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EFFECT OF PREDATORS' BEHAVIOR ON PREY-PREDATOR INTERACTION

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Abstract. In this paper, we study the effect of the behavior of predators on prey-predator interaction. We assume that there is one prey population that suffers from the fear of predators which could force it to move in to the refuge; and three are two predator populations, one of them is aggressive in its attack and the other one using sit-and-wait procedure, which is less aggressive. All the model's equilibrium points have been found and their stability was established. The possibility of transcritical and Hopf bifurcation was also investigated and numerical simulations were given. The effect of prey refuge and fear also are detected. The cost of them is to allow the model to reach to double transcritical point. The effect of the competition between prey populations is to convert the model from the stable limit cycle to a spiral stable equilibrium point of afraid prey with predator. When it becomes large it converts the model to the stable trivial solution.

Keywords: predators' behavior; prey refuge; Holling type-II functional response; bifurcation.

2010 AMS Subject Classification: 92B05, 34C23.

1. Preliminaries

Theory predicts that organisms should modulate behavior to maximize their expected fitness [1]. Since death is the most definite negative effect on fitness, it is very important for individuals to avoid predation and therefore a strong selection for proper anti-predator response is to be

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expected. Behavioral adjustments to predator presence are very widespread responses used by organisms in many taxa [10]. The optimal behaviorally mediated anti-predator response is of course highly dependent on what kind of predator an organism is encountering. For example, [11] showed that damselfly larvae that swim to escape fish predators have a very low chance of survival, while the same escape response in the presence of invertebrate predators resulted in much higher chances of survival. Their study illustrates that a good anti-predator response to one type of predator can be maladaptive in the presence of another predator type. It is therefore vital for the prey to be able to identify and correctly assess the risk linked to a specific predator. Since many organisms encounter several different types of predators, plasticity in the behavioral response to predators is expected to be beneficial for prey populations [4, 13]. In order to study the effects of preys' behavior changes due to the fear of predators and the effect of predator populations, we developed a mathematical model with one prey population and two predator populations. It is assumed that the prey population fears the predator and each predator follows a different type of attack; one of them is very aggressive and the other one follows the sit-and-wait strategy which considered to be less aggressive in its nature [2, 3, 5].

2. Mathematical Mode Building and Analysis

To develop our model we assume that there is one prey population and two predator populations. It is assumed that the prey population suffers from fear of predator which affect its reproduction and cause it to move inside a refuge for sometimes which affects the availability of food and causes the rise of inter-species competition [7], therefore the reproduction term of the prey population takes the form

$$\frac{bN}{1+e_1P_1+e_2P_2} - dN - sN^2$$

Where the term

$$f(e;P_1,P_2) = \frac{1}{1+e_1P_1+e_2P_2}$$

represents the fear of the prey from the predator. It is also assumed that one predator is aggressive in nature and therefore its attack follows Holling type-II functional response, and the other one is less aggressive in its behavior and therefore it follows a modified Holling type-II functional response in its attack [8]. Taking these assumption into consideration, our model is represented by the following set of differential equations

$$
\begin{aligned}
\frac{dN}{dt} &= \frac{bN}{1+e_1P_1+e_2P_2} - dN - sN^2 - \frac{b_1N(1-m_1)P_1}{1+\alpha_1 N} - \frac{b_2N(1-m_2)P_2}{1+\alpha_2 N+\gamma P_2} \\
\frac{dP_1}{dt} &= -c_1P_1 + \frac{b_1\lambda_3 N(1-m_1)P_1}{1+\alpha_1 N} \qquad (1)\\
\frac{dP_2}{dt} &= -c_2P_2 + \frac{b_2\lambda_4 N(1-m_2)P_2}{1+\alpha_2 N+\gamma P_2}
\end{aligned}
$$

where N represents the prey which afraid and stay in refuges, P_1 and P_2 represent bold and aggressive predator respectively. The next table is demonstrated the meaning of each parameter:

2.1. Equilibrium Points.

The system (1) has the following equilibrium points:

(i) $E_0(0,0,0)$, the trivial equilibrium point.

(ii) $E_1(\frac{b-d}{s},0,0)$, representing the existence of prey which stay in refuge.

(iii) $E_2(\frac{c_1}{b_1\lambda_3(1-m_1)-\alpha_1 c_1},P_1^*,0)$, representing the existence of afraid pray and first predator. Where P_1^* is the root of equations $H_2X^2+H_1X+H_0=0$, where, $H_2 = e_1(\alpha_1c_1-b_1\lambda_3(1-m_1))^2$, $H_1 = b_1\lambda_3^2(1-m_1)(b_1(1-m_1)+e_1d)-2\alpha_1b_1(1-m_1)+c_1e_1\lambda_3(\alpha_1d-s)+\alpha_1^2c_1^2$, $H_0=\lambda_3(b_1(1-m_1)(d-b)\lambda_3+c_1(\alpha_1(b-d)+s))$

(iv) $E_3(N^*,0,P_2^*)$, representing the existence of afraid pray and second predator which is faster than first predator. Where N^* represents the solution of equation $N^*=AX^3+BX^2+CX+D=0$, $A=\lambda_4\gamma e_2 s(\alpha_2c_2-b_2\lambda_4(1-m_2))$, $B=-(b_2(1-m_2)+\gamma d)(1-m_2)e_2b_2\lambda_4^2+c_2\lambda_4([2\alpha_2(1-m_2)+\gamma(\alpha_2d+s)]e_2-s\gamma^2)-\alpha_2^2c_2^2e_2$,

$$C = c_2\left[\lambda_4((b-d)\gamma^2-\gamma(b_2(1-m_2)-e_2d)+2b_2e_2(1-m_2))-2\alpha_2c_2(e_2-\frac{1}{2}\gamma)\right]$$

, $D=c_2^2(\gamma-e_2)$. Consequently we can choose all factors to be positive or negative. We will consider the equation $N^*=AX_2^{*3}+BX_2^{*2}+CX_2^*+D=0$. Consequently by using Descartes's Sign Rule it can be chosen $D>0$ with $B<0$ and $C>0$ or $D<0$ with $B>0$ and $C>0$ to get positive solution and, it can be taken $D>0$, with any sign of b and c except $B>0$ and $C<0$ to get a unique positive solution.

Parameters	The meaning
b	The birth rate of prey.
d	The death rate of prey.
s	The competition rate between prey.
b_1, α_1	(units; 1/time), (units; 1/prey) respectively are describe the effect of capture rate and handling time on the feeding rate of the bold predator.
b_2, α_2	(units; 1/time), (units; 1/prey) respectively are describe the effect of capture rate and handling time on the feeding rate of the aggressive predator.
γ	measures the effect of anti-bold predator behaviour of prey.
c_1, c_2	The death rate of bold predators and aggressive predators respectively.
e_1, e_2	fear parameter of prey from bold predator and aggressive predator respectively.
m_1, m_2	refuge parameter of prey according of fear from bold predator and aggressive predator respectively.
$b_1\lambda_3$	The increment rate of the bold predator.
$b_2\lambda_4$	The increment rate of the aggressive predator.

TABLE 1. Table of meaning of parameters

(v) $E_6(N^*, P_1^*, P_2^*)$, representing the coexistence of all populations, where, $N^* = \dfrac{c_1}{b_1\lambda_3(1-m_1)-\alpha_1 c_1}$, $P_1^* = \dfrac{-N^*}{c_1} * f(X)$, where $f(X)$ is the root of equation $M_2X^2 + M_1X + M_0 = 0$, such that,

$$M_2 = e_1c_1c_2\lambda_4\gamma^2(\alpha_1c_1 - b_1\lambda_3(1-m_1)),$$

$$M_1 \;=\; -\gamma\left[-(c_2(\alpha_1(\gamma-e_2)+\alpha_2e_2-b_2\lambda_4e_2(1-m_2))\,\alpha_1\lambda_4c_1^3+\lambda_3(\alpha_1e_1(\alpha_1-\alpha_2)c_2^2+\left(2b_1c_1^2(1-m_1)((\gamma-e_2)\alpha_1+\tfrac{1}{2}\alpha_2e_2)\right]\;+\right.$$

$$-\gamma\left[\left(\alpha_1(((1-m_1)+\gamma d)-s\gamma e_1\lambda_4c_2-\lambda_4^2b_1b_2e_1(1-m_1)(1-m_2)\right)c_1^2+\lambda_3^2\left(2e_1c_2(\alpha_1-\tfrac{1}{2}\alpha_2)+((1-m_1)(\gamma-e_2)b_1)\right]\right.$$

$$-\lambda_3^2\gamma[e_1(b_2(1-m_2)+\gamma d)\lambda_4]\,(m_1-1)b_1c_1c_2-\gamma e_1\lambda_3^3b_1^2c_2^2(1-m_2)^2$$

