Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 6168-6192
https://doi.org/10.28919/jmcs/6164
ISSN: 1927-5307

COUPLED BEST PROXIMITY POINT THEOREMS FOR MIXED g-MONOTONE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

LAKSHMI NARAYAN MISHRA[1], VINITA DEWANGAN[2], VISHNU NARAYAN MISHRA[3,*], HANIF AMRULLOH[4]

[1]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, T.N., India

[2]Department of Mathematics, Govt. Girls Polytechnic College Raipur, Chhattishgarh, India

[3]Department of Mathematics, Faculty of Science, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India

[4]Department of Mathematics Education, Faculty of Tarbiya, Institute for Islamic Studies Ma'arif NU (IAIMNU) Metro Lampung, Metro-34111, Indonesia

Abstract. In this paper, we establish existence and uniqueness of coupled best proximity point for a mixed g-monotone mapping satisfying the proximally coupled weak (ψ, ϕ) contraction in partially ordered metric spaces. The results presented in this paper generalize the results of Kumam et al. [Fixed point theory and applications 2014, 2014:107]. Also some examples and applications of the main results in this paper are given.

Keywords: partially ordered set; coupled fixed point; coupled best proximity points; g-monotone property.

2010 AMS Subject Classification: 41A65, 47H05, 47H09, 47H10, 90C30.

1. Introduction and Preliminaries

The classical contraction mapping principal of Banach is one of the most useful and fundamental results in fixed point theory. Several authors studied and extended it in many directions.

*Corresponding author
E-mail addresses: vnm@igntu.ac.in, vishnunarayanmishra@gmail.com
Received May 29, 2021

Existence and uniqueness of a fixed points for self mappings was extended to partially ordered metric spaces has been considered recently in Ran and Reurings [1], Bhaskar and Lakshmikantham [2], Nieto and Lopez [3], Agarwal, El-Gebeily and O'Regan [4] and Lakshmikantham and Ciric [5] (see also [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]).

The study of the existence and convergence of best proximity points is an interesting field of optimization which recently attracted the attention of several researchers (see [24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 33]). One can also find the existence of best proximity point in the setting of partially ordered metric spaces in [19, 20, 21, 22, 23, 24].

In the sequel, we will use the following notations. Set A and B are nonempty subsets of a metric space X,

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\},$$

$$A_0 = \{x \in A : d(x,y) = d(A,B) \quad for\ some\ y \in B\}$$

$$B_0 = \{y \in B : d(x,y) = d(A,B) \quad for\ some\ x \in A\}$$

Definition 1.1. *An element $x \in A$ is said to be a best proximity point of the non-self mapping $T : A \to B$ if*

$$d(x,Tx) = d(A,B).$$

Because of the fact that $d(x,Tx) \geq d(A,B)$ for all $x \in A$, the global minimum of the mapping $x \mapsto (x,Tx)$ is attained at a best proximity point. Moreover, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point.

Definition 1.2. [2] *Let X be a non-empty set. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping T if $T(x,y) = x$ and $T(y,x) = y$.*

Definition 1.3. [5] *Let X be a nonempty set and let $T : X \times X \to X$ and $g : X \to X$ be two given mappings. An element $(x,y) \in X \times X$ is said to be a coupled coincidence point of if $T(x,y) = g(x)$ and $T(y,x) = g(y)$ for all $x,y \in X$.*

Definition 1.4. [5] *Let X be a nonempty set and let $T : X \times X \to X$ and $g : X \to X$ be two given mappings. We call T and g are commutative if $g(T(x,y)) = T(g(x),g(y))$ for all $x,y \in X$.*

Definition 1.5. [21] *Let A, B be subsets of a metric space X. An element $(x,y) \in A \times A$ is called a coupled best proximity point of the mapping $T : A \times A \to B$ if $d\big(x, T(x,y)\big) = d(A,B)$ and $d\big(x, T(y,x)\big) = d(A,B)$.*

Here, if we take A = B, then this definition reduced to Definition 1.2.

Definition 1.6. [2] *Let $(X, \leq)$ be a partially ordered set and $T : X \times X \to X$. We say that T has the mixed monotone property if T(x,y) is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$*

$$x_1, x_2 \in X \quad x_1 \leq x_2 \Rightarrow T(x_1, y) \leq T(x_2, y)$$

$$y_1, y_2 \in X \quad y_1 \leq y_2 \Rightarrow T(x, y_1) \geq T(x, y_2).$$

Definition 1.7. [20] *A mapping $T : A \times A \to B$ is said to be the proximal mixed monotone property if T(x, y) is proximally nondecreasing in x and is proximally nonincreasing in y, that is*

$$\begin{cases} x_1 \leq x_2 \\ d(u_1, T(x_1, y)) = d(A,B) & \Rightarrow u_1 \leq u_2, \\ d(u_2, T(x_2, y)) = d(A,B) \end{cases}$$

and

$$\begin{cases} y_1 \leq y_2 \\ d(v_1, T(x, y_1)) = d(A,B) & \Rightarrow v_2 \leq v_1, \\ d(v_2, T(x, y_2)) = d(A,B) \end{cases}$$

where $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2 \in A$.

If we take A = B in the above definition, then proximal mixed monotone property reduces to mixed monotone property.

Definition 1.8. [18] *Let Φ denote all functions $\phi : [0,\infty) \to [0,\infty)$ which satisfy*

(i) ϕ is continuous and nondecreasing,

(ii) $\phi(t) = 0$ if and only if t = 0,

(iii) $\phi(t+s) \leq \phi(t) + \phi(s), \forall t, s \in (0,\infty]$.

Definition 1.9. [18] *Let ψ denote all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy* $\lim_{t\to r}\psi(t) > 0$ *for all $r > 0$ and* $\lim_{t\to 0^+}\psi(t) = 0$.

In [18] Luong and Thuan obtained a result of coupled fixed. Following is the main theoretical results of Luong and Thuan.

Theorem 1.10. [18] *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $T : X\times X \to X$ be mapping having the mixed monotone property on X such that*

$$\phi\big(d(T(x,y),T(u,v))\big) \leq \frac{1}{2}\phi\big(d(x,u)+d(y,v)\big) - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big) \tag{1.1}$$

for all $x,y,u,v \in X$ with $x \geq u$ and $y \leq v$, where $\psi \in \Psi$ and $\phi \in \Phi$. If there exist $x_0,y_0 \in X$ such that $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$. Suppose either

(a) T is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \geq y_n$ for all n.

Then there exist $x,y \in X$ such that $T(x,y) = x$ and $T(y,x) = y$.

Recently, Kumam et al.[20] extended the results of Luong and Thuan [18]. They also introduced the concept of the proximal mixed monotone property and established coupled best proximity point theorem. Following is the main results of Kumam et al.[20].

Theorem 1.11. [20] *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A\times A \to B$ satisfy the following conditions:*

(i) T is a continuous proximally coupled weak (ψ,ϕ) contraction on A having the proximal mixed monotone property on A such that $T(A_0,A_0) \subseteq B_0$;.

