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STRONGLY CONTINUOUS BICOMPLEX SEMIGROUPS

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Abstract. In this paper, we study the semigroups of operator algebras with bicomplex scalars and also investigate the $*$-derivation.

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1. INTRODUCTION

Bicomplex numbers are a generalization of complex numbers which form a ring under the usual addition and multiplication. Moreover, this ring is a module over itself. The set of bicomplex numbers do not form a field because every non-zero bicomplex number does not possesses its multiplicative inverse. Zero divisors of bicomplex numbers play a significant role in the idempotent representation of bicomplex numbers. For basics of bicomplex numbers and their properties one can refer to [1], [8], [9] and references therein.

Hyperbolic numbers are a natural replacement for the real number system. However, it was

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seen that the hyperbolic numbers are subset of bicomplex numbers. The role of hyperbolic for bicomplex numbers is analogous to the role of real numbers for complex numbers.

In the last several years, the theory of bicomplex numbers and hyperbolic numbers has found many applications in different areas of mathematics and theoretical physics (cf. [3], [5], [6]), and references there in provide more information on these applications.

The work in this paper can be seen as a continuation of work in [11].

2. PRELIMINARIES

In this section, we summarize some known results about bicomplex numbers.

The set of bicomplex numbers is denoted by $\mathscr{BC}$ and is defined as the commutative ring whose elements are of the form

$$Z = z_1 + jz_2, \tag{1}$$

where $z_1 = x_1 + iy_1 \in \mathbb{C}(i)$ and $z_2 = x_2 + iy_2 \in \mathbb{C}(i)$ are complex numbers with imaginary unit i and where i and $j \neq i$ are commutating imaginary units i.e., $ij = ji$. Note that $i^2 = j^2 = -1$.

In equation (1), if $z_1 = x$ is real and $z_2 = iy$ is purely imaginary with $i.j = k$, then we obtain the ring of hyperbolic numbers.

$$\mathscr{D} = \{x + ky : k^2 = 1 \; and \; x, y \in \mathbb{R} \; with \; k \notin \mathbb{R}\}.$$

A hyperbolic number α can be written as

$$\alpha = \alpha_1 e + \alpha_2 e^{\dagger}, \; where \; \alpha_1, \; \alpha_2 \geq 0.$$

We denote by $\mathscr{D}_+$, the set of all "positive" hyperbolic numbers. The set

$$\mathscr{D}_+ = \{x + ky \setminus x^2 - y^2 \geq 0, \; x \geq 0\},$$

is called the set of positive hyperbolic numbers. (cf. [1], page 5). For $P, Q \in \mathscr{D}$, (set of hyperbolic numbers) we define a relation $\preceq'$ on $\mathscr{D}$ by $P \preceq' Q \iff Q - P \in \mathscr{D}_+$.

This relation is reflexive, anti-symmetric as well as transitive and hence defines a partial order on $\mathscr{D}$, (cf. [1]).

Further details about bicomplex numbers are discussed in [1], [5] and [7].

3. Main Results

In this section, we study the semigroups on operator algebras with bicomplex scalars and also investigate the $*$-derivation. The work of this section is based on [2, page 22-23].

DEFINITION 3.1. A one parameter family $\{\Phi(p); p \succeq' 0\}$ of bounded bicomplex linear operators on $\mathscr{BC}$ Banach module satisfying :

$$\Phi(p+q) = \Phi(p)\Phi(q) \quad p,q \succ' 0$$

$$\Phi(0) = I$$

$$\lim_{p\to 0^+} \|\Phi(p) - I\|_{\mathscr{D}} = 0$$

is called a uniformly continuous semigroup of bounded linear operators.

The one parameter semigroups on $\mathscr{BC}$ Banach module have been introduced in [11, Section 3].