$$
\begin{aligned}
M_0 \;=\;& \lambda_3\left[\alpha_1(\alpha_2-\alpha_1)((\alpha_2-\alpha_1)+\alpha_1\gamma)c_2^2+e_2\left(2(\alpha_2-\alpha_1)(\gamma(\tfrac{1}{2}s-\tfrac{1}{2}\alpha_1 d)-\alpha_1 b_2(1-m_2)\right)+\alpha_1\left(\gamma((b-d)\alpha_1+s)\right]\lambda_4 c_2\right. - \\
&(\lambda_3\alpha_1 b_2(1-m_2)\gamma)\lambda_4 c_2 - b\lambda_3 e_2\lambda_4^2 c_1^3\left((\alpha_1 d-s)\gamma+b_2\alpha_1(1-m_2)\right)(m_2-1)-2\lambda-3^2(1-m_1)c_2^2\left[(\tfrac{3}{2}(\alpha_1-\alpha_2)(\alpha_1-\tfrac{1}{3}\alpha_2)e_2\right] - \\
&2\lambda-3^2(1-m_1)c_2^2\left[-\tfrac{3}{2}\alpha_1(\alpha_1\alpha-\tfrac{2}{3}\alpha_2)\gamma\right]+e_2\lambda_3\left((\alpha_1 d-\tfrac{1}{2}\alpha_2 d-\tfrac{1}{2}s)\gamma+2b_2(\alpha_1-\tfrac{1}{2}\alpha_2)(1-m_2)\right)+\lambda_3\lambda_4 c_2\gamma\left[(b-d)\alpha_1+\tfrac{1}{2}s\right] + \\
&\lambda_3\lambda_4 c_2\gamma[-b_2\alpha_1(1-m_2)]+\tfrac{1}{2}\lambda_3 b_1 b_2 e_2 c_1^2\lambda_4^2(\gamma d+(1-m_2)b_2)(1-m_2)+b_1c_1c_2\lambda_3^3(1-m_1)^2\left[(3\alpha_1-2\alpha_2)-3\gamma(\alpha_1-\tfrac{1}{3}\alpha_2)c_2\right] + \\
&b_1c_1c_2\lambda_3^3(1-m_1)^2\left[((\gamma d+2b_2(1-m_2))e_2+(b-d)\gamma-b_2(1-m_2)\gamma)\lambda_4+\lambda_3^3 b_1^3 c_2^2(1-m_1)^3(\gamma-e_2)\right].
\end{aligned}
$$

Notice that it is need to take the negative solution of $f(x)$ to get P_1^* is positive. So $b_1\lambda_3(1-m_1)-\alpha_1 c_1$ is positive, this implies M_2 is negative. To get unique negative solution this need to take M_0 is positive. In addition it can be chosen M_0 is negative with M_1 is negative and $M_1^2-4M_2M_0>0$ to get two negative solution.

2.2. Local stability of the equilibrium points.

Theorem 1. *The stability of the system (1)is given by:*

(i) $E_0(0,0,0)$ *is locally asymptotically stable if* $b-d<0$.

(ii) $E_1(\dfrac{b-d}{s},0,0)$ *is locally asymptotically stable if* $-c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)}<0$ *and* $-c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}<0$.

(iii) $E_2(\dfrac{c_1}{b_1\lambda_3(1-m_1)-\alpha_1 c_1},P_1^*,0)$ *is locally asymptotically stable if* $-c_2+\frac{b_2\lambda_4 N^*(1-m_2)}{(1+\alpha_2 N^*)}<0$ *and* $\alpha_1^2 sN^{*2}+\alpha_1(2sN^*-b_1P_1^*(1-m_1)+s>0$.

(iv) $E_3(N^*,0,P_2^*)$ *is locally asymptotically stable if* $-c_1+\frac{b_1\lambda_3 N^*(1-m_1)}{(1+\alpha_1 N^*)}<0$ *and* $\left[\gamma^2 sP_2^{*2}+(s\gamma(2\alpha_2N^*+2)+b_2(\gamma\lambda_4-\alpha_2)(1-m_2))P_2^*+s(1+\alpha_1N^*)^2\right]>0$.

(v) $E_4(N^*,P_1^*,P_2^*)$*, is locally asymptotically stable by using Qualitative Matrix Stability method.*

Proof

From equations (1), the Jacobian matrix of the system is given by:

$$
J(E_i)=\begin{bmatrix}
S_1^* & S_2^* & S_3^* \\
\frac{b_1\lambda_3 P_1(1-m_1)}{(1+\alpha_1 N)^2} & -c_1+\frac{b_1\lambda_3 N(1-m_1)}{(1+\alpha_1 N)^2} & 0 \\
\frac{b_2\lambda_4 P_2(1-m_2)(1+\gamma P_2)}{(1+\alpha_2 N+\gamma P_2)^2} & 0 & -c_2+\frac{b_2\lambda_4 N(1-m_2)(1+\alpha_2 N)}{(1+\alpha_2 N+\gamma P_2)^2}
\end{bmatrix}.
$$

Where, $S_1^* = \frac{b}{1+e_1P_1+e_2P_2} - d - sN - \frac{b_1P_1(1-m_1)}{1+\alpha_1 N} - \frac{b_2P_2(1-m_2)}{1+\alpha_2 N+\gamma P_2} + N(-s + \frac{b_1\alpha_1P_1(1-m_1)}{(1+\alpha_1 N)^2} +$

$\frac{b_2\alpha_2 P_2(1-m_2)}{(1+\alpha_2 N+\gamma P_2)^2}$), $S_2^* = -N(\frac{b_1(1-m_1)}{(1+\alpha_1 N)} + \frac{be_1}{(1+e_1P_1+e_2P_2)^2})$, $S_3^* = -N(\frac{b_2(1-m_2)(1+\alpha_2 N)}{(1+\alpha_2 N+\gamma P_2)^2} + \frac{be_2}{(1+e_1P_1+e_2P_2)^2})$.

By evaluating the Jacobian matrix at each equilibrium points, we get:

(i) The Jacobian matrix at $E_0(0,0,0)$ is given by:

$$J(E_0) = \begin{bmatrix} b-d & 0 & 0 \\ 0 & -c_1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix},$$

clearly the eigenvalues of the matrix are $b-d$, $-c_1$ and $-c_2$ and it is obvious that $E_0(0,0,0)$ is asymptotically stable if $b-d<0$.

(ii) The Jacobian matrix at $E_1\left(\frac{b-d}{s},0,0\right)$ is given by:

$$J(E_1) = \begin{bmatrix} d-b & \frac{b-d}{s}\left(\frac{be_1\alpha_1(d-b)-s(be_1+b_1(1-m_1))}{s+\alpha_1(b-d)}\right) & \frac{b-d}{s}\left(\frac{be_2\alpha_2(d-b)-s(be_2+b_2(1-m_2))}{s+\alpha_2(b-d)}\right) \\ 0 & -c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} & 0 \\ 0 & 0 & -c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)} \end{bmatrix},$$

clearly this is upper triangular matrix, so the eigenvalues of the matrix are $d-b$, $-c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)}$ and $-c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}$ which implies that $E_1(\frac{b-d}{s},0,0)$ is asymptotically stable if $b-d>0$, $-c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)}<0$ and $-c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}<0$.

(iii) The Jacobian matrix at $E_2(N^*,P_1^*,0)$ is given by:

$$J(E_2) = \begin{bmatrix} N^*(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}) & -N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2}) & -N^*(\frac{b_2(1-m_2)}{(1+\alpha_2N^*)}+\frac{be_2}{(1+e_1P_1^*)^2}) \\ \frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2} & 0 & 0 \\ 0 & 0 & -c_2+\frac{b_2\lambda_4N^*(1-m_2)}{(1+\alpha_2N^*)} \end{bmatrix}.$$

It is obvious that $-c_2+\frac{b_2\lambda_4N^*(1-m_2)}{(1+\alpha_2N^*)}$ is one of the eigenvalue of the Jacobian matrix. The other eigenvalues is gotten from reduced matrix, which is:

$$\begin{bmatrix} N^*(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}) & -N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2}) \\ \frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2} & 0 \end{bmatrix}.$$

The characteristic polynomial is:

$$P(\lambda) = \lambda^2 + \frac{(\alpha_1^2 sN^{*2}+\alpha_1(2sN^*-b_1P_1^*(1-m_1)+s)}{(1+\alpha_1N^*)^3}N^*\lambda + \frac{b_1(1-m_1)(1+e_1P_1^*)^2+e_1b(1+\alpha_1N^*)}{(1+\alpha_1N^*)^3(1+e_1P_1^*)^2}.$$

Clearly, this point is locally asymptotically stable if

$(\alpha_1^2 sN^{*2} + \alpha_1(2sN^* - b_1P_1^*(1-m_1)+s) > 0.$

(iv) The Jacobian matrix at $E_3(N^*,0,P_2^*)$ is given by:

$$J(E_3) = \begin{bmatrix} N^*(-s+\frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}) & -N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_2P_2^*)^2}) & -N^*(\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*)}+\frac{be_2}{(1+e_2P_2^*)^2}) \\ 0 & -c_1+\frac{b_1\lambda_3N^*(1-m_1)}{(1+\alpha_1N^*)} & 0 \\ \frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2} & 0 & \frac{-\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2} \end{bmatrix}.$$

It is clearly that $-c_1+\frac{b_1\lambda_3N^*(1-m_1)}{(1+\alpha_1N^*)}$ is one of the eigenvalue of the Jacobian matrix. The other eigenvalues is gotten from reduced matrix, which is:

$$\begin{bmatrix} N^*(-s+\frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}) & -N^*(\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*)}+\frac{be_2}{(1+e_2P_2^*)^2}) \\ \frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2} & \frac{-\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2} \end{bmatrix}.$$