(ii) There exist elements (x_0,y_0) and $(x_1,y_1) \in A_0\times A_0$ such that

$$d(x_1,T(x_0,y_0)) = d(A,B) \quad \textit{with} \quad x_0 \leq x_1,$$

and

$$d(y_1, T(y_0, x_0)) = d(A, B) \quad with \quad y_0 \geq y_1.$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(x, T(x,y)) = d(A,B) \text{ and } d(y, T(y,x)) = d(A,B).$$

Motivated by the results of [18] and [20], we present the coupled best proximity point and coupled fixed point, and by defining the concept of proximal mixed g-monotone mapping and proximally coupled weak (ψ, ϕ) contraction on A. The existence and uniqueness of coupled best proximity points are obtained in partially ordered metric spaces. We also provide an example to support of our results.

2. Main Results

In this section we first present following definitions.

Definition 2.1. *Let* $(X, d, \leq)$ *be a partially ordered metric space. Let A, B be nonempty subsets of X, and* $T : A \times A \to B$ *and* $g : A \to A$ *be two given mappings. We say that T has the proximal mixed* g*-monotone property provided that for all* $x, y \in A$*, if*

$$\begin{cases} g(x_1) \leq g(x_2), \\ d(g(u_1), T(g(x_1), g(y))) = d(A,B) \\ d(g(u_2), T(g(x_2), g(y))) = d(A,B) \end{cases} \implies g(u_1) \leq g(u_2),$$

and

$$\begin{cases} g(y_1) \leq g(y_2), \\ d(g(u_3), T(g(x), g(y_1))) = d(A,B) \\ d(g(u_4), T(g(x), g(y_2))) = d(A,B) \end{cases} \implies g(u_4) \leq g(u_3),$$

where $x_1, x_2, y_1, y_2, u_1, u_2, u_3, u_4 \in A$.

Definition 2.2. *Let* $(X, d, \leq)$ *be a partially ordered metric space and A, B are nonempty subsets of X. Let* $T : A \times A \to B$ *and* $g : A \to A$ *be two given mappings. T is said to be proximally coupled* *weak* (ψ,ϕ) *contraction on A, whenever*

$$\begin{cases} g(x_1) \leq g(x_2) \quad g(y_1) \geq g(y_2), \\ d(g(u_1), T(g(x_1), g(y_1)) = d(A,B) \\ d(g(u_2), T(g(x_2), g(y_2)) = d(A,B) \end{cases}$$

$$\begin{aligned} \Longrightarrow \phi\big(d(g(u_1), g(u_2))\big) \quad &\leq \quad \frac{1}{2}\phi\Big(d(g(x_1), g(x_2)) + d(g(y_1), g(y_2))\Big) \\ &\quad -\psi\Big(\frac{d(g(x_1), g(x_2)) + d(g(y_1), g(y_2))}{2}\Big), \end{aligned} \tag{2.1}$$

where $x_1, x_2, y_1, y_2, u_1, u_2 \in A$.

Lemma 2.3. *Let* $(X, d, \leq)$ *be a partially ordered metric space and A, B be nonempty subsets of X,* $A_0 \neq \emptyset$ *and* $T : A \times A \to B$ *and* $g : A \to A$ *be two given mappings. If T has the proximal mixed* g*-monotone property, with* $g(A_0) = A_0$*,* $T(A_0, A_0) \subseteq B_0$

$$\begin{cases} g(x_1) \leq g(x_2) \quad g(y_1) \geq g(y_2), \\ d(g(x_2), T(g(x_1), g(y_1)) = d(A,B) \\ d(g(u), T(g(x_2), g(y_2)) = d(A,B) \end{cases} \Longrightarrow g(x_2) \leq g(u) \tag{2.2}$$

where $x_1, x_2, y_1, y_2, u \in A_0$.

Proof. Since $g(A_0) = A_0$, $T(A_0, A_0) \subseteq B$, it follows that $T(g(x_2), g(y_1)) \in B_0$. Hence there exists $g(u_1^*) \in A_0$ such that

$$d(g(u_1^*), T(g(x_2), g(y_1)) = d(A,B). \tag{2.3}$$

Using the fact that T has the proximal mixed g-monotone property, together with 2.2 and 2.3, we get

$$\begin{cases} g(x_1) \leq g(x_2) \\ d(g(x_2), T(g(x_1), g(y_1)) = d(A,B) \\ d(g(u_1^*), T(g(x_2), g(y_1)) = d(A,B) \end{cases} \Longrightarrow g(x_2) \leq g(u_1^*). \tag{2.4}$$

Also, from the proximal mixed g-monotone property of T with 2.2 and 2.4, we get

$$\begin{cases} g(y_2) \le g(y_1) \\ d(g(u), T(g(x_2), g(y_2)) = d(A,B) \\ d(g(u_1^*), T(g(x_2), g(y_1)) = d(A,B) \end{cases} \implies g(u_1^*) \le g(u). \tag{2.5}$$

From 2.4 and 2.5, one can conclude the $g(x_2) \le g(u)$. Hence the proof is complete. □

Lemma 2.4. *Let $(X, d, \le)$ be a partially ordered metric space and A, B be nonempty subsets of X, $A_0 \ne \emptyset$ and $T : A \times A \to B$ and $g : A \to A$ be two given mappings. Let T have the proximal mixed* g*-monotone property, with $g(A_0) = A_0$, $T(A_0, A_0) \subseteq B_0$ If*

$$\begin{cases} g(x_1) \le g(x_2) \quad g(y_1) \ge g(y_2), \\ d(g(y_2), T(g(y_1), g(x_1)) = d(A,B) \\ d(g(u), T(g(y_2), g(x_2)) = d(A,B) \end{cases} \implies g(x_2) \le g(u) \tag{2.6}$$

where $x_1, x_2, y_1, y_2, u \in A_0$.

Proof. The proof is the same as Lemma 2.3, so we omit the details. □

Following is the main result of this paper.

Theorem 2.5. *Let $(X, d, \le)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \ne \emptyset$. Let $T : A \times A \to B$ and $g : A \to A$ be two given mappings satisfying the following conditions:*

(a) T and g are continuous;

(b) T has the proximal mixed g*-monotone property on A such that $g(A_0) = A_0$, $T(A_0, A_0) \subseteq B_0$;*

(c) T is a proximally coupled weak (ψ, ϕ) contraction on A;

(d) there exist elements (x_0, y_0) and $(x_1, y_1) \in A_0 \times A_0$ such that

$$d(g(x_1), T(g(x_0), g(y_0))) = d(A,B) \quad with \quad g(x_0) \le g(x_1),$$

and

$$d(g(y_1), T(g(y_0), g(x_0))) = d(A,B) \quad with \quad g(y_0) \ge g(y_1).$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(g(x),T(g(x),g(y))) = d(A,B) \text{ and } d(g(y),T(g(y),g(x))) = d(A,B).$$