Let H be bicomplex Hilbert module and let $A \in L(H)$ be a bicomplex skew-adjoint operator and $(e^{pA})_{p\in\mathscr{D}}$ be a unitary group. Then $(e^{pA})_{p\in\mathscr{D}}$ defines an operator $\mathscr{U}(p)$ on the $\mathscr{BC}$ operator algebra $L(H)$.

Let $\mu_1(p)$ and $\mu_2(p)$ are operators with complex scalars defined by $(e^{pA_1})_{p\in\mathscr{D}}$ and $(e^{pA_2})_{p\in\mathscr{D}}$ respectively.

Define a $\mathscr{BC}$ linear-map $\mathscr{U}(p) : L(H) \to L(H)$ by

$$\begin{aligned}\mathscr{U}(p)(\Phi) &= e.\mu_1(p)\Phi_1 + e^{\dagger}.\mu_2(p)\Phi_2 \\ &= e.e^{pA_1}.\,\Phi_1\,.\,e^{pA_1^*} + e^{\dagger}.e^{pA_2}.\,\Phi_2\,.\,e^{pA_2^*} \\ &= e^{pA}.\,\Phi\,.\,e^{pA^*},\ \forall\,\Phi \in L(H).\end{aligned}$$

Then

(i) each $\mathscr{U}(p)$ is a $*$-automorphism of the $\mathscr{BC}$ Banach $*$-algebra L(H),

(ii) the system $(\mathscr{U}(p))_{p\in\mathscr{D}}$ is one parameter group with bicomplex scalars on $L(H)$ and

(iii) this operator group is uniformly continuous.

As in Theorem 3.7 in [2, Chapter 1], there exists an operator
$\Gamma : L(H) \to L(H)$ such that

$$\mathscr{U}(p) = e^{p\Gamma} \ for\ each\ p \in \mathscr{D}.$$

And the differentation of the map

$$p \longmapsto e^{pA}\Phi e^{pA^*}$$

at $p = 0$ gives that $\Gamma(\Phi) = \left(e.A_1\Phi_1 + e^{\dagger}.A_2\Phi_2\right) + \left(e.\Phi_1 A_1^* + e^{\dagger}.\Phi_2 A_2^*\right).$
Since $A_1^* = -A_1$ and $A_2^* = -A_2$, we have

$$\begin{aligned}\Gamma(\Phi) &= \Big(e.A_1\Phi_1 + e^{\dagger}.A_2\Phi_2\Big) - \Big(e.\Phi_1 A_1 + e^{\dagger}.\Phi_2 A_2\Big)\\ &= A\Phi - \Phi A\ \forall\ \Phi\ \in L(H).\end{aligned}$$

PROPOSITION 3.2. The uniformly continuous group $(\mathscr{U}(p))_{p\in\mathscr{D}}$ of $*$-automorphisms on $L(H)$ given by the unitary group $(e^{pA})_{p\in\mathscr{D}}$ has generator Γ given by

$$\Gamma(\Phi) = A.\Phi - \Phi.A,\ \forall\ \Phi \in L(H). \tag{2}$$

We now study uniformly continuous groups consisting of $*$-automorphism.
The proof of the following results is based on [[2], Chapter 1].
LEMMA 3.3. Let $(e^{p\mathfrak{D}})_{p\in\mathscr{D}}$ be uniformly continuous group on a $\mathscr{BC}$ Banach $*$-algebra $\mathscr{A}$ with unit $e \in \mathscr{A}$. The following are equivalent:

(a) $(e^{p\mathfrak{D}})_{p\in\mathscr{D}}$ is a group of $*$-automorphisms
(b) $\mathfrak{D}$ is a $*$-derivation, that is,

$$\mathfrak{D}(ab^*) = (\mathfrak{D}a)b^* + a(\mathfrak{D}b)^*,\ \forall\ a, b \in \mathscr{A}. \tag{3}$$