The characteristic polynomial is:

$$(2)\quad P(\lambda) = \lambda^2 + \frac{\gamma^2 sP_2^{*2} + (s\gamma(2\alpha_2N^*+2) + b_2(\gamma\lambda_4-\alpha_2)(1-m_2))P_2^* + s(1+\alpha_1N^*)^2}{(1+\alpha_2N^*+\gamma P_2^*)^2}N^*\lambda + \frac{b_2\lambda_4(1-m_2)N^*P_2^*\left(s\gamma\alpha_2(1+e_2P_2^*)N^* + s(1+e_2P_2^*)N^*\left[s\gamma(1+e_2P_2^*)^2+be_2\alpha_2\right]\right)}{(1+e_2P_2^*)^2(1+\alpha_2N^*+\gamma P_2^*)^3} + \frac{b_2\lambda_4(1-m_2)N^*P_2^*(2be_2\gamma P_2^* + be_2^2(1-m_2)P_2^* + 2b_2e_2(1-m_2)P_2^* + be_2 + b_2(1-m_2))}{(1+e_2P_2^*)^2(1+\alpha_2N^*+\gamma P_2^*)^3}.$$

Clearly, this point is locally asymptotically stable if

$\left[\gamma^2 sP_2^{*2} + (s\gamma(2\alpha_2N^*+2) + b_2(\gamma\lambda_4-\alpha_2)(1-m_2))P_2^* + s(1+\alpha_1N^*)^2\right] > 0.$

(v) The Jacobian matrix at $E_4(N^*,P_1^*,P_2^*)$ is given by:

$$J(E_4) = \begin{bmatrix} N^*(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}+\frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}) & -N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*+e_2P_2^*)^2}) & B^* \\ \frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2} & 0 & 0 \\ \frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2} & 0 & \frac{-\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2} \end{bmatrix},$$

where $B^* = -N^*(\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*)}+\frac{be_2}{(1+e_1P_1^*+e_2P_2^*)^2})$ To prove the stability of the coexistence equilibrium point we will use the Qualitative matrix stability method.

$$Q_3 = sign(J_4) = \begin{bmatrix} - & - & - \\ + & 0 & 0 \\ + & 0 & - \end{bmatrix}.$$

Then:

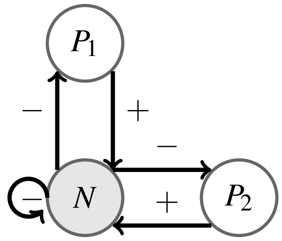


FIGURE 1. The signed digraph with the color test.

(i) From Q_3 we have $q_{11} = -$, $q_{22} = 0$, $q_{33} = -$. Thus condition (1) and (2) of theorem are satisfied.

(ii) Since, $q_{12} = -q_{21} =$, $q_{13} = -q_{31}$, $q_{23} = q_{32} = 0$ which means every pair of interacting nodes have opposite sign and this implies that condition (3) of theorem is satisfied.

(iii) From it is clear that condition (4) of theorem is satisfied.

(iv) $\det(J_3) \neq 0$ since the solution of $Jx = 0$ is only the trivial solution.

Thus from figure there are two predation links between N and P_1 and between N and P_2. since these links are connected by node P_1 and P_2 , then the entire signed digraph forms a predation community.

Color test

We color each node with a negative feedback loop with gray and the other nodes by white as in figure .

Applying the color test to our model we find that:

* The node N is gray and the other nodes P_1 and P_2 are white.
* However, condition (ii) is not satisfied because there is no predation link between white nodes.
* The gray node N is connected by a predation link to both white nodes P_1 and P_2. In conclusion, the signed matrix Q_3 fails the color test. Hence the Jacobian matrix J_4 corresponding to the equilibrium point $E_4(N^*, P_1^*, P_2^*)$,where $Q_3 = sign(J_4)$ is qualitative stable. Then the equilibrium is locally asymptotically stable.

2.3. Global stability of the equilibrium points. The global stability of stability points will be analysed by transforming the system of equations (1) into a linear system and then choosing a suitable Lyapunov function to analyse each equilibrium point or by using Dulac's criteria, following the same strategy as [**?**].

Theorem 2. *The global stability of the system of equations (1) is given by:* $E_1 = (N^*, 0, 0) = (\frac{b-d}{s}, 0, 0)$ *is global asymptotically stable.*

Proof

By letting $N = N^* + n$, $P_1 = P_1^* + p_1$ and $P_2 = P_2^* + p_2$, where n, p_1 and p_2 are small perturbations about N^*, P_1^* and P_2^* respectively, the system of equations (1) is turned into a linear system which is of the form $\dot{n_i} = J(E_i)n_i$, where $J(E_i)$ is the Jacobian matrix of equations (1). Thus, the linear system of equations (1) is,

$$\frac{dn}{dt} = \left[(-s + \frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2} + \frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2})\right]N^*n - \left[\frac{b_1(1-m_1)}{(1+\alpha_1N^*)} + \frac{be_1}{(1+e_1P_1^*+e_2P_2^*)^2}\right]N^*p_1$$
$$- \left[\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*} + \frac{be_2}{(1+e_1P_1^*+e_2P_2^*)^2})\right]N^*p_2$$

$$\frac{dp_1}{dt} = \frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}n \tag{3}$$

$$\frac{dp_2}{dt} = \left[\frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2}\right]n - \left[\frac{\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}\right]p_2$$

Global stability of $E_1(N_1^*, 0, 0) = (\frac{b-d}{s}, 0, 0)$. We define a Lyapunov function as $V(n, p_1, p_2) = \frac{n^2}{2N^*} + \frac{p_1^2}{2} + \frac{p^2}{2}$. It is obvious that $V(n, p_1, p_2)$ is a positive definite function. Differentiating V with respect to time t we get, $\dot{V}(n, p_1, p_2) = \frac{n\dot{n}}{N^*} + p_1\dot{p_1} + p_2\dot{p_2}$. By substituting for $\dot{n}$, $\dot{p_1}$ and $\dot{p_2}$ in equations of system (3) gives,
$\dot{V}(n, p_1, p_2) = -(\frac{r}{\kappa}n_2 + a_2p)n_2$, which is negative semi-definite. Therefore, $E_1(N^*, 0, 0)$ is globally asymptotically stable.

2.3.1. *Hopf bifurcation.*

Theorem 3. *The system (1) undergoes a Hopf bifurcation at the positive equilibrium:*

(i) $E_2(N^*, P_1^*, 0)$ *when* $s = s_0 = \frac{b_1(1-m_1)\alpha_1P_1^*}{(1+\alpha_1N^*)^2}$

(ii) $E_3(N^*,0,P_2^*)$ *when* $s = s_0 = \dfrac{b_2(1-m_2)(\alpha_2-\gamma\lambda_4)P_2^*}{(1+\alpha_2 N^*+\gamma P_2^*)^2}$

Proof

(i) The eigenvalues of the linearized system around the equilibrium point E_2 are:

$\mu_{1,2} = \alpha(s) \pm i\beta(s)$

where:

$$\begin{aligned} \alpha(s) &= \frac{1}{2} trac(J) \\ \beta(s) &= \sqrt{det(J)-(\alpha(s))^2} \end{aligned}$$

where J is the Jacobian of the linearized system at the equilibrium point E_2. From 2.2, Trase $J(E_2) = N^*\left(-s+\dfrac{b_1(1-m_1)\alpha_1 P_1^*}{(1+\alpha_1 N^*)^2}\right)$, and determinant $J(E_2) = \frac{b_1\lambda_3(1-m_1)N^*P_1^*\left(b_1(1-m_1)(1+e_1P_1^*)^2+be_1(1+\alpha_1N^*)\right)}{(1+\alpha_1N^*)^3(1+e_1P_1^*)^2}$, Now at s_0, $\alpha(s_0) = 0$,

$$\beta(s_0) = \frac{b_1\lambda_3(1-m_1)N^*P_1^*\left(b_1(1-m_1)(1+e_1P_1^*)^2+be_1(1+\alpha_1N^*)\right)}{(1+\alpha_1N^*)^3(1+e_1P_1^*)^2} \neq 0$$

and $\dfrac{d\alpha}{ds}|_{s=s_0} = \dfrac{-N^*}{2} \neq 0$.
Therefore from Hopf Theorem the proof is concluded.

(ii) The eigenvalues of the linearized system around the equilibrium point E_3 are:

$\mu_{1,2} = \phi(s) \pm i\psi(s)$

where:

$$\begin{aligned} \phi(s) &= \frac{1}{2} trac(J) \\ \psi(s) &= \sqrt{det(J)-(\phi(s))^2} \end{aligned}$$

where J is the Jacobian of the linearized system at the equilibrium point E_3. From 2.2, Trase $J(E_3) = N^*\left(-s+\frac{b_2(1-m_2)(\alpha_2-\gamma\lambda_4)P_2^*}{(1+\alpha_2N^*+\gamma P_2^*)^2}\right)$, and determinant $J(E_3) = \frac{b_2(1-m_2)N^*\left[e_2^2P_2^*\left(\gamma^2 sN^*P_2^*+\alpha_2\gamma sN^*+\gamma sN^*+b_2(1-m_2)\right)+R^*\right]}{(1+\alpha_2N^*+\gamma P_2^*)^3(1+e_2P_2^*)^2}$, where R^* is equal $\left(e_2\gamma P_2^{*2}(2sN^*+b)+P_2^*\left(2\alpha_2\gamma sN^{*2}+\gamma N^*(\alpha_2 b+2s)+2b\gamma+2b_2(1-m_2)\right)+b(1+\alpha_2N^*)\right) \quad + \gamma^2 sN^*P_2^*+\alpha_2\gamma sN^{*2}+\gamma sN^*+b_2(1-m_2)$, Now at s_0,
$\phi(s)(s_0) = 0$, $\psi(s_0) > 0$ if $(\alpha_2-\gamma\lambda_4) > 0$ and $\dfrac{d\phi}{ds}|_{s=s_0} = \dfrac{-N^*}{2} \neq 0$.