Proof. Let $(x_0,y_0),(x_1,y_1) \in A_0 \times A_0$ be such that $d(g(x_1),T(g(x_0),g(y_0))) = d(A,B)$ with $g(x_0) \leq g(x_1)$, and $d(g(y_1),T(g(y_0),g(x_0))) = d(A,B)$ with $g(y_0) \geq g(y_1)$. Since $T(A_0,A_0) \subseteq B_0$ and $g(A_0) = A_0$, there exists an element $(x_2,y_2) \in A_0 \times A_0$ such that $d(g(x_2),T(g(x_1),g(y_1))) = d(A,B)$ and $d(g(y_2),T(g(y_1),g(x_1))) = d(A,B)$. Hence from Lemma 2.3 and Lemma 2.4 we obtain $g(x_1) \leq g(x_2)$ and $g(y_1) \geq g(y_2)$. Continuing this process, we can construct the sequences $\{x_n\},\{yn\} \in A_0$ such that

$$d\big(g(x_{n+1}),T(g(x_n),g(y_n))\big) = d(A,B) \; for \; all \; n \geq 0,$$

with

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \cdots \leq g(x_n) \leq g(x_{n+1}) \leq \cdots \tag{2.7}$$

and

$$d\big(g(y_{n+1}),T(g(y_n),g(x_n))\big) = d(A,B) \; for \; all \; n \geq 0,$$

with

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \cdots \geq g(y_n) \geq g(y_{n+1}) \geq \cdots. \tag{2.8}$$

Then $d\big(g(x_n),T(g(x_{n-1}),g(y_{n-1}))\big) = d(A,B)$ and $d\big(g(x_{n+1}),T(g(x_n),g(y_n))\big) = d(A,B)$ $g(x_{n-1}) \leq g(x_n)$ and $g(y_{n-1}) \geq g(y_n)$. Now using the fact that T is a proximally coupled weak (ψ,ϕ) contraction on A, we get

$$\begin{aligned} \phi\big(d(g(x_n),g(x_{n+1}))\big) \;\leq\; & \frac{1}{2}\phi\big(d(g(x_{n-1}),g(x_n)) + d(g(y_{n-1}),g(y_n))\big) \\ & -\psi\Big(\frac{d(g(x_{n-1}),g(x_n)) + d(g(y_{n-1}),g(y_n))}{2}\Big). \end{aligned} \tag{2.9}$$

Similarly,

$$\begin{aligned} \phi\big(d(g(y_n),g(y_{n+1}))\big) \;\leq\; & \frac{1}{2}\phi\big(d(g(y_{n-1}),g(y_n)) + d(g(x_{n-1}),g(x_n))\big) \\ & -\psi\Big(\frac{d(g(y_{n-1}),g(y_n)) + d(g(x_{n-1}),g(x_n))}{2}\Big). \end{aligned} \tag{2.10}$$

Adding (2.9) and (2.10), we get

$$\begin{aligned}\phi\big(d(g(x_n),g(x_{n+1}))\big)+\phi\big(d(g(y_n),g(y_{n+1}))\big) &\leq \phi\big(d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))\big)\\ &\quad -2\psi\Big(\frac{d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))}{2}\Big).\end{aligned} \tag{2.11}$$

By the definition of ϕ, we have

$$\phi\big(d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\big)\leq \phi\big(d(g(x_n),g(x_{n+1}))\big)+\phi\big(d(g(y_n),g(y_{n+1}))\big). \tag{2.12}$$

From (2.11) and (2.12), we get

$$\begin{aligned}\phi\big(d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\big) &\leq \phi\big(d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))\big)\\ &\quad -2\psi\Big(\frac{d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))}{2}\Big).\end{aligned} \tag{2.13}$$

Since ϕ is nondecreasing, we get

$$d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\leq d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n)). \tag{2.14}$$

Putting $\delta_n=d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))$ then the sequence (δ_n) is decreasing. Therefore, there is some $\delta\geq 0$such that

$$\lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\big[d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\big]=\delta. \tag{2.15}$$

We shall show that $\delta=0$. Suppose, to the contrary, that $\delta>0$. Then taking the limit as $n\to\infty$ both sides of (2.13) and having in mind that we assume $\lim_{t\to r}\psi(t)>0$ for all $r>0$ and ϕ is continuous, we have

$$\phi(\delta)=\lim_{n\to\infty}\phi(\delta_n)\leq\lim_{n\to\infty}\phi(\delta_{n-1})-2\psi\Big(\frac{\delta_{n-1}}{2}\Big)=\phi(\delta)-2\lim_{n\to\infty}\psi\Big(\frac{\delta_{n-1}}{2}\Big)<\phi(\delta), \tag{2.16}$$

a contradiction. Thus $\delta=0$, that is,

$$\lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\big[d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\big]=0. \tag{2.17}$$

Now, we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Suppose that at least one of the sequences $\{g(x_n)\}$ or $\{g(y_n)\}$ is not a Cauchy sequence. This implies that $\lim_{n,m\to\infty} d(g(x_n), g(x_m)) \nrightarrow 0$ or $\lim_{n,m\to\infty} d(g(y_n), g(y_m)) \nrightarrow 0$, and, consequently

$$\lim_{n,m\to\infty} \left[d(g(x_n), g(x_m)) + d(g(y_n), g(y_m))\right] \nrightarrow 0. \tag{2.18}$$

Then there exists an $\varepsilon > 0$ for which we can find sub-sequences $\{g(x_{n(k)})\}$, $\{g(x_{m(k)})\}$ of $\{g(x_n)\}$ and $\{g(y_{n(k)})\}$, $\{g(y_{m(k)})\}$ of$\{g(y_n)\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$\left[d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))\right] \geq \varepsilon. \tag{2.19}$$

This means that

$$\left[d(g(x_{n(k)-1}), g(x_{m(k)})) + d(g(y_{n(k)-1}), g(y_{m(k)}))\right] < \varepsilon. \tag{2.20}$$

Therefore by using (2.19), (2.20) and the triangle inequality, we obtain

$$\begin{aligned} \varepsilon &\leq d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \\ &\leq d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(x_{n(k)-1}), g(x_{m(k)})) \\ &\quad + d(g(y_{n(k)}), g(y_{n(k)-1})) + d(g(y_{n(k)-1}), g(y_{m(k)})) \\ &\leq d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(y_{n(k)}), g(y_{n(k)-1})) + \varepsilon. \end{aligned}$$

On taking the limit $k \to \infty$ and using (2.17), we obtain

$$\lim_{k\to\infty} \left[g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))\right] = \varepsilon. \tag{2.21}$$

By the triangle inequality

$$\begin{aligned} & d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \\ &\leq d(g(x_{n(k)}), g(x_{n(k)+1})) + d(g(x_{n(k)+1}), g(x_{m(k)+1})) \\ &\quad + d(g(x_{m(k)+1}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{n(k)+1})) \\ &\quad + d(g(y_{n(k)+1}), g(y_{m(k)+1})) + d(g(y_{m(k)+1}), g(y_{m(k)})) \\ &= \delta_{n(k)} + \delta_{m(k)} + d(g(x_{n(k)+1}), g(x_{m(k)+1})) + d(g(y_{n(k)+1}, g(y_{m(k)+1})). \end{aligned}$$