LEMMA 3.4. Let $\mathfrak{D}$ be a bounded $*$-derivation on $\mathscr{A} = L(H)$. Then there exists a bicomplex skew-adjoint operator $A \in L(H)$ such that

$$\mathfrak{D}(\Phi) = A.\Phi - \Phi.A,\ for\ each\ \Phi \in L(H).$$

Proof. Let $x = e.x_1 + e^{\dagger}.x_2$ and $y = e.y_1 + e^{\dagger}.y_2 \in H$, We define the rank-one operator $x \otimes y$ by

$$z \mapsto x \otimes y(z) = e.x_1 \otimes y_1(z_1) + e^{\dagger}.x_2 \otimes y_2(z_2) = e.(x_1, z_1)y_1 + e^{\dagger}.(x_2, z_2)y_2 = (x, z)y.$$

Let $z = e.z_1 + e^\dagger z_2 \in H, \|z\| = 1$, and define $A \in L(H)$ by

$$Ay = e.A_1y_1 + e^\dagger A_2y_2 = e.\mathfrak{D}(z_1 \otimes y_1)(z_1) + e^\dagger.\mathfrak{D}(z_2 \otimes y_2)(z_2) = \mathfrak{D}(z \otimes y)(z), \ \forall\, y \in H.$$

Thus for $\Phi \in L(H)$ and for each $y \in H$, we have

$$\begin{aligned}
(A\Phi - \Phi A)(y) &= e.(A_1\Phi_1 - \Phi_1A_1)(y_1) + e^\dagger.(A_2\Phi_2 - \Phi_2A_2)(y_2)\\
&= e.\mathfrak{D}(z_1 \otimes \Phi_1y_1)(z_1) - e.\Phi_1(\mathfrak{D}(z_1 \otimes y_1)(z_1))\\
&+ e^\dagger.\mathfrak{D}(z_2 \otimes \Phi_2y_2)(z_2) - e^\dagger\Phi_2(\mathfrak{D}(z_2 \otimes y_2)(z_2))\\
&= e.\mathfrak{D}(\Phi_1(z_1 \otimes y_1))(z_1) - e(\Phi_1.\mathfrak{D}(z_1 \otimes y_1))(z_1)\\
&+ e^\dagger.\mathfrak{D}(\Phi_2(z_2 \otimes y_2))(z_2) - e^\dagger(\Phi_2.\mathfrak{D}(z_2 \otimes y_2))(z_2)\\
&= e.(\mathfrak{D}\Phi_1(z_1 \otimes y_1))(z_1) + e(\Phi_1.\mathfrak{D}(z_1 \otimes y_1))(z_1)\\
&- e(\Phi_1.\mathfrak{D}(z_1 \otimes y_1))(z_1) + e^\dagger(\mathfrak{D}\Phi_2(z_2 \otimes y_2))(z_2)\\
&+ e^\dagger.(\Phi_2.\mathfrak{D}(z_2 \otimes y_2))(z_2) - e^\dagger.(\Phi_2.\mathfrak{D}(z_2 \otimes y_2))(z_2)\\
&= e.(\mathfrak{D}\Phi_1(z_1 \otimes y_1))(z_1) + e^\dagger(\mathfrak{D}\Phi_2(z_2 \otimes y_2))(z_2)\\
&= e.\mathfrak{D}\Phi_1(y_1) + e^\dagger.\mathfrak{D}\Phi_2(y_2)\\
&= \mathfrak{D}\Phi(y),\ for\ each\ y \in H.
\end{aligned}$$

Finally, we show that the operator $A \in L(H)$ satisfying the property $A\Phi - \Phi A$, for all $\Phi \in L(H)$ can be taken as a bicomplex skew-adjoint operator. Since $\mathfrak{D}$ is a $*$-derivation, we have

$$\begin{aligned}
A\Phi^* - \Phi^*A &= e.(A_1\Phi_1^* - \Phi_1^*A_1) + e^\dagger.(A_2\Phi_2^* - \Phi_2^*A_2)\\
&= \mathfrak{D}(e.\Phi_1^* + e^\dagger\Phi_2^*)\\
&= \mathfrak{D}(e.\Phi_1 + e^\dagger\Phi_2)^*\\
&= e.(-A_1^*\Phi_1^* + \Phi_1^*A_1^*) + e^\dagger.(-A_2^*\Phi_2^* + \Phi_2^*A_2^*)\\
&= -A^*\Phi^* + \Phi^*A^*,
\end{aligned}$$

