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ON AN UPPER BOUND FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Liman, Mohopatra and Shah proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$,

$$\left| zD_\alpha p(z) + n\beta \left(\frac{|\alpha|-1}{2} \right) p(z) \right| \leq \left[\left\{ \left| \alpha + \beta \left(\frac{|\alpha|-1}{2} \right) \right| + \left| z + \beta \left(\frac{|\alpha|-1}{2} \right) \right| \right\} \max_{|z|=1} |p(z)| \right.$$
$$\left. - \left\{ \left| \alpha + \beta \left(\frac{|\alpha|-1}{2} \right) \right| - \left| z + \beta \left(\frac{|\alpha|-1}{2} \right) \right| \right\} \min_{|z|=1} |p(z)| \right],$$

where $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ is the polar derivative of $p(z)$ with respect to the point α. We extend and generalize this inequality for the polynomial $p(z)$ which does not vanish in $|z| < k$, $k \leq 1$. Our result also generalizes other known inequalities as well.

Keywords: Bernstein inequality; polar derivative; polynomial; zero.

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1. Introduction

Bernstein [4] established an estimate of the derivative of a polynomial $p(z)$ of degree n in terms of the maximum modulus of $p(z)$ on the unit circle by proving

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

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In (1.1) equality is attained if $p(z)$ is of the form αz^n, where α is a non zero constant. Erdös conjectured that if we restrict $p(z)$ to the polynomials of degree n having no zero in $|z|<1$, then (1.1) can be sharpened and replaced by

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

Inequality (1.2) was proved later by Lax [9]. Equality is attained in (1.2) for $p(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$. For the same class of polynomials as considered by Erdös and Lax, Aziz and Dawood [1] involved $\min|p(z)|$ on the unit circle and proved a refinement of (1.2). In fact, they proved

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=1}|p(z)|\right\}. \tag{1.3}$$

Dewan and Hans [6] improved (1.3) by proving that if $p(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for any $\beta\in\mathbb{C}$ with $|\beta|\leq 1$ and $|z|=1$

$$\begin{aligned}\left|zp'(z)+\frac{n\beta}{2}p(z)\right| \leq & \frac{n}{2}\left\{\left(\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right)\max_{|z|=1}|p(z)|\right.\\ & \left.-\left(\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|p(z)|\right\}.\end{aligned} \tag{1.4}$$

Let α be any real or complex number and let $p(z)$ be a polynomial of degree n. We define the polar derivative [11] of $p(z)$ with respect to α, denoted by $D_\alpha p(z)$, as

$$D_\alpha p(z) = np(z)+(\alpha-z)p'(z).$$

$D_\alpha p(z)$ is a polynomial of degree at most $n-1$. Since,

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z),$$

therefore, $D_\alpha p(z)$ is considered as a generalized form of the ordinary derivative of $p(z)$.

Aziz and Shah [2] extended (1.1) to polar derivative and proved that if $p(z)$ is a polynomial of degree n, then for every α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \leq n|\alpha|\max_{|z|=1}|p(z)|. \tag{1.5}$$

Aziz and Shah [3] refined and extended their result (1.5) by considering that the polynomial $p(z)$ of degree n having no zeros in $|z| < 1$ and for every real or complex number α satisfying $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} \left\{ (|\alpha|+1) \max_{|z|=1} |p(z)| - (|\alpha|-1) \min_{|z|=1} |p(z)| \right\}. \tag{1.6}$$

Considering the more general class of polynomials of degree n, namely, $p(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, we find some generalizations of (1.6) in the literature (see Dewan et al. [7] and Bidkham et al. [5]). The next result was proved by Liman et al. [10]. It generalizes inequalities (1.4) and (1.6) proved by Dewan and Hans [6] and Aziz and Shah [3] respectively.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for all α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$,*

$$\begin{aligned} \left| zD_\alpha p(z) + n\beta \frac{|\alpha|-1}{2} p(z) \right| \leq & \frac{n}{2} \left\{ \left(\left| \alpha + \beta \frac{|\alpha|-1}{2} \right| + \left| z + \beta \frac{|\alpha|-1}{2} \right| \right) \max_{|z|=1} |p(z)| \right. \\ & \left. - \left(\left| \alpha + \beta \frac{|\alpha|-1}{2} \right| - \left| z + \beta \frac{|\alpha|-1}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}. \end{aligned} \tag{1.7}$$

2. MAIN RESULTS

In this paper, by involving some coefficients of the polynomial $p(z)$, we generalize and extend inequality (1.7). The result also generalizes other inequalities mentioned in the preceding section. More precisely, we prove the following result.

