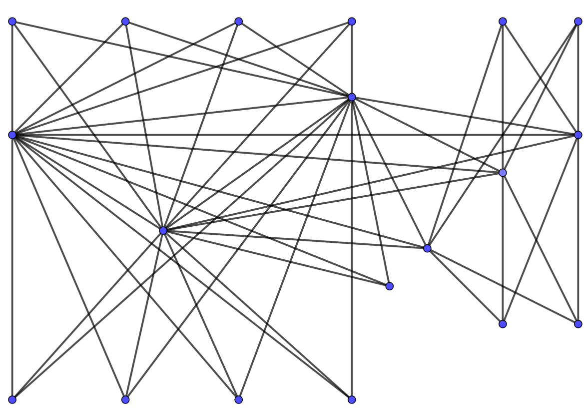

FIGURE 8. $\Gamma(F_4 \times Z_4[A]/(2A, A^2))$ or $\Gamma(F_4 \times F_2[A_1, A_2]/(A_1, A_2)^2)$

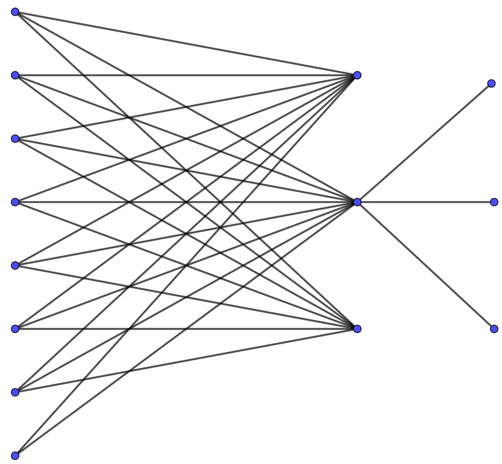

FIGURE 9. $\Gamma(F_9 \times Z_4)$ or $\Gamma(F_9 \times F_2[A]/(A^2)$

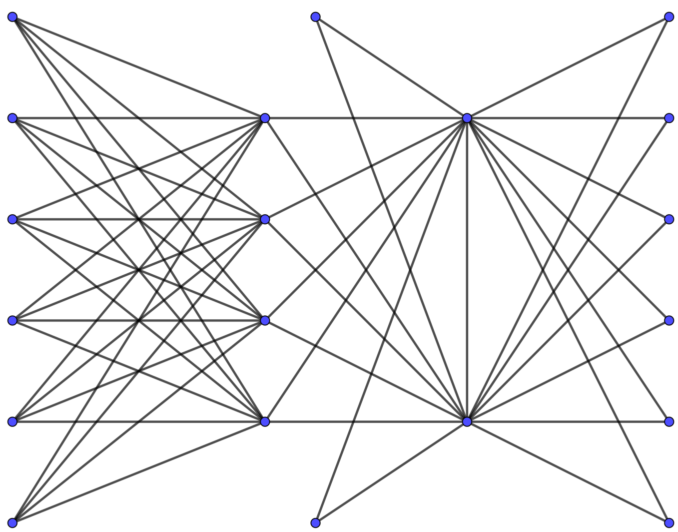

FIGURE 10. $\Gamma(F_5 \times Z_9)$ or $\Gamma(F_5 \times F_3[A]/(A^2))$

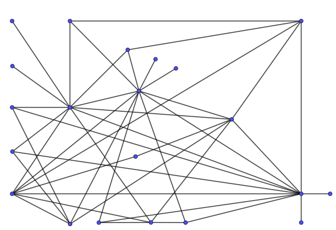

FIGURE 11. $\Gamma(F_2 \times F_2 \times F_2 \times F_3)$

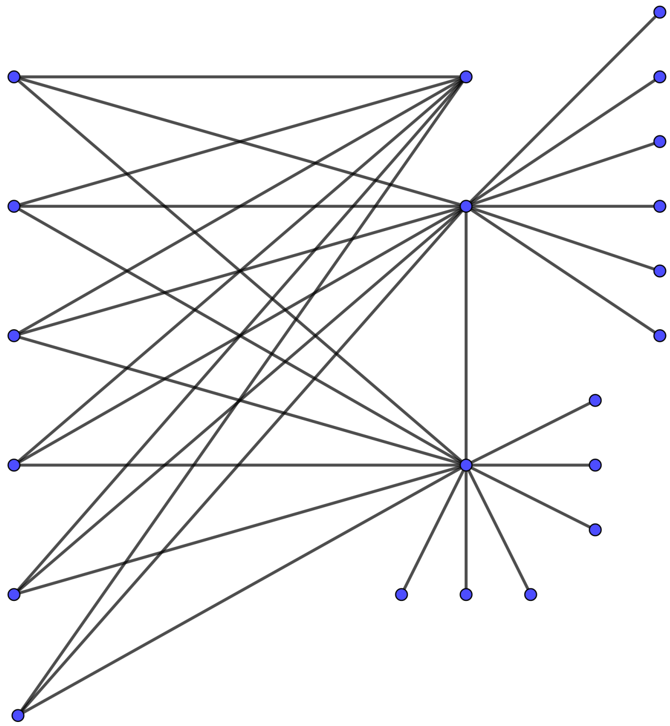

FIGURE 12. $\Gamma(F_2 \times F_2 \times F_7)$

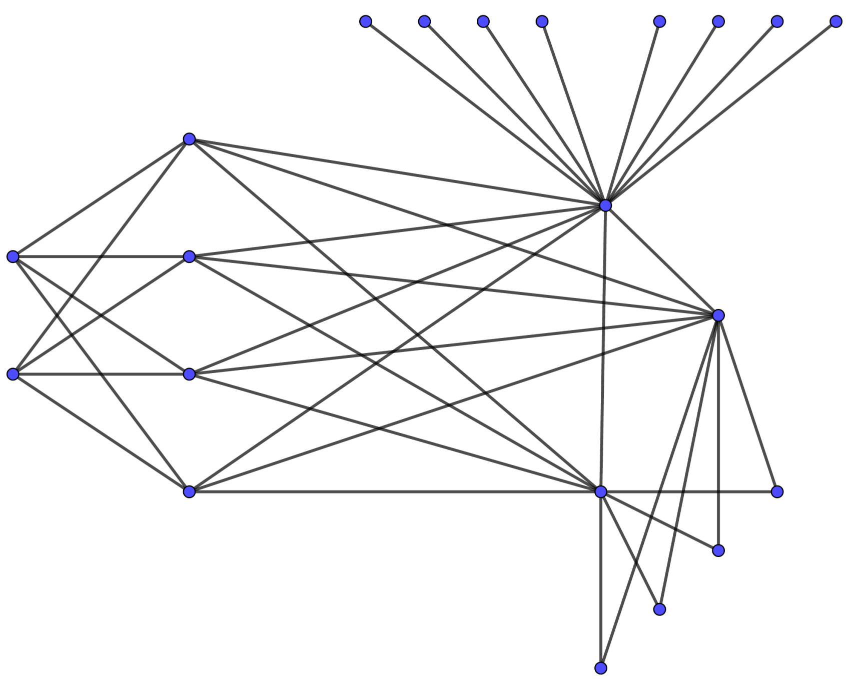

FIGURE 13. $\Gamma(F_2 \times F_3 \times F_5)$

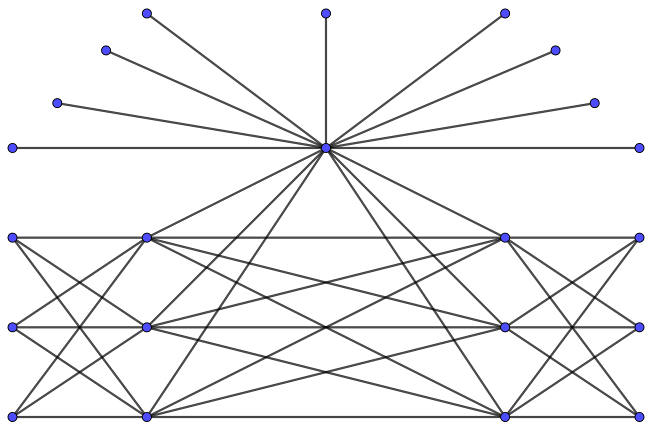

FIGURE 14. $\Gamma(F_2 \times F_4 \times F_4)$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring. J. Algebra, 217(2) (1999), 434-447.

[2] D. F. Anderson, S. B. Mulay, On the diameter and girth of a zero-divisor graph. J. Pure Appl. Algebra, 210(2) (2007), 543-550.

[3] D. F. Anderson, D. Weber, The zero-divisor graph of a commutative ring without identity. Int. Electron. J. Algebra, 23(23) (2018), 176-202.

[4] M. Behboodi, R. Beyranvand, On the Structure of Commutative Rings with ${p_1}^{k_1} \cdots {p_n}^{k_n}$ $(1 \leq k_1 \leq 7)$ Zero-Divisors. Eur. J. Pure Appl. Math. 3(2) (2010), 303-316.

[5] I. Beck, Coloring of commutative rings. J. Algebra, 116(1) (1988), 208-226.