Therefore from Hopf Theorem the proof is concluded.

2.3.2. *Transcritical bifurcation.*

Theorem 4. *The system (1) undergoes a Transcritical bifurcation at the positive equilibrium*

(i) $E_1(N^*,0,0)=(\frac{b-d}{s},0,0)$ *when (a)* $m_1=\dfrac{b_1\lambda_3(b-d)-c_1(\alpha_1(b-a)+s)}{b_1\lambda_3(b-d)}$ *or*
(b) $m_2=\dfrac{b_2\lambda_4(b-d)-c_2(\alpha_2(b-a)+s)}{b_2\lambda_4(b-d)}$

(ii) $E_2(N^*,P_1^*,0)$ *when* $m_2=\dfrac{N^*(b_1\lambda_3-\alpha_1c_1)-c_1}{b_1\lambda_3N^*}$

(iii) $E_3(N^*,0,P_2^*)$ *when* $m_1=\dfrac{N^*(b_2\lambda_4-\alpha_2c_2)-c_2}{b_2\lambda_4N^*}$

Proof

(i)(a) The eigenvalues of the linearized system around the equilibrium point E_1 are:

$$\mathscr{J}(\mathscr{E}_1)=\begin{bmatrix} d-b & \frac{b-d}{s}\left(\frac{be_1\alpha_1(d-b)-s(be_1+b_1(1-m_1))}{s+\alpha_1(b-d)}\right) & \frac{b-d}{s}\left(\frac{be_2\alpha_2(d-b)-s(be_2+b_2(1-m_2))}{s+\alpha_2(b-d)}\right)\\ 0 & -c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} & 0\\ 0 & 0 & -c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}\end{bmatrix},$$

$$\mathscr{J}(\mathscr{E}_1,m_{1_0})=\begin{bmatrix} d-b & \frac{b-d}{s}\left(\frac{be_1\alpha_1(d-b)-s(be_1+b_1(1-m_1))}{s+\alpha_1(b-d)}\right) & \frac{b-d}{s}\left(\frac{be_2\alpha_2(d-b)-s(be_2+b_2(1-m_2))}{s+\alpha_2(b-d)}\right)\\ 0 & 0 & 0\\ 0 & 0 & -c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)}\end{bmatrix}. \tag{4}$$

Let us define $v=(v_1,v_2,v_3)^T$ and $w=(w_1,w_2,w_3)^T$ are the right and left eigenvectors of $\lambda_2=0$. From (4) and $\mathscr{J}(\mathscr{E}_1,m_{1_0})v=0$ as well as $\mathscr{J}^T(\mathscr{E}_1,m_{1_0})w=0$, then,

$$\left((d-b)v_1+H_3^{**},\frac{(\lambda_4b_2(1-m_2)(b-d)-(\alpha_2(b-d)+s)c_2)v_3}{s+\alpha_2(b-d)}v_3,0\right)=(0,0,0)^T$$

where $H_3^{**}=\frac{(be_1(d-b)\lambda_3-c_1s)v_2}{s\lambda_3}+\frac{(\alpha_2b^2e_2-e_2(\alpha_2d-s)b+b_2(1-m_2)s)(b-d)v3}{s(\alpha_2b-\alpha_2d+s)}$ and

$$\left((d-b)w_1,\frac{be_1\lambda_3((d-b)-c_1s)w_1}{s\lambda_3},H_2**\right)^T=(0,0,0)^T$$

where $H_2**=\frac{(\alpha_2b^2e_2-e_2(\alpha_2d-s)b+b_2(1-m_2)s)(d-b)w_1}{s(\alpha_2(b-d)+s)}+\frac{(\lambda_4b_2(1-m_2)(b-d)-(\alpha_2(b-d)+s)c_2)w_3}{s+\alpha_2(b-d)}w_3$, So the left eigenvector is $(0,w_2,0)^T$ and the right eigenvector is

$\left(\frac{(b^2e_1\lambda_3-bde_1\lambda_3+c_1s)v_2}{s\lambda_3(d-b)},v_2,0\right)^T$. Here, w_2 and v_2 are any non-zero real numbers. Now, system (1) can be rewritten as in the following vector form:

$$\dot{X}=f(X),$$

where $X=(N,P_1,P_2)^T$ and

$$f(X)=\begin{pmatrix} \dfrac{bN}{1+e_1P_1+e_2P_2}-dN-sN^2-\dfrac{b_1N(1-m_1)P_1}{1+\alpha_1N}-\dfrac{b_2N(1-m_2)P_2}{1+\alpha_2N+\gamma P_2} \\ -c_1P_1+\dfrac{b_1\lambda_3N(1-m_1)P_1}{1+\alpha_1N} \\ -c_2P_2+\dfrac{b_2\lambda_4N(1-m_2)P_2}{1+\alpha_2N+\gamma P_2} \end{pmatrix}.$$

Taking derivative on $f(X)$ with respect to m_1, we get

$$f_{m_1}(X)=\begin{pmatrix} \frac{b_1NP_1}{1+\alpha_1N} \\ -\frac{b_1\lambda_3NP_1}{1+\alpha_1N} \\ 0 \end{pmatrix}, \text{ then } f_{\mathscr{E}_1,m_{1_0}}(X)=\begin{pmatrix} 0\\0\\0 \end{pmatrix}.$$

Hence, $w^Tf_{\mathscr{E}_1,m_{1_0}}(X)=0$.

Next, taking derivative on $f_{m_1}(X)$ with respect to $X=(N,P_1,P_2)^T$, then,

$$Df_{\mathscr{E}_1,m_{1_0}}(X)=\begin{pmatrix} 0 & \dfrac{b_1N^*}{1+\alpha_1N^*} & 0 \\ 0 & -\dfrac{b_1\lambda_3N^*}{1+\alpha_1N^*} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have, $w^T\left(Df_{\mathscr{E}_1,m_{1_0}}(X)\cdot v\right)=\frac{b_1\lambda_3N^*v_1w_2}{1+\alpha_1N^*}\neq 0$ Furthermore, we find $\mathscr{D}^2f(X)=D(J(E_i))=D(D(f(X)))$ and then subistitute the value of m_{1_0} and E_1, so we end by

$$w^T[D^2f_{\mathscr{E}_1,m_{1_0}}(X)(v,v)] \;=\; \frac{2c_1(\alpha_1(b-d)+s)v_1v_2w_2}{(b-d)(1+\alpha_1N^*)}\neq 0.$$

where, (v,v) is a Kronecker product of $(v_1,v_2,v_3)^T$. Therefore, according to the Sotomayor's theorem for local bifurcation [9], system (1) has a transcritical bifurcation at steady state $\mathscr{E}_1$ when the parameter m_1 passes through the bifurcation value m_{1_0}.

(i)(b) The eigenvalues of the linearized system around the equilibrium point E_1 are:

$$\mathscr{J}(\mathscr{E}_1) = \begin{bmatrix} d-b & \frac{b-d}{s}\left(\frac{be_1\alpha_1(d-b)-s(be_1+b_1(1-m_1))}{s+\alpha_1(b-d)}\right) & \frac{b-d}{s}\left(\frac{be_2\alpha_2(d-b)-s(be_2+b_2(1-m_2))}{s+\alpha_2(b-d)}\right) \\ 0 & -c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} & 0 \\ 0 & 0 & -c_2+\frac{b_2\lambda_4(b-d)(1-m_2)}{s+\alpha_2(b-d)} \end{bmatrix},$$

$$\mathscr{J}(\mathscr{E}_1, m_{2_0}) = \begin{bmatrix} d-b & \frac{b-d}{s}\left(\frac{be_1\alpha_1(d-b)-s(be_1+b_1(1-m_1))}{s+\alpha_1(b-d)}\right) & \frac{b-d}{s}\left(\frac{be_2\alpha_2(d-b)-s(be_2+b_2(1-m_2))}{s+\alpha_2(b-d)}\right) \\ 0 & -c_1+\frac{b_1\lambda_3(b-d)(1-m_1)}{s+\alpha_1(b-d)} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Let us define $v=(v_1,v_2,v_3)^T$ and $w=(w_1,w_2,w_3)^T$ are the right and left eigenvectors of $\lambda_3=0$. From (5) and $\mathscr{J}(\mathscr{E}_1,m_{2_0})v=0$ as well as $\mathscr{J}^T(\mathscr{E}_1,m_{1_0})w=0$, then,

$$\left((d-b)v_1+H_4^{**}, \frac{(\lambda_3 b_1(1-m_1)(b-d)-(\alpha_1(b-d)+s)c_1)v_2}{s+\alpha_1(b-d)}v_2, 0\right) = (0,0,0)^T$$

where $H_4^{**} = \frac{(be_2(d-b)\lambda_4-c_2s)v_3}{s\lambda_4} + \frac{(\alpha_1 b^2 e_1 - e_1(\alpha_1 d-s)b+b_1(1-m_1)s)(b-d)v2}{s(\alpha_1 b-\alpha_1 d+s)}$ and