Theorem 2.1. *Let $p(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$. Then, for all α, $\beta \in \mathbb{C}$ with $|\alpha| \geq A$, $|\beta| \leq 1$ and $|z| = 1$,*

$$\begin{aligned} \left| zD_\alpha p(z) + n\beta \frac{|\alpha|-A}{1+A} p(z) \right| \leq & \frac{n}{2} \left\{ \left(k^{-n} \left| \alpha + \beta \frac{|\alpha|-A}{1+A} \right| + \left| z + \beta \frac{|\alpha|-A}{1+A} \right| \right) \max_{|z|=1} |p(z)| \right. \\ & \left. - \left(k^{-n} \left| \alpha + \beta \frac{|\alpha|-A}{1+A} \right| - \left| z + \beta \frac{|\alpha|-A}{1+A} \right| \right) \min_{|z|=k} |p(z)| \right\}, \end{aligned} \tag{2.1}$$

where

$$A = \frac{\mu |a_{n-\mu}| k^{\mu-1} + n|a_n| k^{2\mu}}{\mu |a_{n-\mu}| + n|a_n| k^{\mu-1}}. \tag{2.2}$$

Remark 2.2. Under the assumptions of Theorem 2.1, we can verify that $A = 1$ when $k = 1$, whereas, for $k < 1$ we can verify that $A \leq k$ as shown below. Using inequality (3.4), we have

$$\mu|a_{n-\mu}|k^{\mu} \leq n|a_n|k^{2\mu}$$

$$\implies \quad \mu|a_{n-\mu}|k^{\mu}\left(\frac{1}{k}-1\right) \leq n|a_n|k^{2\mu}\left(\frac{1}{k}-1\right),$$

$$\text{or} \quad \mu|a_{n-\mu}|k^{\mu-1} + n|a_n|k^{2\mu} \leq n|a_n|k^{2\mu-1} + \mu|a_{n-\mu}|k^{\mu},$$

$$\text{i.e.} \quad \frac{\mu|a_{n-\mu}|k^{\mu-1} + n|a_n|k^{2\mu}}{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}} \leq k^{\mu} \leq k \quad \text{as } k \leq 1 \quad \textit{and} \quad \mu \geq 1,$$

$$\text{i.e.} \quad A \leq k.$$

Remark 2.3. Taking $k = 1$ (so that $A = 1$) in Theorem 2.1, inequality (2.1) for $\mu = 1$ reduces to (1.7) due to Liman[10]. Thus, Theorem 2.1 is an extension and a generalization of Theorem 1.1 for the lacunary polynomial $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$.

If we take $\beta = 0$ in Theorem 2.1, it takes the following simplified form.

Corollary 2.4. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *and* $p(z) \neq 0$ *in* $|z| < k$, $k \leq 1$, *then for all* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geq 1$

$$\max_{|z|=1} |zD_\alpha p(z)| \leq \frac{n}{2}\left\{\left(k^{-n}|\alpha|+1\right)\max_{|z|=1}|p(z)| - \left(k^{-n}|\alpha|-1\right)\min_{|z|=k}|p(z)|\right\}, \tag{2.3}$$

where A *is given by* (2.2).

Remark 2.5. If we take $\mu = 1$ and $k = 1$ in Corollary 2.4, then (2.3) reduces to (1.6) due to Aziz and Shah [3] and therefore Theorem 2.1 extends and generalizes (1.6) to lacunary polynomials of the type $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$.

Dividing both sides of (2.1) by $|\alpha|$ and taking the limit as $|\alpha| \to \infty$, we have the following result.