Using the property of ϕ, we obtain

$$\begin{aligned}\phi(\alpha_k) &= \phi\Big(\delta_{n(k)}+\delta_{m(k)}+d(g(x_{n(k)+1}),g(x_{m(k)+1}))+d(g(y_{n(k)+1},g(y_{m(k)+1}))\Big)\\ &\leq \phi\big(\delta_{n(k)}\big)+\phi\big(\delta_{m(k)}\big)+\phi\big(d(g(x_{n(k)+1}),g(x_{m(k)+1}))\big)\\ &\quad +\phi\big(d(g(y_{n(k)+1},g(y_{m(k)+1}))\big).\end{aligned} \tag{2.22}$$

where $\alpha_k = d(g(x_{n(k)}),g(x_{m(k)}))+d(g(y_{n(k)}),g(y_{m(k)}))$. Since $g(x_{n(k))} \geq g(x_{m(k)})$ and $g(y_{n(k)}) \leq g(y_{m(k)})$, using the fact that T is a proximally coupled weak (ψ,ϕ) contraction on A, we get

$$\begin{aligned}\phi\big(d(g(x_{n(k)+1}),g(x_{m(k)+1}))\big) &\leq \frac{1}{2}\phi\big(d(g(x_{n(k)}),g(x_{m(k)}))+d(g(y_{n(k)}),g(y_{m(k)}))\big)\\ &\quad -\psi\Big(\frac{d(g(x_{n(k)}),g(x_{m(k)}))+d(g(y_{n(k)}),g(y_{m(k)}))}{2}\Big)\\ &\leq \frac{1}{2}\phi(\alpha_k)-\psi\Big(\frac{\alpha_k}{2}\Big).\end{aligned} \tag{2.23}$$

Similarly, we also have

$$\begin{aligned}\phi\big(d(g(y_{n(k)+1}),g(y_{m(k)+1}))\big) &\leq \frac{1}{2}\phi\big(d(g(y_{n(k)}),g(y_{m(k)}))+d(g(x_{n(k)}),g(x_{m(k)}))\big)\\ &\quad -\psi\Big(\frac{d(g(y_{n(k)}),g(y_{m(k)}))+d(g(x_{n(k)}),g(x_{m(k)}))}{2}\Big)\\ &\leq \frac{1}{2}\phi(\alpha_k)-\psi\Big(\frac{\alpha_k}{2}\Big).\end{aligned}$$

(2.24)

From (2.22),(2.23), (2.24), we obtain

$$\phi(\alpha_k) \leq \phi\big(\delta_{n(k)}+\delta_{m(k)}\big)+\phi(\alpha_k)-2\psi\big(\frac{\alpha_k}{2}\big).$$

On taking the limit $k\to\infty$ using (2.17) and (2.21), we have

$$\phi(\varepsilon) \leq \phi(0)+\phi(\varepsilon)-2\lim_{k\to\infty}\psi\big(\frac{\alpha_k}{2}\big) = \phi(\varepsilon)-2\lim_{k\to\infty}\psi\big(\frac{\alpha_k}{2}\big) < \phi(\varepsilon). \tag{2.25}$$

Which is a contradiction. This shows that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since A is a closed subset of a complete metric space X, there exist $x',y' \in A$ such that $g(x_n)\to x'$ and $g(y_n)\to y'$ as $n\to\infty$. Here $x_n,y_n \in A_0$, $g(A_0)=A_0$ so that $g(x_n),g(y_n)\in A_0$. Since A_0 is closed, we conclude that $x', y' \in A_0 \times A_0$, i.e., there exist $x, y \in A_0$ such that $g(x) = x'$, $g(y) = y'$. Therefore

$$g(x_n) \to g(x) \textit{ and } g(y_n) \to g(y). \tag{2.26}$$

Since $\{g(x_n)\}$ is monotone increasing and $\{g(y_n)\}$ is monotone decreasing, we have $g(x_n) \leq g(x)$ and $g(y_n) \geq g(y)$. From (2.7) and (2.8), we have

$$d\big(g(x_{n+1}), T(g(x_n), g(y_n))\big) = d(A, B), \tag{2.27}$$

and

$$d\big(g(y_{n+1}), T(g(y_n), g(x_n))\big) = d(A, B). \tag{2.28}$$

Since T is continuous, we have, from (2.26),

$$T(g(x_n), g(y_n)) \to T(g(x), g(y)),$$

and

$$T(g(y_n), g(x_n)) \to T(g(y), g(x)).$$

Thus, the continuity of the metric d implies that

$$d\big(g(x_{n+1}), T(g(x_n), g(y_n))\big) \to d\big(g(x), T(g(x), g(y))\big), \tag{2.29}$$

and

$$d\big(g(y_{n+1}), T(g(y_n), g(x_n))\big) \to d\big(g(y), T(g(y), g(x))\big). \tag{2.30}$$

Therefore from (2.27), (2.28), (2.29),(2.31)

$$d\big(g(x), T(g(x), g(y))\big) = d(A, B) \qquad d\big(g(y), T(g(y), g(x))\big) = d(A, B).$$

$\square$

If g is assumed to be the identity mappings in Theorem 2.5.

Corollary 2.6. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \times A \to B$ given mappings satisfying the following conditions:*

(a) T be continuous;

(b) T has the proximal mixed monotone property on A such that $T(A_0,A_0) \subseteq B_0$*;*

(c) T is a proximally coupled weak (ψ,ϕ) *contraction on A;*

(d) there exist elements (x_0,y_0) *and* $(x_1,y_1) \in A_0 \times A_0$ *such that*

$$d(x_1,T(x_0,y_0)) = d(A,B) \text{ with } x_0 \le x_1 \text{ and}$$

$$d(y_1,T(y_0,x_0)) = d(A,B) \text{ with } y_0 \ge y_1.$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(x,T(x,y)) = d(A,B) \text{ and } d(y,T(y,x)) = d(A,B).$$

Corollary 2.7. *Let* $(X,d,\le)$ *be a partially ordered complete metric space. Let A be a nonempty closed subsets of the metric space (X, d). Let* $T : A \times A \to A$ *and* $g : A \to A$ *be two given mappings satisfying the following conditions:*

(a) T and g are continuous;

(b) T has the mixed g*-monotone property on A such that* $g(A) = A$ *and* $T(A,A) \subseteq A$*;*

(c) T is a coupled weak (ψ,ϕ) *contraction on A;*

(d) There exist elements (x_0,y_0) *and* $(x_1,y_1) \in A \times A$ *such that*

$$g(x_1) = T(g(x_0),g(y_0)) \quad \text{with} \quad g(x_0) \le g(x_1) \text{ and}$$

$$g(y_1) = T(g(y_0),g(x_0)) \quad \text{with} \quad g(y_0) \ge g(y_1).$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(g(x),T(g(x),g(y))) = 0 \text{ and } d(g(y),T(g(y),g(x))) = 0.$$

Theorem 2.8. *Let* $(X,d,\le)$ *be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that* $A_0 \neq \emptyset$*. Let* $T : A \times A \to B$ *and* $g : A \to A$ *be two given mappings satisfying the following conditions:*

(a) g is continuous;

(b) T has the proximal mixed g*-monotone property on A such that* $g(A_0) = A_0$*,* $T(A_0,A_0) \subseteq B_0$*;*