and so

$$(A + A^*)\Phi^* - \Phi^*(A + A^*) = 0,\ \forall\ \Phi \in L(H).$$

Therefore, $\tilde{A} = A - \frac{(A^*+A)}{2} = \frac{A-A^*}{2}$ is a skew-adjoint operator such that

$$\tilde{A}\Phi - \Phi\tilde{A} = A\Phi - \Phi A,\ \forall\ \Phi \in H.$$

□

According to Lemma 3.3, the generator Γ of a uniformly continuous group of $*$-automorphisms on L(H) is a $*$-derivation, which is implemented by some bicomplex skew-adjoint operator $A \in L(H)$ according to Lemma 3.4. Further, this operator A generates a uniformly group $(e^{pA})_{p\in\mathscr{D}}$ that implements an automorphism group whose generator coincides with Γ.

The proof of the following theorem is similar to the proof of [[2], Chapter 1, Theorem 3].

THEOREM 3.5. Let H be a $\mathscr{BC}$-Hilbert module and take $(\mathscr{U}(p))_{p\in\mathscr{D}}$ to be a uniformly continuous group on L(H). Then the following are equivalent:

(a) $(\mathscr{U}(p))_{p\in\mathscr{D}}$ is a group of $*$-automorphism on the $\mathscr{BC}$ Banach $*$-algebra L(H).

(b) There exists a bicomplex skew-adjoint operator $A \in L(H)$ and a unitary group $(e^{pA})_{p\in\mathscr{D}}$ on H such that

$$\mathscr{U}(p)\Phi = e^{pA}\Phi e^{pA^*},\ \forall\ \Phi \in L(H).$$

Consider the $\mathscr{BC}$ Banach module $\mathbf{B} = C_o(\Omega, \mathscr{BC})$, where Ω is $\mathscr{BC}$ locally compact module, defined as

$$\mathbf{B} = C_o(\Omega, \mathscr{BC}) = \{ f : \Omega \to \mathscr{BC} : f \textit{ is continuous}, \textit{for each } \varepsilon \succ' 0 \textit{ and}$$
$$\exists \textit{ compact } K \subseteq \Omega \textit{ such that } \|f(x)\|_{\mathscr{D}} \prec' \varepsilon \}.$$

For each $f \in \mathbf{B}$, $f(x) = ef_1(x) + e^{\dagger}f_2(x), x \in \Omega$, the corresponding hyperbolic norm on $\mathbf{B}$ is defined as

$$\begin{aligned}\|f\|_{\mathscr{D}} &= \|ef_1(x) + e^{\dagger}f_2(x)\|_{\mathscr{D}}\\ &= e\|f_1(x)\|_1 + e^{\dagger}\|f_2(x)\|_2\\ &= e\ sup_{x\in\Omega}|f_1(x)|_1 + e^{\dagger}\ sup_{x\in\Omega}|f_2(x)|_2.\end{aligned}$$

One can check that $\mathbf{B}$ is a $\mathscr{BC}$ Banach module under the hyperbolic norm $\|.\|$ defined above.

4. Strongly Continuous Semigroups

In this section, we study strongly continuous semigroup or C_0 semigroup and also discuss some of its basic properties. The results of this section are essentially based on [2, pp. 36-39].

DEFINITION 4.1. Let $\mathbf{B}$ be a $\mathscr{BC}$ Banach module and a family $(\Phi(p))_{p\in\mathscr{D}_+}$ of bounded $\mathscr{BC}$ linear operators in $\mathbf{B}$ is called a strongly continuous (one-parameter) semigroup or $(C_0-$ $semigroup)$ if it satisfies the following conditions:

(1) Functional Equation (F.E) i.e.,

$$\Phi(p+q)=\Phi(p)\Phi(q),\ for\ all\ p,q\in\mathscr{D}_+,\ and\ \Phi(0)=I.$$

(2) strongly continuous i.e., $\Phi(p)x\to x\ as\ p\to 0,\ \forall x\in\mathbf{B}$.