Corollary 2.6. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n *and* $p(z) \neq 0$ *in* $|z| < k$, $k \le 1$, *then for any* $\beta \in \mathbb{C}$ *with* $|\beta| \le 1$ *and* $|z| = 1$,

$$\begin{aligned} \left| zp'(z) + \frac{n\beta}{1+A} p(z) \right| \le \frac{n}{2} \Bigg\{ & \left(k^{-n} \left| 1 + \frac{\beta}{1+A} \right| + \left| \frac{\beta}{1+A} \right| \right) \max_{|z|=1} |p(z)| \\ & - \left(k^{-n} \left| 1 + \frac{\beta}{1+A} \right| - \left| \frac{\beta}{1+A} \right| \right) \min_{|z|=k} |p(z)| \Bigg\}, \end{aligned} \tag{2.4}$$

where A *is given by* (2.2).

Remark 2.7. If we take $k = 1$ (so that $A = 1$) and $\mu = 1$ in Corollary 2.6, inequality (2.4) reduces to (1.4) due to Dewan and Hans [6]. Further more, if $\beta = 0$ along with $k = 1$ and $\mu = 1$, inequality (2.4) becomes (1.3) due to Aziz and Dawood [1].

3. Lemmas

For the proof of Theorem 2.1, we require the following lemmas. The first lemma is due to Laguerre [8, 11]

Lemma 3.1. *If all the zeros of an* n^{th} *degree polynomial* $p(z)$ *lie in a circular region* C, *and* ω *is any zero of* $D_\alpha p(z)$, *where* α *is any real or complex number, then at most one of the points* ω *and* α *may lie outside* C.

Lemma 3.2. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zeros in* $|z| < k, k \ge 1$, *then on* $|z| = 1$

$$|q'(z)| \ge k^{\mu+1} \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} k^{\mu-1} + 1}{1 + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|} k^{\mu+1}} |p'(z)| \tag{3.1}$$

and

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} k^\mu \le 1. \tag{3.2}$$

This lemma is due to Qazi [12].

Lemma 3.3. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in the closed disk* $|z| \le k, k \le 1$, *then for any real or complex number* α *with* $|\alpha| \ge A$

and $|z|=1$

$$|D_\alpha p(z)| \geq n\frac{|\alpha|-A}{1+A}|p(z)|. \tag{3.3}$$

where A is given by (2.2).

Before proving Lemma 3.3, we take note of an important consequence of Lemma 3.2 and Lemma 3.3. If $p(z)$ is polynomial assumed as in Lemma 3.3 and $q(z)=z^n\overline{p(\frac{1}{\bar{z}})}$, then $q(z)$ has no zero in $|z|<\frac{1}{k}$, $\frac{1}{k}\geq 1$. Thus, on applying Lemma 3.2 to $q(z)$, by inequality (3.2) we obtain

$$\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^\mu} \leq 1. \tag{3.4}$$

Proof of Lemma 3.3. Let $q(z)=z^n\overline{p(\frac{1}{\bar{z}})}=\bar{a}_n+\sum\limits_{\nu=\mu}^{n}\bar{a}_{n-\nu}z^\nu$. Then, it can be easily verified that

$$|q'(z)|=|np(z)-zp'(z)| \quad \text{for} \quad |z|=1. \tag{3.5}$$

Since $p(z)$ has all its zeros in $|z|\leq k$, $k\leq 1$, therefore, the polynomial $q(z)$ has no zero in $|z|<\frac{1}{k}$, $\frac{1}{k}\geq 1$. Thus, applying Lemma 3.2 to $q(z)$, we have by (3.1) for $|z|=1$

$$\begin{aligned}
|p'(z)| &\geq \frac{1}{k^{\mu+1}}\,\frac{\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^{\mu-1}}+1}{1+\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\frac{1}{k^{\mu+1}}}\,|q'(z)| \\
&= \frac{\mu|a_{n-\mu}|+n|a_n|k^{\mu-1}}{n|a_n|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}\,|q'(z)|, \\
\text{therefore,} \quad |q'(z)| &\leq \frac{n|a_n|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}|+n|a_n|k^{\mu-1}}\,|p'(z)|.
\end{aligned}$$

Equivalently, for $|z|=1$

$$|q'(z)| \leq A|p'(z)|. \tag{3.6}$$

$$|p'(z)|+|q'(z)| \leq (1+A)|p'(z)|, \tag{3.7}$$

where $A = \dfrac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}$.