(c) T is a proximally coupled weak (ψ,ϕ) *contraction on A;*

(d) *there exist elements* (x_0, y_0) *and* $(x_1, y_1) \in A_0 \times A_0$ *such that*

$$d\big(g(x_1), T(g(x_0), g(y_0))\big) = d(A,B) \quad with \quad g(x_0) \leq g(x_1),$$

and

$$d\big(g(y_1), T(g(y_0), g(x_0))\big) = d(A,B) \quad with \quad g(y_0) \geq g(y_1).$$

(e) *if* $\{x_n\}$ *is a nondecreasing sequence in A such that* $x_n \to x$*, then* $x_n \leq x$ *and if* $\{y_n\}$ *is a nonincreasing sequence in A such that* $y_n \to y$*, then* $y_n \geq y$*.*

Then there exists $(x,y) \in A \times A$ *such that*

$$d\big(g(x), T(g(x), g(y))\big) = d(A,B) \text{ and } d\big(g(y), T(g(y), g(x))\big) = d(A,B).$$

Proof. As in the proof of Theorem 2.5, there exist sequences $\{x_n\}$ and $\{y_n\}$ in A_0 such that

$$d\big(g(x_{n+1}), T\big(g(x_n), g(y_n)\big)\big) = d(A,B) \text{ with } g(x_n) \leq g(x_{n+1}) \text{ for all } n \geq 0, \tag{2.31}$$

and

$$d\big(g(y_{n+1}), T\big(g(y_n), g(x_n)\big)\big) = d(A,B) \text{ with } g(y_n) \geq g(y_{n+1}) \text{ for all } n \geq 0. \tag{2.32}$$

Also, $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$. From (e), we get $g(x_n) \leq g(x)$, and $g(y_n) \geq g(y)$. Since $T(A_0, A_0) \subseteq B_0$, it follows that $T(g(x), g(y))$ and $T(g(y), g(x))$ are in B_0. Therefore, there exists $(x_1^*, y_1^*) \in A_0 \times A_0$ such that $d\big(x_1^*, T(g(x), g(y))\big) = d(A,B)$ and $d(y_1^*, T(g(y), g(x))) = d(A,B)$. Since $g(A_0) = A_0$, there exist $x^*, y^* \in A_0$ such that $g(x^*) = x_1^*$ and $g(y^*) = y_1^*$. Hence,

$$d\big(g(x^*), T(g(x), g(y))\big) = d(A,B) \text{ and} \tag{2.33}$$

$$d(g(y^*), T\big(g(y), g(x))\big) = d(A,B). \tag{2.34}$$

Since $g(x_n) \leq g(x)$, and $g(y_n) \geq g(y)$ and T is a proximally coupled weak (ψ, ϕ) contraction on A for (2.31) and (2.33), and also for (2.32) and (2.34), we get

$$\begin{aligned}\phi\left(d\big(g(x_{n+1}), g(x^*)\big)\right) \;&\leq\; \frac{1}{2}\phi\left(d(g(x_n), g(x)) + d(g(y_n), g(y))\right) \\ &\quad -\psi\Big(\frac{d(g(x_n), g(x)) + d(g(y_n), g(y))}{2}\Big),\end{aligned}$$

$$\begin{aligned}\phi\big(d\big(g(y_{n+1}),g(y^*)\big)\big) &\leq \frac{1}{2}\phi\big(d(g(y_n),g(y))+d(g(x_n),g(x))\big)\\ &\quad -\psi\Big(\frac{d(g(y_n),g(y))+d(g(x_n),g(x))}{2}\Big).\end{aligned}$$

By taking the limit of the above two inequalities, we get $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Hence, from (2.33) and (2.34), we get $d\big(g(x),T(g(x),g(y))\big) = d(A,B)$ and $d(g(y),T\big(g(y),g(x)\big)) = d(A,B)$. $\square$

Remark 2.9. *If we replace the continuity of T by the condition (e) of Theorem 2.8 in Corollary 2.6, then the Corollary 2.6 holds true.*

Note that the assumptions in Theorems 2.5 and 2.8 do not guarantee the uniqueness of coupled best proximity point. The next example shows this fact.

Example 2.10. *Let* $X = \{(0,2),(2,0),(-2,0),(0,-2)\} \subset \mathbb{R}^2$ *and consider the usual order* $(x,y) \leq (z,t) \Leftrightarrow x \leq z$ *and* $y \leq t$. *Thus,* $(X,\leq)$ *is a partially ordered set. Besides, (X,d) is a complete metric space considering d the euclidean metric. Let* $A = \{(0,2),(2,0)\}$ *and* $B = \{(0,-2),(-2,0)\}$ *be a closed subset of X. Then,* $d(A,B) = 2\sqrt{2}$, $A = A_0$ *and* $B = B_0$. *Let* $T : A\times A \to B$ *and* $g : A \to A$ *be two mappings defined as* $T((x_1,x_2),(y_1,y_2)) = 2(-x_2,-x_1)$ *and* $g(x) = x$. *Then, it can be seen that T is continuous such that* $T(A_0 \times A_0) \subseteq B_0$ *and* $g(A_0) = A_0$. *The only comparable pairs of points in A are* $gx \leq gx$ *for* $x \in A$, *hence proximal mixed g-monotone property and proximally coupled weak* (ψ,ϕ) *contraction on A are satisfied trivially.*

It can be shown that the other hypotheses of the theorem are also satisfied. However, T has three coupled best proximity points ((0,2),(0,2)),((0,2),(2,0)) and ((2,0),(2,0)).

One can prove that the coupled best proximity point is in fact unique, provided that the product space $A\times A$ endowed with the partial order mentioned earlier has the following property:

$$Every\ pair\ of\ elements\ has\ either\ a\ lower\ bound\ or\ an\ upper\ bound.$$

It is known that this condition is equivalent to the following. For every pair of $(x,y),(z,t) \in A\times A$, there exists $(u,v) \in A\times A$ that is comparable to (x, y) and (z,t).

Theorem 2.11. *Suppose that all the hypotheses of Theorem 2.5 hold and further, for all* $(x,y),(z,t) \in A_0 \times A_0$, *there exists* $(u,v) \in A_0 \times A_0$ *such that (u, v) is comparable to (x, y),*

(z,t) (with respect to the ordering in $A\times A$). Then there exists a unique $(x,y)\in A\times A$ such that $d\big(g(x),T\big(g(x),g(y)\big)\big)=d(A,B)$ and $d\big(g(y),T\big(g(y),g(x)\big)\big)=d(A,B)$.