$$\left((d-b)w_1, \frac{be_2\lambda_4((d-b)-c_2s)w_1}{s\lambda_4}, H_5{**}\right)^T = (0,0,0)^T$$

where $H_5{**} = \frac{(\alpha_1 b^2 e_1 - e_1(\alpha_1 d-s)b+b_1(1-m_1)s)(d-b)w_1}{s(\alpha_1(b-d)+s)} + \frac{(\lambda_3 b_1(1-m_1)(b-d)-(\alpha_1(b-d)+s)c_1)w_2}{s+\alpha_1(b-d)}w_2$, So the left eigenvector is $(0,w_2,0)^T$ and the right eigenvector is $\left(\frac{(b^2e_2\lambda_4-bde_2\lambda_4+c_2s)v_3}{s\lambda_4(d-b)}, 0, v_3\right)^T$. Here, w_2 and v_3 are any non-zero real numbers. Now, system (1) can be rewritten as in the following vector form:

$$\dot{X} = f(X),$$

where $X=(N,P_1,P_2)^T$ and

$$f(X) = \begin{pmatrix} \frac{bN}{1+e_1P_1+e_2P_2} - dN - sN^2 - \frac{b_1N(1-m_1)P_1}{1+\alpha_1 N} - \frac{b_2N(1-m_2)P_2}{1+\alpha_2 N+\gamma P_2} \\ -c_1P_1 + \frac{b_1\lambda_3 N(1-m_1)P_1}{1+\alpha_1 N} \\ -c_2P_2 + \frac{b_2\lambda_4 N(1-m_2)P_2}{1+\alpha_2 N+\gamma P_2} \end{pmatrix}.$$

Taking derivative on $f(X)$ with respect to m_2, then,

$$Df_{\mathscr{E}_1,m_{2_0}}(X) = \begin{pmatrix} 0 & 0 & \frac{b_2N^*}{(1+\alpha_2 N^*)} \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{b_2\lambda_4 N^*}{(1+\alpha_2 N)} \end{pmatrix}.$$

We have, $w^T\left(Df_{\mathscr{E}_1,m_{2_0}}(X)\cdot v\right)=-\frac{b_2\lambda_4N^*v_3w_3}{(1+\alpha_1N^*)}\neq 0.$

Furthermore, we find $\mathscr{D}^2f(X)=D(J(E_i))=D(D(f(X)))$ and then subistitute the value of m_{2_0} and E_1, so we end by

$$w^T[D^2f_{\mathscr{E}_1,m_{2_0}}(X)(v,v)] \;=\; \frac{2c_2(v_1-\gamma N^*v_3)(\alpha_2(b-d)+s)v_3w_3}{(b-d)(1+\alpha_2N^*)^2}\neq 0.$$

where, (v,v) is a Kronecker product of $(v_1,v_2,v_3)^T$. Therefore, according to the Sotomayor's theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state $\mathscr{E}_1$ when the parameter m_2 passes through the bifurcation value m_{2_0}.

(ii) The eigenvalues of the linearized system around the equilibrium point E_2 are:

$$\mathscr{J}(\mathscr{E}_2)=\begin{bmatrix} N^*\left(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}\right) & -N^*\left(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2}\right) & -N^*\left(\frac{b_2(1-m_2)}{(1+\alpha_2N^*)}+\frac{be_2}{(1+e_1P_1^*)^2}\right)\\ \frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2} & 0 & 0\\ 0 & 0 & -c_2+\frac{b_2\lambda_4N^*(1-m_2)}{(1+\alpha_2N^*)}\end{bmatrix}.$$

$$\mathscr{J}(\mathscr{E}_2,m_{2_0})=\begin{bmatrix} N^*\left(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}\right) & -N^*\left(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2}\right) & -N^*\left(\frac{c_2}{\lambda_4N^*}+\frac{be_2}{(1+e_1P_1^*)^2}\right)\\ \frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{6}$$

Let us define $v=(v_1,v_2,v_3)^T$ and $w=(w_1,w_2,w_3)^T$ are the right and left eigenvectors of $\lambda_3=0$. From (6) and $\mathscr{J}(\mathscr{E}_2,m_{2_0})v=0$ as well as $\mathscr{J}^T(\mathscr{E}_2,m_{2_0})w=0$, then,

$$\left(N^*(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2})v_1-N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2})v_2-N^*(\frac{c_2}{\lambda_4N^*}+\frac{be_2}{(1+e_1P_1^*)^2})v_3,\frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}v_1,0\right)=(0,0,0)^T$$

and

$$\left(N^*(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2})w_1+\frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}w_2,-N^*\left(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2}\right)w_1,-N^*\left(\frac{c_2}{\lambda_4N^*}+\frac{be_2}{(1+e_1P_1^*)^2}\right)w_1\right)^T=0$$

So the left eigenvector is $(0,0,w_3)^T$ and the right eigenvector is $\left(0,v_2,-\frac{\lambda_4N^*\left[(1+e_1P_1^*)^2(1-m_1)b_1+e_1b(1+\alpha_1N^*)\right]v_2}{(1+\alpha_1N^*)(c_2(1+e_1P_1^*)^2+\lambda_4e_2bN^*)}\right)^T$. Here, w_3 and v_2 are any non-zero real numbers. Now, system (1) can be rewritten as in the following vector form:

$$\dot{X}=f(X),$$

where $X=(N,P_1,P_2)^T$ and

$$f(X) = \begin{pmatrix} \dfrac{bN}{1+e_1P_1+e_2P_2} - dN - sN^2 - \dfrac{b_1N(1-m_1)P_1}{1+\alpha_1 N} - \dfrac{b_2N(1-m_2)P_2}{1+\alpha_2 N+\gamma P_2} \\ -c_1P_1 + \dfrac{b_1\lambda_3 N(1-m_1)P_1}{1+\alpha_1 N} \\ -c_2P_2 + \dfrac{b_2\lambda_4 N(1-m_2)P_2}{1+\alpha_2 N+\gamma P_2} \end{pmatrix}.$$ Taking derivative on $f(X)$ with respect to m_2, we get

$$f_{m_2}(X) = \begin{pmatrix} \dfrac{b_2NP_2}{1+\alpha_2N+\gamma P_2} \\ 0 \\ -\dfrac{b_2\lambda_4NP_2}{1+\alpha_2N+\gamma P_2} \end{pmatrix}, \text{ then } f_{\mathscr{E}_2,m_{2_0}}(X) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $w^T f_{\mathscr{E}_2,m_{2_0}}(X) = 0$.

Next, taking derivative on $f_{m_2}(X)$ with respect to $X = (N, P_1, P_2)^T$, we get,

$$Df_{\mathscr{E}_2,m_{2_0}}(X) = \begin{pmatrix} 0 & 0 & \dfrac{b_2N^*}{(1+\alpha_2N^*)} \\ 0 & 0 & 0 \\ 0 & 0 & -\dfrac{b_2\lambda_4N^*}{(1+\alpha_2N)} \end{pmatrix}.$$

We have, $w^T\left(Df_{\mathscr{E}_2,m_{2_0}}(X)\cdot v\right) = \frac{\left[b_1(1-m_1)(1+e_1P_1^{*2})^2+be_1(1+\alpha_1N^*)\right]b_2\lambda_4^2N^{*2}v_2w_3}{(1+\alpha_1N^*)(1+\alpha_2N^*)(c_2(1+e_1P_1^*)^2+be_1\lambda_4N^*)} \neq 0$. Furthermore, we find $\mathscr{D}^2 f(X) = D(J(E_i)) = D(D(f(X)))$ and then subistitute the value of m_{2_0} and E_2, so we end by

$$w^T[D^2 f_{\mathscr{E}_2,m_{2_0}}(X)(v,v)] \;=\; \frac{2c_2(v_1-\gamma v_3N^*)v_3w_3}{N^*(1+\alpha_1N^*)} \neq 0.$$

where, (v,v) is a Kronecker of $(v_1,v_2,v_3)^T$. Therefore, according to the Sotomayor's theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state $\mathscr{E}_2$ when the parameter m_2 passes through the bifurcation value m_{2_0}.