Now,

$$\begin{aligned} n|p(z)| &= |np(z) - zp'(z) + zp'(z)| \\ &\le |np(z) - zp'(z)| + |p'(z)| \quad \text{on} \quad |z| = 1, \end{aligned}$$

which on using inequality (3.5) gives for $|z| = 1$

$$n|p(z)| \le |p'(z)| + |q'(z)|. \tag{3.8}$$

combining (3.7) and (3.8), we have for $|z| = 1$

$$n|p(z)| \le (1+A)|p'(z)|.$$

$$\text{i.e.} \quad |p'(z)| \ge \frac{n}{1+A}|p(z)|. \tag{3.9}$$

By definition, if $\alpha \in \mathbb{C}$, particularly for $|\alpha| \ge A$, we have

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

Then,

$$\begin{aligned} |D_\alpha p(z)| &= |np(z) - zp'(z) + \alpha p'(z)| \\ &\ge |\alpha||p'(z)| - |np(z) - zp'(z)|, \end{aligned}$$

which on using inequality (3.5) gives for $|z| = 1$

$$|D_\alpha p(z)| \ge |\alpha||p'(z)| - |q'(z)|. \tag{3.10}$$

Using (3.6) to (3.10), we have for $|z| = 1$

$$\begin{aligned} |D_\alpha p(z)| &\ge |\alpha||p'(z)| - A|p'(z)| \\ &= (|\alpha| - A)\,|p'(z)|, \end{aligned}$$

which in conjunction with (3.9) gives for $|z| = 1$

$$|D_\alpha p(z)| \ge n\frac{|\alpha| - A}{1+A}|p(z)|.$$

□

Lemma 3.4. *Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \ge A$, $|\beta| \le 1$ and $|z| = 1$, we have*

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1+A} p(z) \right| \ge \frac{n}{k^n} \left| \alpha + \beta \frac{|\alpha| - A}{1+A} \right| \min_{|z|=k} |p(z)|, \tag{3.11}$$

where A is given by (2.2).

Proof of Lemma 3.4. If $p(z)$ has a zero on $|z| = k$, then (3.11) follows trivially. Therefore, we assume that $p(z)$ has all its zeros in $|z| < k$. Let $m = \min_{|z|=k} |p(z)|$, then $m > 0$ and $|p(z)| \ge m$, where $|z| = k$. Therefore, for every λ with $|\lambda| < 1$, it follows by Rouche's theorem that the polynomial $G(z) = p(z) - \lambda m \left(\frac{z}{k}\right)^n$ has all its zeros in $|z| < k$. By lemma 3.1, $D_\alpha G(z)$ has all its zeros in $|z| < k$, where

$$\begin{aligned} D_\alpha G(z) &= D_\alpha p(z) - D_\alpha \left(\lambda m \frac{z^n}{k^n} \right) \\ &= D_\alpha p(z) - \left\{ n\lambda m \frac{z^n}{k^n} - (\alpha - z) n\lambda m \frac{z^{n-1}}{k^n} \right\} \\ &= D_\alpha p(z) - \alpha\lambda mn \end{aligned}$$

with $|\alpha| \ge A$.

Applying Lemma 3.3 to the polynomial $G(z)$, we have for $|z| = 1$

$$|D_\alpha G(z)| \ge n \frac{|\alpha| - A}{1+A} |G(z)|,$$

which is equivalent to

$$|zD_\alpha G(z)| \ge n \frac{|\alpha| - A}{1+A} |G(z)| \quad \text{on} \quad |z| = 1. \tag{3.12}$$

Since $zD_\alpha G(z)$ has all its zeros in $|z| < k \le 1$, by using Rouche's theorem, it can be easily verified from (3.12) that the polynomial $zD_\alpha G(z) + \beta n \frac{|\alpha| - A}{1+A} G(z)$ has all its zeros in $|z| < 1$, where $|\beta| < 1$. Then,

$$\begin{aligned} T(z) &= zD_\alpha p(z) - \alpha\lambda mn \frac{z^n}{k^n} + \beta n \frac{|\alpha| - A}{1+A} \left(p(z) - \lambda m \frac{z^n}{k^n} \right) \\ &= zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1+A} p(z) - \lambda mn \frac{z^n}{k^n} \left(\alpha + \beta \frac{|\alpha| - A}{1+A} \right) \end{aligned} \tag{3.13}$$

will have no zeros in $|z| \geq 1$. This implies for every β with $|\beta| < 1$ and $|z| \geq 1$,