If these properties hold for $\mathscr{D}$ instead of $\mathscr{D}_+$, we call $(\Phi(p))_{p\in\mathscr{D}}$ a strongly continuous (one-parameter) group (or $C_0-group$) on $\mathbf{B}$.

DEFINITION 4.2. [2, page 511-512] Let the space of all $\mathscr{D}$ bounded $\mathscr{BC}$ linear operators on $\mathbf{B}$ will be denoted by $L(\mathbf{B})$ and become a $\mathscr{BC}$ Banach module for the norm

$$\|\Phi\|_{\mathscr{D}}=sup\{\|\Phi x\|_{\mathscr{D}}:\|x\|_{\mathscr{D}}\preceq' 1\}.$$

Note that this norm is hyperbolic norm on Φ.

The operator ($\mathscr{BC}$-linear) $\Phi\in L(\mathbf{B})$ satisfying

$$\|\Phi(x)\|_{\mathscr{D}}\preceq'\|x\|_{\mathscr{D}}\ for\ all\ x\in\mathbf{B}$$

is called contraction, while isometry is defined by

$$\|\Phi(x)\|_{\mathscr{D}}=\|x\|_{\mathscr{D}}\ for\ all\ x\in\mathbf{B}.$$

Besides the uniform operator topology on $L(\mathbf{B})$, which is induced by the operator norm $\|\Phi\|_{\mathscr{D}}$, we consider two more topologies on $L(\mathbf{B})$.

For $\mathscr{BC}$ Banach module $\mathbf{B}$ we denote its dual by $\mathbf{B}'$. Let $L_s(\mathbf{B})$ denotes $L(\mathbf{B})$ with strong operator topology, which is the topology of pointwise convergence on $(\mathbf{B},\|.\|)$ and $L_\sigma(\mathbf{B})$ denotes $L(\mathbf{B})$ with weak operator topology, topology of pointwise convergence on $(\mathbf{B},\sigma(\mathbf{B},\mathbf{B}'))$, where

$\sigma(\mathbf{B},\mathbf{B}')$ is the weak topology on $\mathbf{B}$.

Let $(\Phi_\alpha)_{\alpha\in\mathscr{A}}$ be a bicomplex net of bicomplex linear operators on the $\mathscr{BC}$ Banach module $\mathbf{B}$. Consider that $(\Phi_\alpha)_{\alpha\in\mathscr{A}}$ converges to some operator Φ on $\mathbf{B}$. This could have different meanings as follows:

If $\|\Phi_\alpha - \Phi\|_{\mathscr{D}} \to 0$, then we say that Φ_α converges to Φ in the uniform operator topology.

If $\|\Phi_\alpha x - \Phi x\|_{\mathscr{D}} \to 0 \;\forall\, x \in \mathbf{B}$, then we say that Φ_α converges to Φ in the strong operator topology.

Finally, we have $|\langle \Phi_\alpha x - \Phi x, x'\rangle|_{\mathscr{D}} \to 0 \;\forall\, x \in \mathbf{B},\; x' \in \mathbf{B}'$ in the weak topology of $\mathbf{B}$. In this case we say that Φ_α converges to Φ in the weak operator topology.

PROPOSITION 4.3. Let $(\Phi(p))_{p\in\mathscr{D}_+}$ be a semigroup on a $\mathscr{BC}$ Banach module $\mathbf{B}$, the following assertions are equivalent:

(a) $(\Phi(p))_{p\in\mathscr{D}_+}$ is strongly continuous.

(b) $lim_{p\to 0}\Phi(p)x = x \; for \; all \; x \in \mathbf{B}$.