(iii) The eigenvalues of the linearized system around the equilibrium point E_3 are:

$$\mathscr{J}(\mathscr{E}_3) = \begin{bmatrix} N^*(-s+\frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}) & -N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_2P_2^*)^2}) & -N^*(\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*)}+\frac{be_2}{(1+e_2P_2^*)^2}) \\ 0 & -c_1+\frac{b_1\lambda_3N^*(1-m_1)}{(1+\alpha_1N^*)} & 0 \\ \frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2} & 0 & \frac{-\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2} \end{bmatrix}.$$

$$(7)\quad \mathscr{J}(\mathscr{E}_3, m_{1_0}) = \begin{bmatrix} N^*(-s+\frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}) & -N^*(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_2P_2^*)^2}) & -N^*(\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*)}+\frac{be_2}{(1+e_2P_2^*)^2}) \\ 0 & 0 & 0 \\ \frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2} & 0 & \frac{-\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2} \end{bmatrix}$$

Let us define $v=(v_1,v_2,v_3)^T$ and $w=(w_1,w_2,w_3)^T$ are the right and left eigenvectors of $\lambda_2=0$. From (7) and $\mathscr{J}(\mathscr{E}_3,m_{1_0})v=0$ as well as $\mathscr{J}^T(\mathscr{E}_3,m_{1_0})w=0$, then, $\left(N^*\left[(-s+\frac{b_2\alpha_2P_2^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2})v_1-(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_2P_2^*)^2})v_2-(\frac{b_2(1-m_2)(1+\alpha_2N^*)}{(1+\alpha_2N^*+\gamma P_2^*)}+\frac{be_2}{(1+e_2P_2^*)^2}v_3\right],0,G_1^*\right) = (0,0,0)^T$, where $G_1^*=\frac{b_2\lambda_4P_2^*(1-m_2)(1+\gamma P_2^*)}{(1+\alpha_2N^*+\gamma P_2^*)^2}v_1+\frac{-\gamma b_2\lambda_4P_2^*N^*(1-m_2)}{(1+\alpha_2N^*+\gamma P_2^*)^2}v_3$ and

$$\left(N^*(-s+\frac{b_1\alpha_1P_1^*(1-m_1)}{(1+\alpha_1N^*)^2})w_1+\frac{b_1\lambda_3P_1^*(1-m_1)}{(1+\alpha_1N^*)^2}w_2,-N^*\left(\frac{b_1(1-m_1)}{(1+\alpha_1N^*)}+\frac{be_1}{(1+e_1P_1^*)^2}\right)w_1,-N^*\left(\frac{c_2}{\lambda_4N^*}+\frac{be_2}{(1+e_1P_1^*)^2}\right)w_1\right)^T=0$$

So the left eigenvector is $(0,w_2,0)^T$ and the right eigenvector is

$$\left(v_1,\frac{\lambda_3v_1\left[e_2^2P_2^2(\gamma^2sN^*P_2^*+s\gamma N^*(1+\alpha_2N^*)+b_2(1-m_2))+G_2^*\right]}{\gamma(1+\alpha_2N^*+\gamma P_2^*)},\frac{(1+\gamma P_2)v_1}{\gamma N^*}\right)^T$$

, where $G_2^*=P_2^*(\gamma^2P_2^{*2}(2sN^*+b)+\gamma(2\alpha_2sN^{*2}+N^*(2s+\alpha_2b))+2b_2(1-m_2))+e_2b(1+\alpha_2N^*)+\gamma^2sN^*P_2^*+\gamma sN^*(1+\alpha_2N^*)+b_2(1-m_2)$. Here, w_2 and v_1 are any non-zero real numbers. Now, similarly system (1) can be rewritten as in the following vector form:

$$\dot{X}=f(X),$$

where $X=(N,P_1,P_2)^T$ and

$$f(X)=\begin{pmatrix}\frac{bN}{1+e_1P_1+e_2P_2}-dN-sN^2-\frac{b_1N(1-m_1)P_1}{1+\alpha_1N}-\frac{b_2N(1-m_2)P_2}{1+\alpha_2N+\gamma P_2}\\ -c_1P_1+\frac{b_1\lambda_3N(1-m_1)P_1}{1+\alpha_1N}\\ -c_2P_2+\frac{b_2\lambda_4N(1-m_2)P_2}{1+\alpha_2N+\gamma P_2}\end{pmatrix}.$$

Taking derivative on $f(X)$ with respect to m_1, we get

$$f_{m_1}(X)=\begin{pmatrix}\frac{b_1NP_1}{1+\alpha_1N}\\ \\ -\frac{b_1\lambda_3NP_1}{1+\alpha_1N}\\ 0\end{pmatrix},\ \text{then}\ f_{\mathscr{E}_3,m_{1_0}}(X)=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence, $w^Tf_{\mathscr{E}_3,m_{1_0}}(X)=0$.

Next, taking derivative on $f_{m_1}(X)$ with respect to $X=(N,P_1,P_2)^T$. Then,

$$Df_{\mathscr{E}_3,m_{1_0}}(X)=\begin{pmatrix}0 & \frac{b_1N^*}{1+\alpha_1N^*} & 0\\ 0 & -\frac{b_1\lambda_3N^*}{1+\alpha_1N^*} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

We have, $w^T\left(Df_{\mathscr{E}_1,m_{1_0}}(X)\cdot v\right)=\frac{\left(e_2^2P_2^{*2}\left[\gamma^2 sN^*P_2^*+\gamma sN^*(1-m_2)\right]+G_3^*\right)b_1\lambda_3^2N^*v_1w_2}{\gamma(1+\alpha_1N^*)(1+\alpha_2N^*+\gamma P_2^*)(c_1(1+e_2P_2^*)^2+b_1e_1\lambda_3N^*)}\neq 0$, where $G_3^* = e_2\left[\gamma_2^2P_2^{*2}(2sN^*+b)+P_2^*\left(\gamma(2s\alpha_2N^{*2}+N^*(2s+\alpha_2 b)+2b\right)+2b_2(1-m_2)+b(1+\alpha_2N^*)\right] + \gamma^2 sN^*P_2^* + \gamma sN^*(1+\alpha_2N^*)+b_2(1-m_2)$ Furthermore, we find $\mathscr{D}^2 f(X)=D(J(E_i))=D(D(f(X)))$ and then subistitute the value of m_{1_0} and E_3, so we end by

$$w^T[D^2 f_{\mathscr{E}_3,m_{1_0}}(X)(v,v)] \quad = \quad \frac{c_1(v_1+v_2)v_2w_2}{N^*(1+\alpha_1N^*)}\neq 0.$$

where, (v,v) is a Kronecker product of $(v_1,v_2,v_3)^T$. Therefore, according to the Sotomayor's theorem for local bifurcation, system (1) has a transcritical bifurcation at steady state $\mathscr{E}_3$ when the parameter m_1 passes through the bifurcation value m_{1_0}.

3. Numerical solution

In this section we will present some numerical simulations in order to show the theoretically established results. The values of the parameters are taken from the following table, and some of them will be varied in order to see their effect.

Parameters	value
b	0.4
d	0.2
s	0.0009
b_1, α_1	0.05, 0.002
b_2, α_2	0.05, 0.002
γ	0.002
c_1, c_2	0.1, 0.1
e_1, e_2	0.002, 0.002
m_1, m_2	0.6, 0.6
λ_3 0.1	
λ_4	0.1

TABLE 2. Table of value of parameters

value of m_1	The existence point
[0,0.6)	$(N^*, P_1^*, 0)$
0.6	(N^*, P_1^*, P_2^*)
(0.6,1]	$(N^*, 0, P_2^*)$

value of m_2	The existence point
[0,0.6)	$(N^*, 0, P_2^*)$
0.6	(N^*, P_1^*, P_2^*)
(0.6,1]	$(N^*, P_1^*, 0)$

As we have see in figure 2, which demonstrate the effect stability of the trivial solution when the inter-species competition is very big and $b - d < 0$, i.e. when the death rate is greater than the birth rate. In Figure 2 (i) $s = 0.0009$ and in (ii) $s = 0.8$ with $b > d$. While in Figure 2(iii) $s = 0.0009, b < d$. Figure 3 illustrate the stability of the existence of prey only $(N^*, 0, 0)$ under some conditions. The stability of the solution which reach the point $(N^*, P_1^*, 0)$, which represents the existence of prey and the first predator, is shown in figure 4. Figure 5 shows the stability of the solutions when it reach the point $(N^*, 0, P_2^*)$ which represents the existence of prey and second predator. The stability of the solutions when it reaches the coexistence point is illustrated in figure 6. The limit cycles are illustrated in figures 7,8 and 9 for the solutions of existence of prey with first predator, with the second predator and when it reaches the coexistence point, respectively. Figure 10 shows the effect of transcritical parameter which is refuge parameters m_1 and m_2 as illustrated in next table:

value of s	The existence point	The situation
[0,0.000314)	$(N^*, P_1^*, 0)$	limit cycle.
(0.000314,0.0036]	$(N^*, P_1^*, 0)$	Local asymptotically stable point.
(0.0036,1)	$(N^*, 0, 0)$	Local asymptotically stable point.

value of s	The existence point	The situation
(0,0.000279995]	$(N^*, 0, P_2^*)$	limit cycle.
(0.000279995,0.0036]	$(N^*, 0, P_2^*)$	Local asymptotically stable point.
(0.0036,1]	$(N^*, 0, 0)$	Local asymptotically stable point.

The effect of hopf bifurcation parameter which is the inter-species competition parameter is demonstrated in figures 11 and 12. The next table explains this result. We can see that when the value of this parameter changes the solutions changes from a stable limit cycle to an asymptotically stable point of the prey with one predator where we have the phenomenon of hopf bifurcation. When it further changes the solutions moves towards the point where the prey exists alone, where we have a transcritical bifurcation.

Figures 15-18 show the effect of fear parameters e_1 and e_2. As we can see when e_1 is small the solution reach the point of existence of prey with the first predator. If the value of fear parameter e_1 increases, the solution reaches same point but with decreasing in amount of predator population. If the value of the fear continues to increase, then the solution will reach to the point of existence prey only. Same situation happens if the value e_2 increased but here the solution reach the point of existence of prey with the second predator.