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right| \geq nm \left| \frac{z}{k} \right|^n \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right|. \tag{3.14}$$

If (3.14) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left| z_0 D_\alpha p(z_0) + n\beta \frac{|\alpha| - A}{1 + A} p(z_0) \right| < nm \left| \frac{z_0}{k} \right|^n \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right|.$$

Take

$$\lambda = \frac{z_0 D_\alpha p(z_0) + n\beta \frac{|\alpha| - A}{1 + A} p(z_0)}{nm \left(\frac{z_0}{k} \right)^n \left(\alpha + \beta \frac{|\alpha| - A}{1 + A} \right)},$$

then $|\lambda| < 1$ and with this choice of λ, we have $T(z_0) = 0$ from (3.13). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. Thus, for $\beta \in \mathbb{C}$ with $|\beta| < 1$ inequality (3.14) holds and for $|\beta| = 1$, it follows by continuity. Hence,

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right| \geq n \left| \frac{z}{k} \right|^n \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| \min_{|z|=k} |p(z)|.$$

□

Lemma 3.5. *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n, then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k \geq A$, where $k \leq 1$, we have for $|z| = 1$*

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right| \leq \frac{n}{k^n} \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| \max_{|z|=k} |p(z)|, \tag{3.15}$$

where A is given by (2.2).

Proof of Lemma 3.5. Let $M = \max_{|z|=k} |p(z)|$. If $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$, then $|\lambda p(z)| < \left| M \left(\frac{z}{k} \right)^n \right|$ for $|z| = k$. Therefore, it follows by Rouche's Theorem that $G(z) = M \frac{z^n}{k^n} - \lambda p(z)$ has all its zeros in $|z| < k$. Thus, by using Lemma 3.1,

$$D_\alpha G(z) = \alpha M n \left(\frac{z^{n-1}}{k^n} \right) - \lambda D_\alpha p(z)$$

has all its zeros in $|z| < k$ for $|\alpha| \geq A$.

On applying Lemma 3.3 to the polynomial $G(z)$, we have for $|z| = 1$

$$|zD_\alpha G(z)| \geq n \frac{|\alpha| - A}{1 + A} |G(z)|. \tag{3.16}$$

Following a similar argument as used in the proof of Lemma 3.4, the result follows. □

Lemma 3.6. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree n, then for all* $\alpha, \beta \in \mathbb{C}$ *with* $|\beta| \le 1$ *and* $|\alpha| \ge k \ge A$, *where* $k \le 1$, *we have for* $|z| = 1$

$$\left|zD_\alpha p(z) + n\beta \frac{|\alpha|-A}{1+A} p(z)\right| + \left|zD_\alpha Q(z) + n\beta \frac{|\alpha|-A}{1+A} Q(z)\right|$$
$$\le n\left\{k^{-n}\left|\alpha + \beta \frac{|\alpha|-A}{1+A}\right| + \left|z + \beta \frac{|\alpha|-A}{1+A}\right|\right\} \max_{|z|=1} |p(z)|, \tag{3.17}$$

where A is given by (2.2) *and* $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\bar{z}}\right)}$.

Proof of Lemma 2.6. Let $M = \max_{|z|=k} |p(z)|$. For λ with $|\lambda| > 1$, it follows by Rouche's theorem that the polynomial $G(z) = p(z) - \lambda M$ has no zeros in $|z| < k$. Consequently, the polynomial

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\bar{z}}\right)} \tag{3.18}$$

has all its zeros in $|z| \le k$, also $|G(z)| = |H(z)|$ for $|z| = k$. Since all the zeros of $H(z)$ lie in $|z| \le k$, therefore, for δ with $|\delta| > 1$, by Rouche's Theorem all the zeros of $G(z) + \delta H(z)$ lie in $|z| \le k$. Hence, by Lemma 3.3 for every α with $|\alpha| \ge A$, and $|z| = 1$, we have

$$n\frac{|\alpha|-A}{1+A} |G(z) + \delta H(z)| \le |zD_\alpha (G(z) + \delta H(z))| \tag{3.19}$$