Proof. From Theorem2.5, there exists an element $(x,y)\in A\times A$ such that

$$d\big(g(x),T\big(g(x),g(y)\big)\big)=d(A,B) \tag{2.35}$$

and

$$d\big(g(y),T\big(g(y),g(x)\big)\big)=d(A,B). \tag{2.36}$$

Now, suppose that there exists an element $z,t\in A\times A$ such that

$$d(g(z),T(g(z),g(t)))=d(A,B) \tag{2.37}$$

and

$$d(g(t),T(g(t),g(z)))=d(A,B) \tag{2.38}$$

Case 1: Let (g(x), g(y)) be comparable to $(g(z),g(t))$ with respect to the ordering in $A\times A$. Since $d\big(g(x),T(g(x),g(y))\big)=d(A,B)$ and $d\big(g(z),T(g(z),g(t))\big)=d(A,B)$ it follows from the fact that T is a proximally coupled weak (ψ,ϕ) contraction on A, we get

$$\begin{aligned}\phi\big(d\big(g(x),g(z)\big)\big) &\leq \frac{1}{2}\phi\big(d(g(x),g(z))+d(g(y),g(t))\big)\\ &\quad -\psi\Big(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\Big),\end{aligned} \tag{2.39}$$

$$\begin{aligned}\phi\big(d\big(g(y),g(t)\big)\big) &\leq \frac{1}{2}\phi\big(d(g(y),g(t))+d(g(x),g(z))\big)\\ &\quad -\psi\Big(\frac{d(g(y),g(t))+d(g(x),g(z))}{2}\Big).\end{aligned} \tag{2.40}$$

Adding (2.39) and (2.40), we get

$$\begin{aligned}\phi\big(d\big(g(x),g(z)\big)\big)+\phi\big(d\big(g(y),g(t)\big)\big) &\leq \phi\big(d(g(x),g(z))+d(g(y),g(t))\big)\\ &\quad -2\psi\Big(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\Big).\end{aligned} \tag{2.41}$$

By the definition of ϕ, we have

$$\phi\big(d(g(x),g(z))+d(g(y),g(t))\big) \leq \phi\big(d\big(g(x),g(z)\big)\big)+\phi\big(d\big(g(y),g(t)\big)\big). \tag{2.42}$$

From (2.41) and (2.42), we have

$$\begin{aligned}\phi\big(d(g(x),g(z))+d(g(y),g(t))\big) \;&\leq\; \phi\big(d(g(x),g(z))+d(g(y),g(t))\big)\\ &\quad -2\psi\Big(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\Big).\end{aligned}$$

(2.43)

this implies that $2\psi\Big(\frac{d(g(x),g(z))+d(g(y),g(t))}{2}\Big) \leq 0$ and using the property of ψ, we get $d(g(x),g(z))+d(g(y),g(t)) = 0$, hence $gx = gz$ and $gy = gt$.

Case 2: let $(g(x),g(y))$ is not comparable to $(g(z),g(t))$, then there exists $(g(u_1),g(v_1)) \in A_0 \times A_0$ which is comparable to $(g(x),g(y))$ and $(g(z),g(t))$. Since $T(A_0,A_0) \subseteq B_0$ and $g(A_0) = A_0$, there exists $(g(u_2),g(v_2)) \in A_0 \times A_0$ such that $d(g(u_2),T(g(u_1),g(v_1))) = d(A,B)$ and $d(g(v_2),T(g(v_1),g(u_1))) = d(A,B)$.

We assume, without loss of generality, that $(g(u_1),g(v_1)) \leq (g(x),g(y))$, i.e., $g(u_1) \leq g(x)$ and $g(v_1) \geq g(y)$. Therefore $(g(y),g(x)) \leq (g(v_1),g(u_1))$. From Lemma 2.3 and Lemma 2.4, we get

$$\begin{cases} g(u_1) \leq g(x), \quad g(v_1) \geq g(y), \\ d(g(u_2),T(g(u_1),g(v_1)) = d(A,B) \\ d(g(x),T(g(x),g(y)) = d(A,B) \end{cases} \Longrightarrow g(u_2) \leq g(x)$$

$$\begin{cases} g(u_1) \leq g(x), \quad g(v_1) \geq g(y), \\ d(g(v_2),T(g(v_1),g(u_1)) = d(A,B) \\ d(g(y),T(g(y),g(x)) = d(A,B) \end{cases} \Longrightarrow g(v_2) \geq g(y).$$

On continuing this process, we construct sequences $\{u_n\}$ and $\{v_n\}$ such that

$$d\big(g(u_{n+1}),T(g(u_n),g(v_n))\big) = d(A,B)$$

and

$$d\big(g(v_{n+1}),T(g(v_n),g(u_n))\big) = d(A,B)$$

with $(g(u_n),g(v_n)) \leq (g(x),g(y))$. By using the fact that T is a proximally coupled weak (ψ,ϕ) contraction on A, we get

$$\begin{cases} g(u_n) \leq g(x), \quad g(v_n) \geq g(y), \\ d\big(g(u_{n+1}),T(g(u_n),g(v_n))\big) = d(A,B) \\ d(g(x),T(g(x),g(y)) = d(A,B) \end{cases}$$

$$\Longrightarrow \phi\big(d\big(g(u_{n+1}),g(x)\big)\big) \leq \frac{1}{2}\phi\big(d(g(u_n),g(x)) + d(g(v_n),g(y))\big) \tag{2.44}$$
$$-\psi\Big(\frac{d(g(u_n),g(x)) + d(g(v_n),g(y))}{2}\Big).$$

Similarly, we have

$$\begin{cases} g(u_n) \leq g(x), \quad g(v_n) \geq g(y), \\ d(g(v_{n+1}),T(g(v_n),g(u_n)) = d(A,B) \\ d(g(y),T(g(y),g(x)) = d(A,B) \end{cases}$$

$$\Longrightarrow \phi\big(d\big(g(v_{n+1}),g(y)\big)\big) \leq \frac{1}{2}\phi\big(d(g(v_n),g(y)) + d(g(u_n),g(x))\big) \tag{2.45}$$
$$-\psi\Big(\frac{d(g(v_n),g(y)) + d(g(u_n),g(x))}{2}\Big).$$

Adding (2.44) and (2.45), we obtain

$$\phi\big(d\big(g(u_{n+1}),g(x)\big)\big) + \phi\big(d\big(g(v_{n+1}),g(y)\big)\big) \leq \phi\big(d(g(u_n),g(x)) + d(g(v_n),g(y))\big)$$
$$-2\psi\Big(\frac{d(g(u_n),g(x)) + d(g(v_n),g(y))}{2}\Big).$$

But

$$\phi\big(d\big(g(u_{n+1}),g(x)\big) + d\big(g(v_{n+1}),g(y)\big)\big) \leq \phi\big(d\big(g(u_{n+1}),g(x)\big)\big) + \phi\big(d\big(g(v_{n+1}),g(y)\big)\big),$$

hence

$$\phi\big(d\big(g(u_{n+1}),g(x)\big) + d\big(g(v_{n+1}),g(y)\big)\big) \leq \phi\big(d(g(u_n),g(x)) + d(g(v_n),g(y))\big)$$
$$-2\psi\Big(\frac{d(g(u_n),g(x)) + d(g(v_n),g(y))}{2}\Big).$$

(2.46)

Using the fact that ϕ is nondecreasing, we get

$$d\big(g(u_{n+1}),g(x)\big)+d\big(g(v_{n+1}),g(y)\big)\leq d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big). \tag{2.47}$$

Therefore $d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big)$ is a decreasing sequence. Hence there exists $r\geq 0$ such that

$$\lim_{n\to\infty}\big[d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big)\big]=r.$$

We shall show that $r=0$. Suppose, to the contrary, that $r>0$. On taking the limit as $n\to\infty$ in (2.46), we have

$$\phi(r)\leq\phi(r)-2\lim_{n\to\infty}\psi\Big(\frac{d(g(u_n),g(x))+d(g(v_n),g(y))}{2}\Big)<\phi(r),$$

which is a contradiction. Hence, $r=0$, that is,

$$\lim_{n\to\infty}\big[d\big(g(u_n),g(x)\big)+d\big(g(v_n),g(y)\big)\big]=0,$$

so that $g(u_n)\to g(x)$ and $g(v_n)\to g(y)$. Analogously, one can prove that $g(u_n)\to g(z)$ and $g(v_n)\to g(t)$. Therefore, $g(x)=g(z)$ and $g(y)=g(t)$. Hence the proof is complete. □

Considering g is assumed to be the identity mappings in Theorem 2.11 then we obtained the following result.