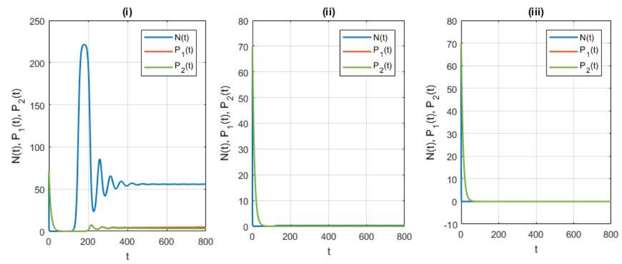


FIGURE 2. The stability of trivial point $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 == \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.6, b = 0.4, d = 0.41$

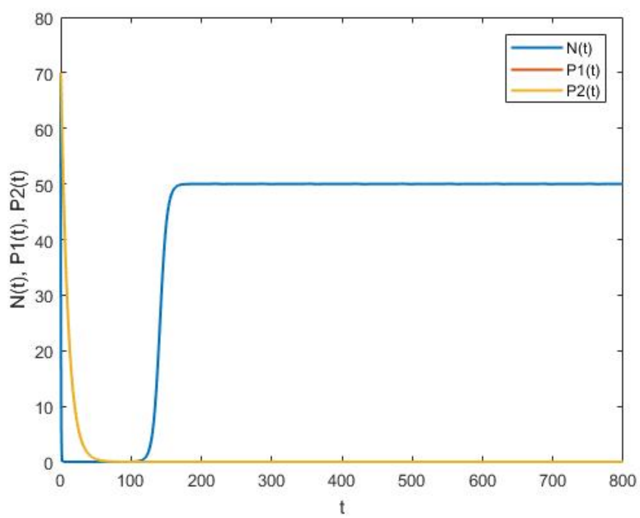


FIGURE 3. The stability of prey $N = P_1 = P_2 = 70, s = 0.004, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.6, b = 0.4, d = 0.2, b_1 = b_2 = 0.05$

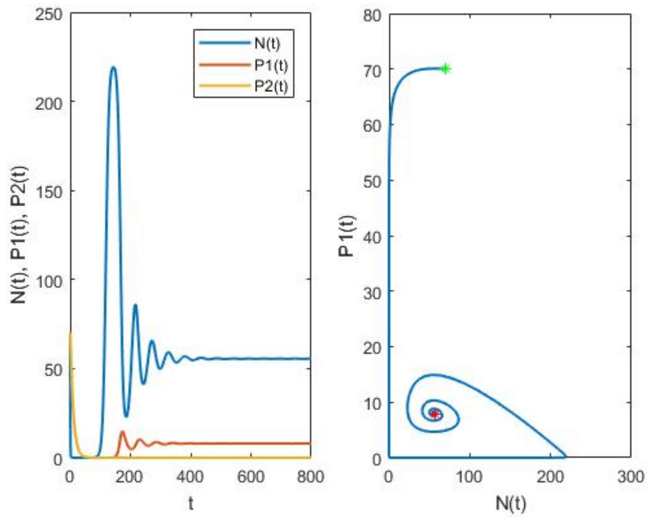


FIGURE 4. The stability of prey with first predator $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.8$

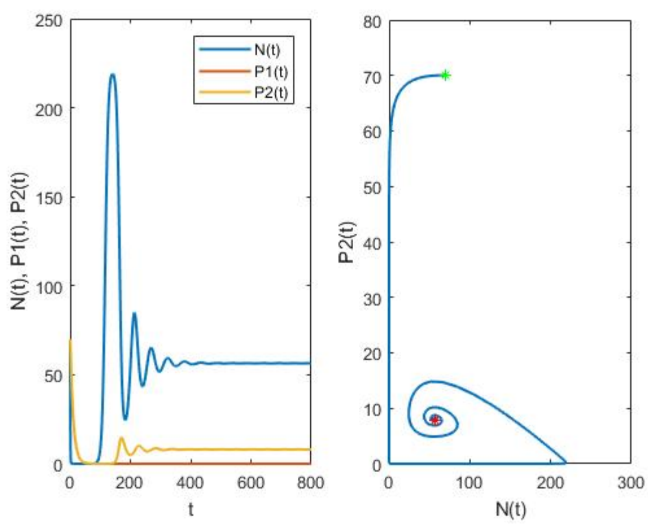


FIGURE 5. The stability of prey with second predator $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m2 = 0.6$

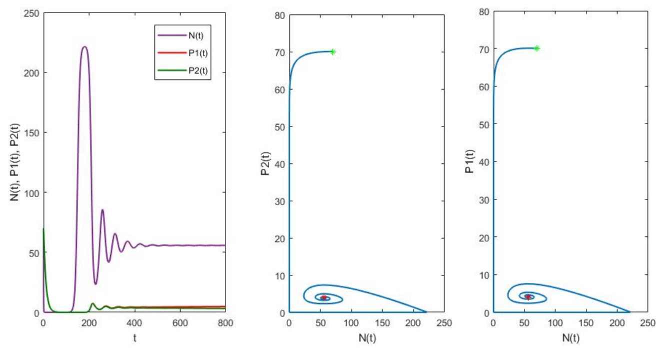


FIGURE 6. The stability of coexistence point $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 == \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.6$

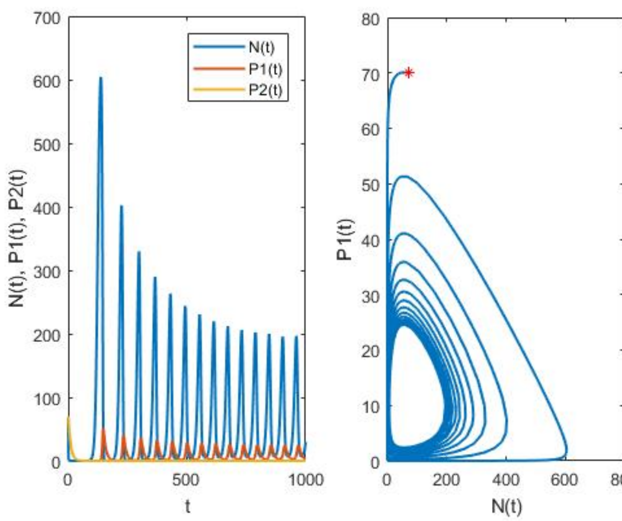


FIGURE 7. The limit cycle of first predator with prey $N = P_1 = P_2 = 70, s = 0.0003, e_1 = e_2 = \alpha_1 = \alpha_2 == \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.8, b = 0.4, d = 0.2$

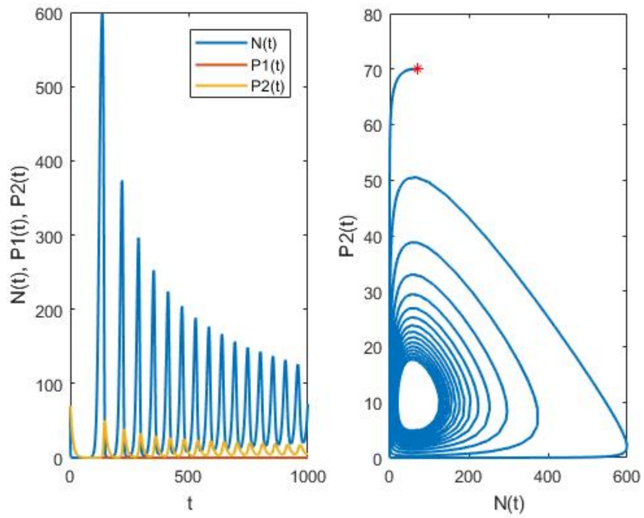


FIGURE 8. The limit cycle of second predator with prey $N = P_1 = P_2 = 70, s = 0.0003, e_1 = e_2 = \alpha_1 = \alpha_2 == \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m2 = 0.6, b = 0.4, d = 0.2$

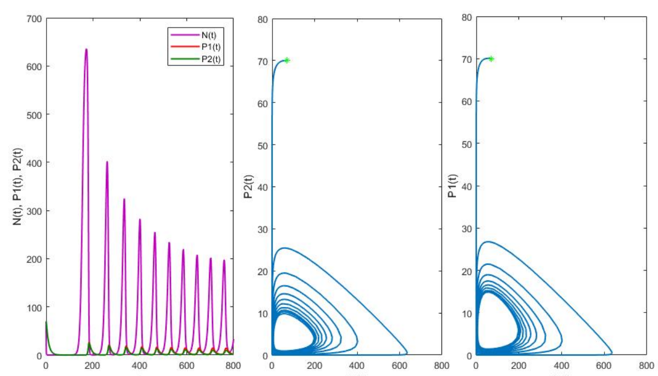
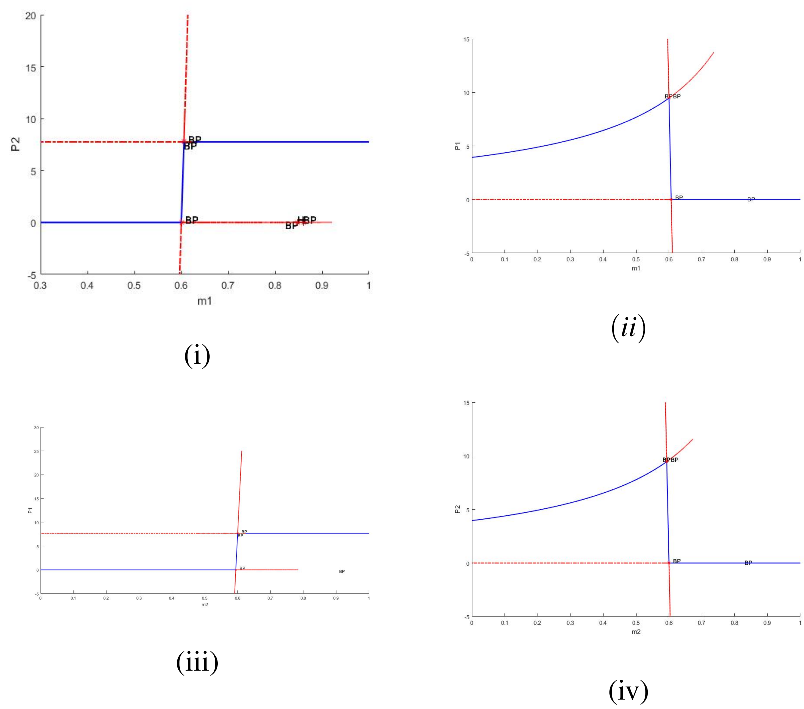