On the other hand, by Lemma 3.1, all the zeros of $D_\alpha(G(z) + \delta H(z))$ lie in $|z| < k < 1$, where $|\alpha| \ge A$. Therefore, for any β with $|\beta| \le 1$, Rouche's theorem implies that all the zeros of $zD_\alpha(G(z) + \delta H(z)) + \beta n \frac{|\alpha|-A}{1+A}(G(z) + \delta H(z))$ lie in $|z| < 1$. This means that the polynomial

$$T(z) = zD_\alpha G(z) + n\beta \frac{|\alpha|-A}{1+A} G(z) + \delta\left(zD_\alpha H(z) + n\beta \frac{|\alpha|-A}{1+A} H(z)\right) \tag{3.20}$$

will have no zeros in $|z| \ge 1$. Now, using a similar argument as used in the proof of Lemma 3.4, we get for $|z| \ge 1$,

$$\left|zD_\alpha G(z) + n\beta \frac{|\alpha|-A}{1+A} G(z)\right| \le \left|zD_\alpha H(z) + n\beta \frac{|\alpha|-A}{1+A} H(z)\right|. \tag{3.21}$$

Therefore, by the equalities

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\bar{z}}\right)} = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\bar{z}}\right)} - \bar{\lambda} M \left(\frac{z}{k}\right)^n = Q(z) - \bar{\lambda} M \left(\frac{z}{k}\right)^n, \tag{3.22}$$

and substituting for $G(z)$ and $H(z)$ in (3.21), we get

$$\left|\left(zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right)-\lambda nM\left(z+\beta\frac{|\alpha|-A}{1+A}\right)\right|$$

$$\leq\left|\left(zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right)-\overline{\lambda}nM\left(\frac{z}{k}\right)^n\left(\alpha+\beta\frac{|\alpha|-A}{1+A}\right)\right|. \tag{3.23}$$

This implies that

$$\left|\left(zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right)\right|-\left|\lambda nM\left(z+\beta\frac{|\alpha|-A}{1+A}\right)\right|$$

$$\leq\left|\left(zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right)-\overline{\lambda}nM\left(\frac{z}{k}\right)^n\left(\alpha+\beta\frac{|\alpha|-A}{1+A}\right)\right|. \tag{3.24}$$

As $|p(z)|=|Q(z)|$ for $|z|=k$, that is, $\max\limits_{|z|=k}|p(z)|=\max\limits_{|z|=k}|Q(z)|=M$, by Lemma 3.5 for $Q(z)$, we obtain

$$\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right|<|\lambda|nMk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|. \tag{3.25}$$

Thus, taking a suitable choice of the argument of λ,

$$\left|\left(zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right)-\overline{\lambda}nM\left(\frac{z}{k}\right)^n\left(\alpha+\beta\frac{|\alpha|-A}{1+A}\right)\right|$$

$$=|\lambda|nMk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|-\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right|. \tag{3.26}$$

By combining the right hand sides of (3.24) and (3.26) for $|z|=1$ and $|\beta|\leq 1$, we get

$$\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right|-\left|\lambda nM\left(z+\beta\frac{|\alpha|-A}{1+A}\right)\right|$$

$$\leq|\lambda|nMk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|-\left|zD_\alpha q(z)+n\beta\frac{|\alpha|-A}{1+A}q(z)\right|.$$

i.e,

$$\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right|+\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right|$$

$$\leq|\lambda|\left\{\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|k^{-n}+\left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}nM.$$

Taking $|\lambda|\to 1$, we have

$$\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right|+\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right|$$

$$\leq n\left\{\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|k^{-n}+\left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}M.$$

□

Lemma 3.7. *Let* $H(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *be a polynomial of degree n having all its zeros in* $|z| \le k, k \le 1$, *and* $G(z)$ *be a polynomial of degree not exceeding that of* $H(z)$. *If* $|G(z)| \le |H(z)|$ *for* $|z| = k$, *then for all* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \ge A$, $|\beta| \le 1$ *and* $|z| = 1$, *we have*

$$\left| zD_\alpha G(z) + n\beta \frac{|\alpha| - A}{1+A} G(z) \right| \le \left| zD_\alpha H(z) + n\beta \frac{|\alpha| - A}{1+A} H(z) \right|, \tag{3.27}$$

where A is given by (2.2).