Corollary 2.12. *Suppose that all the hypotheses of Corollary 2.6 hold and further, for all* $(x,y),(z,t)\in A_0\times A_0$*, there exists* $(u,v)\in A_0\times A_0$ *such that (u, v) is comparable to (x, y),* (z,t) *(with respect to the ordering in* $A\times A$*). Then there exists a unique* $(x,y)\in A\times A$ *such that* $d\big(x,T(x,y)\big)=d(A,B)$ *and* $d\big(y,T(y,x)\big)=d(A,B)$*.*

If A = B in Theorem 2.11, we obtained the following result.

Corollary 2.13. *Suppose that all the hypotheses of Corollary 2.7 hold and further, for all* $(x,y),(z,t)\in A\times A$*, there exists* $(u,v)\in A\times A$ *such that (u, v) is comparable to (x, y),* (z,t) *(with respect to the ordering in* $A\times A$*). Then there exists a unique* $(x,y)\in A\times A$ *such that* $d\big(g(x),T(g(x),g(y))\big)=0$ *and* $d\big(g(y),T(g(y),g(x))\big)=0$*.*

We shall illustrate our results by the following example.

Example 2.14. *Let $X = \mathbb{R}$ and $d(x,y) =\mid x-y \mid$ be the usual metric on X and let the usual ordering $(x,y) \leq (u,v) \Leftrightarrow x \leq u, y \geq v$. Assume that $A = [1,\infty)$ and $B = (0,-1]$ and A, B are nonempty closed subsets of X. We also have $A_0 = \{1\}$ and $B_0 = \{-1\}$ and d(A,B) = 2.*
Let $T : A \times A \to B$ and $g : A \to A$ be two mappings such that $T(x,y) = -\frac{x+y}{2}$ and $g(x) = x^2$. Then T and g are continuous and T(1, 1) = -1 and g(1) = 1, i.e., $T(A_0,A_0) \subseteq B_0$ and $g(A_0) = A_0$. We now define functions $\phi, \psi : [0,\infty] \to [0,\infty]$ by $\phi(t) = t$ and $\psi(t) = t$ with these ϕ and ϕ, it is easy to see that T satisfy the inequality 2.1. Hence T satisfies all the hypotheses of Theorem 2.11 then there exists a unique point $(1,1) \in A \times A$ such that $d\big(g(1), T(g(1), g(1))\big) = 2 = d(A,B)$.

3. APPLICATION

Now we present some applications of the main results in the previous Section.

Theorem 3.1. *Let $(X,d,\leq)$ be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \times A \to B$ and $g : A \to A$ be two given mappings satisfying the following conditions:*

(a) T and g are continuous;

(b) T has the proximal mixed g-monotone property on A such that $g(A_0) = A_0$, $T(A_0,A_0) \subseteq B_0$;

(c) T is a proximally coupled weak (ψ,ϕ) contraction on A, that is

$$\begin{cases} g(x_1) \leq g(x_2) \quad g(y_1) \geq g(y_2), \\ d(g(u_1), T(g(x_1), g(y_1)) = d(A,B) \\ d(g(u_2), T(g(x_2), g(y_2)) = d(A,B) \end{cases}$$

$$\begin{aligned} \Longrightarrow \int_0^{\left(d(g(u_1),g(u_2))\right)} \phi(t)dt \quad &\leq \quad \frac{1}{2}\int_0^{[d(g(x_1),g(x_2))+d(g(y_1),g(y_2))]} \phi(t)dt \\ &\quad -\int_0^{\frac{1}{2}[d(g(x_1),g(x_2))+d(g(y_1),g(y_2))]} \psi(t)dt, \end{aligned} \tag{3.1}$$

where $x_1,x_2,y_1,y_2,u_1,u_2 \in A$ and ϕ and ψ are locally integrable function from $[0,\infty]$ into itself satisfying the following condition $\int_0^s \phi(t)dt > 0$ $\int_0^s \psi(t)dt > 0 \quad \forall s > 0$,

(d) there exist elements (x_0,y_0) and $(x_1,y_1) \in A_0 \times A_0$ such that

$$d(g(x_1),T(g(x_0),g(y_0))) = d(A,B) \quad with \quad g(x_0) \leq g(x_1),$$

and

$$d(g(y_1),T(g(y_0),g(x_0))) = d(A,B) \quad with \quad g(y_0) \geq g(y_1).$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(g(x),T(g(x),g(y))) = d(A,B) \text{ and } d(g(y),T(g(y),g(x))) = d(A,B).$$

Remark 3.2. *This results also true if we replace the continuity of T by the condition (e) of Theorem 2.8.*

Corollary 3.3. *Let* $(X,d,\leq)$ *be a partially ordered complete metric space. Let A, B be nonempty closed subsets of the metric space (X, d) such that* $A_0 \neq \emptyset$*. Let* $T : A \times A \to B$ *given mappings satisfying the following conditions:*

(a) T be continuous;

(b) T has the proximal mixed monotone property on A such that $T(A_0,A_0) \subseteq B_0$*;*

(c) T is a proximally coupled weak (ψ,ϕ) *contraction on A, that is*

$$\begin{cases} x_1 \leq x_2 \quad y_1 \geq y_2, \\ d(u_1,T(x_1,y_1) = d(A,B) \\ d(u_2,T(x_2,y_2) = d(A,B) \end{cases}$$

$$\Longrightarrow \int_0^{\left(d(u_1,u_2)\right)} \phi(t)dt \leq \frac{1}{2}\int_0^{[d(x_1,x_2)+d(y_1,y_2)]} \phi(t)dt - \int_0^{\frac{1}{2}[d(x_1,x_2)+d(y_1,y_2)]} \psi(t)dt, \tag{3.2}$$

where $x_1,x_2,y_1,y_2,u_1,u_2 \in A$ *and* ϕ *and* ψ *are locally integrable function from* $[0,\infty]$ *into itself satisfying the following condition* $\int_0^s \phi(t)dt > 0$ $\int_0^s \psi(t)dt > 0 \quad \forall s > 0$*,*

(d) there exist elements (x_0,y_0) *and* $(x_1,y_1) \in A_0 \times A_0$ *such that*

$$d(x_1,T(x_0,y_0)) = d(A,B) \text{ with } x_0 \leq x_1 \text{ and}$$

$$d(y_1,T(y_0,x_0)) = d(A,B) \text{ with } y_0 \geq y_1.$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(x,T(x,y)) = d(A,B) \text{ and } d(y,T(y,x)) = d(A,B).$$