FIGURE 9. The limit cycle of coexistence point $N = P_1 = P_2 = 70, s = 0.0003, e_1 = e_2 = \alpha_1 = \alpha_2 == \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.6, b = 0.4, d = 0.2$



(i) (ii)

(iii) (iv)

FIGURE 10. Effect of transcriptional refuge parameter on model, $N = P_1 = P_2 = 70, s = 0.0004, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m2 = 0.6$

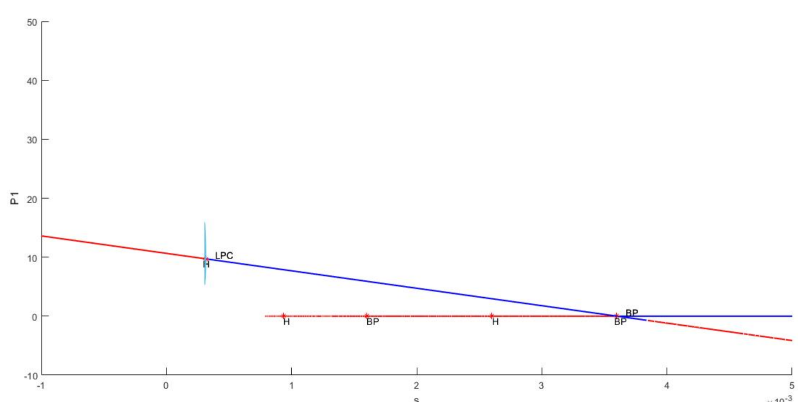


FIGURE 11. Effect of hopf bifurcation parameter on model on the point of first predator with prey.

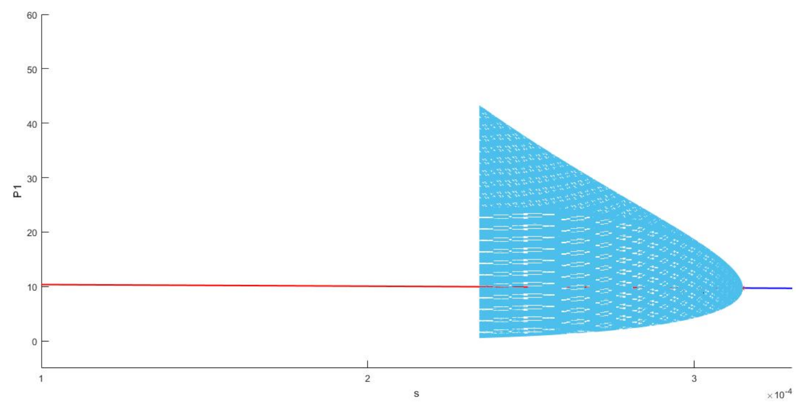


FIGURE 12. Hopf bifurcation diagram of model on the point of first predator with prey.

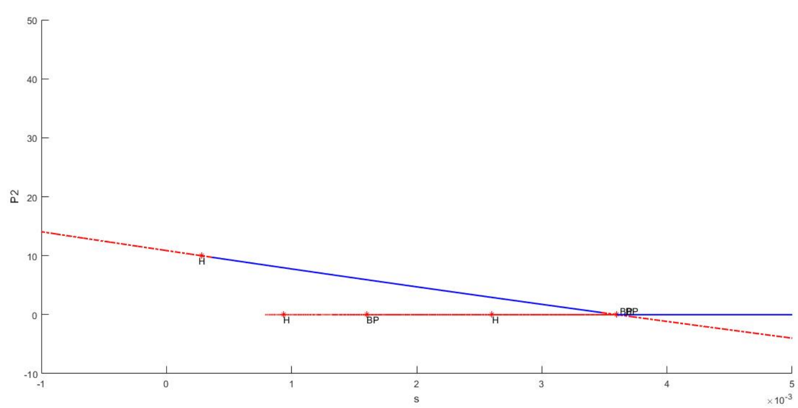


FIGURE 13. Effect of hopf bifurcation parameter on model on the point of second predator with prey.

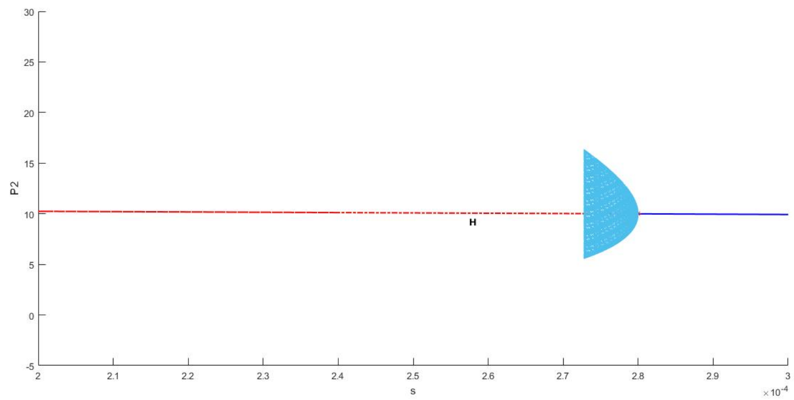


FIGURE 14. Hopf bifurcation diagram of the point of second predator with prey.

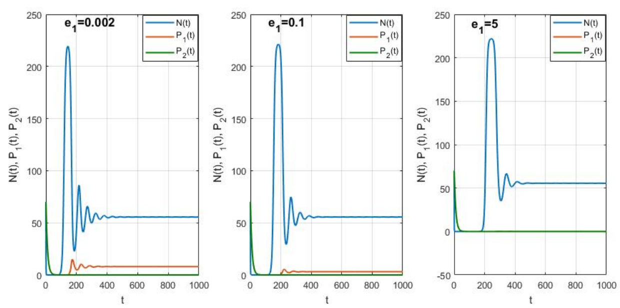


FIGURE 15. Effect of fear parameter on model $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.8$

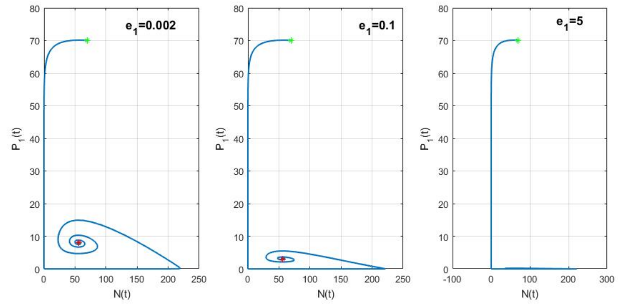


FIGURE 16. Phase plane of effect of fear parameter on model $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.6, m2 = 0.8$

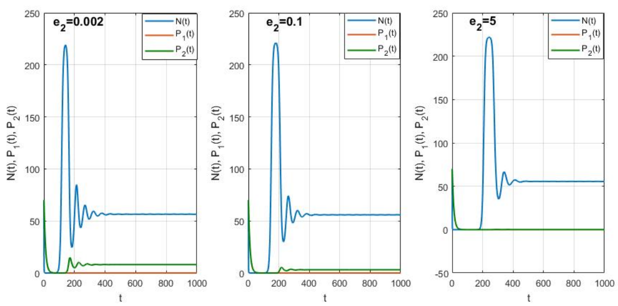


FIGURE 17. Effect of fear parameter on model, $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m2 = 0.6$

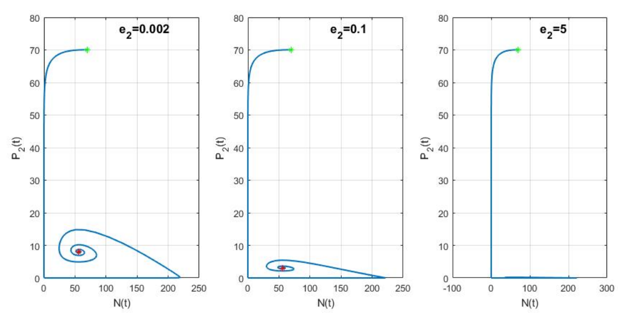


FIGURE 18. Phase plane of effect of fear parameter on model, $N = P_1 = P_2 = 70, s = 0.0009, e_1 = e_2 = \alpha_1 = \alpha_2 = \gamma = 0.002, c_1 = c_2 = \lambda_3 = \lambda_4 = 0.1, m_1 = 0.8, m2 = 0.6$

4. CONCLUSION

In this paper we formulated model of one prey with two predators population; aggressive predator, bold predator and prey which is afraid and stay in refuge. We use holing type II functional response and Beddington-DeAngeiis functional response[12] . We proved the locally and globally stabilities of equilibria. In addition we hold a hopf and double transcritical bifurcations of some parameters. The cost of fear and prey refuge is allow model to reach to double transcritical. we notice that if fear is good when it is small. The effect of competition of prey population is to convert the model from the stable limit cycle to a spiral stable equilibrium point of prey with predator. When it becomes large it converts model to stable trivial solution.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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