Proof of Lemma 2.7. Since $|\lambda G(z)| \le |G(z)| \le |H(z)|$ for $|z| = 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, then it follows by Rouche's theorem that the polynomials $H(z)$ and $H(z) - \lambda G(z)$ have the same number of zeros in the open disk $|z| < k$. Also, the inequality $|G(z)| \le |H(z)|$ for $|z| = k$ implies that any zero of $H(z)$ on $|z| = k$ is also a zero of $G(z)$. Therefore, $H(z) - \lambda G(z)$ has all its zeros in the closed disk $|z| \le k$, $k \le 1$. Thus, applying Lemma 3.3, we have for all real or complex α with $|\alpha| \le A$ and $|z| = 1$

$$|zD_\alpha (H(z) - \lambda G(z))| \ge n \frac{|\alpha| - A}{1+A} |H(z) - \lambda G(z)|.$$

Following a similar argument as used in the proof of Lemma 3.4, we have for any β with $|\beta| < 1$ and $|z| = 1$

$$\begin{aligned} |zD_\alpha (H(z) - \lambda G(z))| &\ge n \frac{|\alpha| - A}{1+A} |H(z) - \lambda G(z)|. \\ &> n|\beta| \frac{|\alpha| - A}{1+A} |H(z) - \lambda G(z)|. \end{aligned}$$

Thus, for $|z| = 1$

$$T(z) = [zD_\alpha H(z) - \lambda z D_\alpha G(z)] + n|\beta| \frac{|\alpha| - A}{1+A} [H(z) - \lambda G(z)] \ne 0, \tag{3.28}$$

which implies that for $|z| = 1$

$$\left\{ zD_\alpha H(z) + n|\beta| \frac{|\alpha| - A}{1+A} H(z) \right\} - \lambda \left\{ zD_\alpha G(z) + n|\beta| \frac{|\alpha| - A}{1+A} G(z) \right\} \ne 0. \tag{3.29}$$

Hence, we can conclude that for $|z| = 1$

$$\left| zD_\alpha H(z) + n\beta \frac{|\alpha| - A}{1+A} H(z) \right| \ge \left| zD_\alpha G(z) + n\beta \frac{|\alpha| - A}{1+A} G(z) \right|. \tag{3.30}$$

If (3.30) is not true, then there exist a point z_0 on the unit circle that

$$\left|z_0 D_\alpha H(z_0) + n\beta \frac{|\alpha|-A}{1+A} H(z_0)\right| < \left|z_0 D_\alpha G(z_0) + n\beta \frac{|\alpha|-A}{1+A} G(z_0)\right|.$$

If we choose

$$\lambda = \frac{z_0 D_\alpha H(z_0) + n\beta \frac{|\alpha|-A}{1+A} H(z_0)}{z_0 D_\alpha G(z_0) + n\beta \frac{|\alpha|-A}{1+A} G(z_0)},$$

then $|\lambda| < 1$ and hence (3.29) gives $T(z_0) = 0$ for $|z_0| = 1$. This is a contradiction to (3.28). Hence, (3.30) must hold for $\beta \in \mathbb{C}$ with $|\beta| < 1$. For $|\beta| = 1$, (3.30) holds by continuity. This completes the proof of Lemma 2.7.

□

4. Proof of the Theorem

We now prove Theorem 2.1.

Proof of Theorem 2.1. Since $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, does not vanish in $|z| < k$, and if $m = \min_{|z|=k} |p(z)|$, then $m \le |p(z)|$ for $|z| = k$. Now for real or complex λ with $|\lambda| < 1$, we have $|\lambda m| < m \le |p(z)|$ for $|z| = k$. Therefore, it follows by Rouche's theorem that the polynomial $G(z) = p(z) - \lambda m$ has no zero in $|z| < k$. Therefore, the polynomial