Remark 3.4. *This results also true if we replace the continuity of T by the condition (e) of Theorem 2.8.*

Corollary 3.5. *Let* $(X,d,\leq)$ *be a partially ordered complete metric space. Let A be a nonempty closed subsets of the metric space (X, d). Let* $T : A \times A \to A$ *and* $g : A \to A$ *be two given mappings satisfying the following conditions:*

(a) T and g are continuous;

(b) T has the mixed g*-monotone property on A such that* $g(A) = A$ *and* $T(A,A) \subseteq A$*;*

(c) T is a coupled weak (ψ,ϕ) *contraction on A, that is*

$$g(x_1) \leq g(x_2) \quad g(y_1) \geq g(y_2),$$

$$\begin{aligned} \int_0^{\left(d(g(u_1),g(u_2))\right)} \phi(t)dt \quad &\leq \quad \frac{1}{2}\int_0^{[d(g(x_1),g(x_2))+d(g(y_1),g(y_2))]} \phi(t)dt \\ &\quad - \int_0^{\frac{1}{2}[d(g(x_1),g(x_2))+d(g(y_1),g(y_2))]} \psi(t)dt, \end{aligned} \tag{3.3}$$

where $x_1,x_2,y_1,y_2,u_1,u_2 \in A$ *and* ϕ *and* ψ *are locally integrable function from* $[0,\infty]$ *into itself satisfying the following condition* $\int_0^s \phi(t)dt > 0$ $\int_0^s \psi(t)dt > 0 \quad \forall s > 0$*,*

(d) There exist elements (x_0,y_0) *and* $(x_1,y_1) \in A \times A$ *such that*

$$g(x_1) = T(g(x_0),g(y_0)) \quad \text{with} \quad g(x_0) \leq g(x_1) \text{ and}$$

$$g(y_1) = T(g(y_0),g(x_0)) \quad \text{with} \quad g(y_0) \geq g(y_1).$$

Then there exists $(x,y) \in A \times A$ *such that*

$$d(g(x),T(g(x),g(y))) = 0 \text{ and } d(g(y),T(g(y),g(x))) = 0.$$

Remark 3.6. *This results also true if we replace the continuity of T by the condition (e) of Theorem 2.8.*

DATA AVAILABILITY

No data were used to support the study.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally in preparation of this paper. All authors read and approved the final manuscript.

ACKNOWLEDGMENTS

This research work was supported by the Lembaga Penelitian dan Pengabdian Masyarakat (LP2M) Institute for Islamic Studies *Màarif* NU (IAIMNU).
No: 07/0121/IAIMNU/LPM/X/2020

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435-1443.

[2] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. TMA. 65 (2006), 1379-1393.

[3] J.J. Nieto, R.R. Lopez, Contractive mapping theorems in partially ordered sets and application to ordinary differential equations, Order 22 (2005), 223-239.

[4] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008), 109-116.

[5] V. Lakshmikantham, L.B. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. TMA. 70 (2009) 4341-4349.

[6] W. Sintunavarat, Y. Chao, P. Kumam, Coupled fixed point theorems for contraction mapping induced by cone ball-metric in partially ordered spaces without mixed, Fixed Theory Appl. 2012 (2012), 128.

[7] J. Kim, S. Chandok, Coupled common fixed point theorems for nonlinear contractions mappings without the mixed monotone property in partially ordered spaces, Fixed Theory Appl. 2013 (2013), 307.

[8] A. Amini-Harandi, H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Anal. TMA. 72 (2010), 2238-2242.

[9] J. Harjani, K. Sadarangni, Fixed point theorems for weakly contraction mappings in partially ordered sets, Nonlinear Anal. TMA. 71 (2009), 3403-3410.

[10] F. Yan, Y. Su, et al. A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory Appl. 2012 (2012), 152.

[11] J. Harjani, K. Sadarangni, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. TMA. 72 (2010), 1188-1197.

[12] J.J. Nieto, An abstract monotone iterative technique, Nonlinear Anal. TMA. 28 (1997), 1923-1933.

[13] A. Cabada, J.J. Nieto, Fixed points and approximate solutions for nonlinear operator equations, J. Comput. Appl. Math. 113 (2000), 17-25.

[14] J.J. Nieto, R.L. Pouso, R. Rodriguez-Lopez, Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc. 135 (2007), 2505-2517.

[15] D. O'Regan, A. Petrusel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008), 1241-1252.

[16] L. Ciric, N. Cakic, M. Rajovic, J.S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 131294.

[17] I. Altun, H. Simsek, Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. 2010 (2010), 621469.

[18] N. Luong and N. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. TMA. 74 (2011), 983-992.

[19] V. Pragadeeswarar, M. Marudai, Best proximity points: approximation and optimization in partially ordered metric spaces, Optim Lett. 7 (2013), 1883–1892.

[20] P. Kumam, V. Pragadeeswarar, M. Marudai, K. Sitthithakerngkiet, Coupled best proximity points in ordered metric spaces, Fixed Theory Appl. 2014 (2014), 107.

[21] W. Sintunavarat, P. Kumam, Coupled best proximity point theorem in metric spaces, fixed Theory Appl. 2012 (2012), 93.

[22] W. Sintunavarat, P. Kumam, Y. Chao, Coupled best proximity point theorems for nonlinear contractions without mixed monotone property, Fixed Theory Appl. 2012 (2012), 170.

[23] J. Zhang, Y. Su and Q. Cheng, Best proximity point theorems for generalized contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2013 (2013), 83.

[24] A. Abkar, S. Ghods and A. Azizi, Coupled best proximity point theorems for proximally g-Meir-Keeler type mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2015 (2015), 107.

[25] A. Abkar, M. Gabeleh, Best proximity points for cyclic mappings in ordered metric spaces, J. Optim. Theory Appl. 150 (2011), 188-193.

[26] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006), 1001-1006.

[27] M. Jleli, E. Karapinar, B. Samet, A best proximity point result in modular spaces with the Fatou property, Abstr. Appl. Anal. 2013 (2013), Article ID 329451.

[28] M. Jleli, B. Samet, An optimization problem involving proximal quasi-contraction mappings, Fixed Point Theory Appl. 2014 (2014), 141.

[29] E. Karapinar, V. Pragadeeswarar, M. Marudai, Best proximity point for generalized proximal weak contractions in complete metric space, J. Appl. Math. 2014 (2014), Article ID 150941.

[30] W.A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (2003), 851-862.

[31] H.K. Nashine, P. Kumam, C. Vetro, Best proximity point theorems for rational proximal contractions, Fixed Point Theory Appl. 2013 (2013), 95.

[32] V. Pragadeeswarar, M. Marudai, Best proximity points for generalized proximal weak contractions satisfying rational expression on ordered metric spaces, Abstr. Appl. Anal. 2015 (2015), Article ID 361657.

[33] V. Dewangan, A. Banerjee, P. Kaushik, Tripled best proximity point theorem for mixed g-monotone mappings in partially ordered metric spaces, Adv. Fixed Point Theory, 7 (2017), 544-571.

[34] V.S. Raj, A best proximity point theorem for weakly contractive non-self mappings, Nonlinear Anal. TMA. 74 (2011), 4804-4808.