(c) There exists $\delta \succ' 0, M \succeq' 1$, and a dense subset $D \subset \mathbf{B}$ such that

(i) $\|\Phi(p)\|_{\mathscr{D}} \preceq' M \; for \; all \; p \in [0,\delta]_{\mathscr{D}}$,

(ii) $lim_{p\to 0}\Phi(p)x = x \; for \; all \; x \in D$.

Proof. The proof of above proposition is on similar lines as in [[2], Proposition 5.3, page 38]. □

DEFINITION 4.4. Let $\phi_p : \Omega \to \Omega$ be a family of continuous functions. Define $C_{\phi_p} : C_c(\Omega,\mathscr{BC}) \to C_c(\Omega,\mathscr{BC})$ by $C_{\phi_p}(f) = C_{\phi_p}(e.f_1 + e^\dagger.f_2) = e.f_1 o\phi_p + e^\dagger.f_2 o\phi_p \;\forall\, f \in C_c(\Omega,\mathscr{BC})$. Then

$$\begin{aligned}
\|C_{\phi_p} f\|_{\mathscr{D}} &= \|C_{\phi_p}\left(e\, f_1 + e^\dagger\, f_2\right)\|_{\mathscr{D}} \\
&= e\, \|C_{\phi_p} f_1\|_1 + e^\dagger\, \|C_{\phi_p} f_2\|_2 \\
&= e\, sup_{x\in\Omega}|f_1(\phi_p(x))|_1 + e^\dagger\, sup_{x\in\Omega}|f_2(\phi_p(x))|_2 \\
&\preceq' e\, sup_{x\in\Omega}|f_1(x)|_1 + e^\dagger sup_{x\in\Omega}|f_2(x)|_2 \\
&= e\, \|f_1\|_1 + e^\dagger\, \|f_2\|_2 \\
&= \|f\|_{\mathscr{D}}.
\end{aligned}$$

Thus $\|C_{\phi_p}\|_{\mathscr{D}} \preceq' 1$.

Therefore $\{C_{\phi_p}\}_{p\succeq' 0}$ is uniformly bounded. Hence

$$\begin{aligned} lim_{p\to 0}\|C_{\phi_p}f-f\|_{\mathscr{D}} &= e.lim_{p\to 0}\|C_{\phi_p}f_1-f_1\|_1+e^{\dagger}.lim_{p\to 0}\|C_{\phi_p}f_2-f_2\|_2 \\ &= e.lim_{p\to 0}sup_{x\in\Omega}|f_1(\phi_p)(x)-f_1(x)|_1 \\ &+e^{\dagger}.lim_{p\to 0}sup_{x\in\Omega}|f_2(\phi_p)(x)-f_2|_2 \\ &= lim_{p\to 0}|f(\phi_p)(x)-f|_k. \end{aligned}$$

Since the **p**-norms (for functions on bounded intervals) is weaker, we have

$$lim_{p\to 0}\|C_{\phi_p}f-f\|_{\mathbf{p}}=0.$$

Example 4.5. The (left) translation group is strongly continuous on $L^{\mathbf{p}}(\Omega,\mathscr{BC})$ for all $1\leq \mathbf{p}<\infty$.

Remark: For a strongly continuous semigroup $(\Phi(p))_{p\in\mathscr{D}_+}$, the fixed orbits

$$\{\Phi(p)x : p\in[0,p_0]_{\mathscr{D}}, p_0\succ' 0\}$$

are continuous images of a compact interval, hence compact and therefore bounded for each $x\in\mathbf{B}$. So by the Uniform Boundedness Principle [4], each strongly continuous semigroup is uniformly bounded on each compact interval, a fact that implies exponential boundedness on $\mathscr{D}_+$.

5. Conclusion

In this paper, we have concluded that the classical results in semigroups of operator algebras and strongly continuous semigroups with real and complex scalars can be proved in bicomplex framework with idempotent decomposition.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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