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\bar{z}}\right)} = Q(z) - \bar{\lambda} m \left(\frac{z}{k}\right)^n, \tag{4.1}$$

where $Q(z) = \left(\frac{z}{k}\right)^n \overline{p\left(\frac{k^2}{\bar{z}}\right)}$, will have all its zeros in $|z| \le k$, $k \le 1$. Also, $|G(z)| = |H(z)|$ for $|z| = k$. Applying Lemma 3.7 for the polynomial $H(z)$ and $G(z)$, we have

$$\left|z D_\alpha G(z) + n\beta \frac{|\alpha|-A}{1+A} G(z)\right| \le \left|z D_\alpha H(z) + n\beta \frac{|\alpha|-A}{1+A}\right|, \tag{4.2}$$

where $|\alpha| \ge k$, $|\beta \le 1|$ and $|z| = 1$.

Substituting for $G(z)$ and $H(z)$ in (4.2), we conclude that for every α, β with $|\alpha| \ge A$, $|\beta| \le 1$ and $|z| = 1$

$$\left|zD_\alpha p(z) - \lambda nmz + n\beta\frac{|\alpha|-A}{1+A}\left(p(z)-\lambda m\right)\right| \leq \left|zD_\alpha Q(z) - \overline{\lambda}\alpha nm\left(\frac{z}{k}\right)^n + n\beta\frac{|\alpha|-A}{1+A}\left\{Q(z)-\overline{\lambda}m\left(\frac{z}{k}\right)^n\right\}\right|,$$

which implies that

$$\begin{aligned}&\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-A}{1+A}p(z) - \lambda mn\left(z+\beta\frac{|\alpha|-A}{1+A}\right)\right| \\ &\leq \left|zD_\alpha p(z) + n\beta\frac{|\alpha|-A}{1+A}Q(z) - \overline{\lambda}mn\left(\frac{z}{k}\right)^n\left(\alpha+\beta\frac{|\alpha|-A}{1+A}\right)\right|. \end{aligned} \tag{4.3}$$

Since all the zeros of $Q(z)$ lie in $|z| \leq k$ and $|p(z)| = |Q(z)|$ for $|z| = k$, therefore, by applying Lemma 2.4 to $Q(z)$, we get

$$\begin{aligned}\left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-A}{1+A}Q(z)\right| &\leq nk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|\min_{|z|=k}|Q(z)| \\ &= nk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|\min_{|z|=k}|p(z)|. \end{aligned} \tag{4.4}$$

Then, for an appropriate choice of the argument of λ, we have

$$\begin{aligned}&\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-A}{1+A}Q(z) - \overline{\lambda}mn\left(\frac{z}{k}\right)^n\left(\alpha+\beta\frac{|\alpha|-A}{1+A}\right)\right| \\ &= \left|zD_\alpha p(z) + n\beta\frac{|\alpha|-A}{1+A}Q(z)\right| - |\lambda|mnk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|, \quad \text{on } |z|=1. \end{aligned} \tag{4.5}$$

Combining the right hand sides of (4.3) and (4.5), we can rewrite inequality (4.5) as

$$\begin{aligned}\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-A}{1+A}p(z)\right| - |\lambda|mn\left|z+\beta\frac{|\alpha|-A}{1+A}\right| &\leq \left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-A}{1+A}Q(z)\right| \\ &\quad - |\lambda|mnk^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right| \quad \text{for } |z|=1. \end{aligned}$$

Equivalently,

$$\begin{aligned}\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-A}{1+A}p(z)\right| &\leq \left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-A}{1+A}Q(z)\right| \\ &\quad - |\lambda|mn\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right| - \left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}. \end{aligned}$$

As $|\lambda| \to 1$, we have

$$\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right| \le \left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right| - mn\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|-\left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}.$$

It implies for every real or complex number β with $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}2\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right| \le &\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right| \\ &+\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-A}{1+A}Q(z)\right| \\ &- mn\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|-\left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}.\end{aligned} \tag{4.6}$$

Inequality (4.6) in conjunction with Lemma 3.6 gives for $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}2\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-A}{1+A}p(z)\right| \le\, &n\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|+\left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}\max_{|z|=1}|p(z)| \\ &-n\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-A}{1+A}\right|+\left|z+\beta\frac{|\alpha|-A}{1+A}\right|\right\}\min_{|z|=k}|p(z)|,\end{aligned}$$

from which (2.1) follows. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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