[6] B. Corbas, G. D. Williams, Rings of order p5 Part I Nonlocal rings. J. Algebra, 231(2) (2000), 677-690.

[7] P. C. Dey, K. Patra, Zero divisor graph associated to Γ- semigroup of Z_n. J. Math. Comput. Sci. 11(3) (2021), 3767-3774.

[8] N. Ganesan, Properties of rings with a finite number of zero divisors. Math. Ann. 157(3) (1964), 215-218

[9] K. Koh, On Properties of rings with a finite number of zero divisors. Math. Ann. 171(1) (1967), 79-80.

[10] H. Q. Mohammad, N. H. Shuker, L. A. Khaleel, The maximal degree of a zero-divisor graph. Asian-Eur. J. Math. 14(1) (2021), 2050151.

[11] P.M. Magi, Sr. Magie Jose, Anjaly Kishore: Adjacency matrix and eigenvalues of the zero divisor graph $\Gamma(Z_n)$. J. Math. Comput. Sci. 10(4) (2020), 1285-1297.

[12] S. Pirzada, M. Aijaz, S.P. Redmond, Upper dimension and bases of zero-divisor graphs of commutative rings. AKCE Int. J. Graphs Comb. 17 (2020), 168–173.

[13] S. P. Redmond, On zero-divisor graphs of small finite commutative rings. Discrete Math. 307(9-10) (2007), 1155-1166.

[14] N. H. Shuker, H. Q. Mohammad, Classification of Zero Divisor Graphs of a Commutative Ring With Degree Equal 7 and 8. AL-Rafidain J. Computer Sci. Math. 10(2) (2013), 123-127.

[15] N. H. Shuker, H. Q. Mohammad, Classification of Zero Divisor Graphs of Commutative Rings of Degrees 11, 12 and 13. AL-Rafidain J. Computer Sci. Math. 11(1) (2014), 19-24.