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ON A GENERALIZED GAMMA FUNCTION AND ITS PROPERTIES

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Abstract. In this work, we introduce a new generalized gamma function and establish its validity through the Bohr-Mullerup theorem. We also establish the generalized Euler reflection formula and some other properties related to the generalized gamma function.The concept of powers of logarithm was largely used to establish the results.

Keywords: generalized gamma functions; properties; Euler reflection formula.

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1. INTRODUCTION

Euler introduced the gamma function with the goal to generalize the factorial to non integer values. Some prominent mathematicians such as Gauss, Legendre, Weierstrass, among others also studied it. The Gamma function belongs to the class of special functions and some mathematical constants such as the Euler-Mascheroni constant occur in its study.

In studying the Riemann zeta function and other special functions, the gamma function plays a

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vital role and has been the subject of study for over 300 years. The gamma function is still being studied by contemporary mathematicians and yet there seems to be so much to study about it. The gamma function is essential for modeling situations involving continuous change and has applications in calculus, differential equations, statistics, fluid mechanics, quantum physics and complex analysis.

A generalized gamma function $\Gamma_k(z)$ for $k \in \mathbb{N}_0$ was introduced in [1] which connects to the constant γ_k as $\Gamma(z)$ does to γ. Motivated by a series form of the generalized Euler-Mascheroni constants and through the concept of powers of logarithms, some properties of the gamma function were presented in [1].

The aim of this paper is to establish another form of a generalized gamma function and its properties.

2. PRELIMINARIES

The gamma function, a generalization of the factorial to non-integer values, was defined by Euler as

$$\Gamma(z) = \frac{1}{z} \lim_{n\to\infty} \prod_{j=1}^{n} \frac{\left(1+\frac{1}{j}\right)^z}{\left(1+\frac{z}{j}\right)}, z \in \mathbb{C} \setminus \mathbb{Z}^-. \tag{2.1}$$

Gauss rewrote Euler's product representation of the gamma function as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)(z+2)...(z+n)}, z \in \mathbb{C} \setminus \mathbb{R}^-. \tag{2.2}$$

The integral representation of the gamma function was also defined by Euler as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, R(z) > 0. \tag{2.3}$$

Weierstrass also established another product representation of the gamma function as

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{j=1}^{\infty} \left(1+\frac{z}{j}\right)^{-1} e^{\frac{z}{j}}, z \in \mathbb{C} \setminus \mathbb{Z}^-, \tag{2.4}$$

where γ is the Euler Mascheroni constant.

Eventhough Euler pioneered the theory of complex analysis, he did not consider the gamma function of a complex argument. Gauss did by establishing its multiplication theorem. Karl Weierstrass also introduced the product representation as given by (2.4).

In [3] a wide class of generalizations for the gamma function was studied and a special case for this class of generalizations was also studied by Dilcher [1]. In particular, the generalized gamma function $\Gamma_k(z)$ was introduced for $k \in \mathbb{N}_0$ and some basic properties such as product and series expansions of a generalized gamma function were developed in [1]. He also established a series expansion for the generalized Euler constant for $k \in \mathbb{N}$.

It was observed in [4] that an assymptotic expansion of Dilcher's generalized gamma function converges for $k = 1$ and the closed form was unknown for $k > 1$.

Some results in [1] are connected with those in [2] and a question was posed in [2] whether it is possible to extend the gamma function by analytic continuation. This study establishes a new generalized gamma function and its properties.

The generalized Euler-Mascheroni constants are defined as

$$\gamma_k = \lim_{n\to\infty}\left(-\frac{\ln^{k+1} n}{k+1} + \sum_{j=1}^{n}\frac{\ln^k j}{j}\right), k = 0,1,2,... \tag{2.5}$$

and are coefficients of the Laurent expansion of the Riemann zeta function $\zeta(s)$ about $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} A_k (s-1)^k, Re(s) \geq 0,$$

where $A_k = \frac{(-1)^k}{k!}\gamma_k$.

The constants A_k were first defined by Stieltjes in 1885 and have been studied by other authors. It is worth noting that $\gamma_0 = \gamma$ is the Euler constant and is closely related to the gamma function. Observe that if $s = 0$, the above Laurent expansion gives

$$\sum_{k=0}^{\infty}\frac{\gamma_k}{k!} = \frac{1}{2}.$$

Thus, the real part of the nontrivial zeroes of the Riemann zeta function is associated with the generalized Euler-Mascheroni constants.

The Euler's reflection formula is given by

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}. \tag{2.6}$$

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.7}$$

and its derivatives are given by

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{\ln^k n}{n^s}. \tag{2.8}$$

The stirling numbers of the first kind, $s(m, j)$, is defined by the generating function as

$$\ln^j \left(1 + \frac{z}{n}\right) = \sum_{n=j}^{\infty} \frac{j!}{m!} s(m, j) \left(\frac{z}{n}\right)^m, \tag{2.9}$$

where $|z| < 1$.

Alternatively, stirling numbers of the first kind are also defined by

$$\frac{1}{j} \ln^j (1+t) = \sum_{n=j}^{\infty} s(n, j) \frac{t^n}{n!}, \tag{2.10}$$

or

$$\ln^j (1-t) = \sum_{n=j}^{\infty} \frac{(-1)^n}{n!} s(n, j) t^n. \tag{2.11}$$

From (2.9), we see that

$s(m,1) = (-1)^{m-1}(m-1)!$, $s(m,m) = 1$ and

$s(m,2) = (-1)^m (m-1)! \sum_{j=1}^{m-1} \frac{1}{j}$.

For $k = 0, 1, 2, ...,$

Dilcher defined the gamma function as [1]

$$\Gamma_k^*(z) = \lim_{n\to\infty} \frac{\exp\left(\frac{1}{k+1} \ln^{k+1} n^z\right) \prod_{j=1}^{n} \exp\left(\frac{1}{k+1} \ln^{k+1} j\right)}{\prod_{j=0}^{n} \exp\left(\frac{1}{k+1} \ln^{k+1} (j+z)\right)}, \tag{2.12}$$

where $z \in \mathbb{C} \setminus \mathbb{R}$.

This definition excluded non positive real numbers in its domain. That means it is not possible to find $\Gamma_k^*(-\frac{1}{2})$ using Dilcher's definition.

The functional equation was obtained as

$$\Gamma_k^*(z+1) = \exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\Gamma_k^*(z). \tag{2.13}$$

The Weierstrass form of the generalized gamma function was also established as

$$\frac{1}{\Gamma_k^*(z)} = e^{\gamma_k z}\exp\left(\frac{1}{k+1}\ln^{k+1} z\right) \times \prod_{n=1}^{\infty}\left(-\frac{z}{n}\ln^k n\right)\exp\left(\frac{1}{k+1}\left(\ln^{k+1}(n+z)-\ln^{k+1} n\right)\right). \tag{2.14}$$

It was also discovered in [1] that for $|z|<1$,

$$\ln\Gamma_k^*(z+1) = -\gamma_k z + (-1)^k k!\sum_{n=2}^{\infty}\frac{z^n}{n!}\sum_{j=1}^{n}\frac{(-1)^j s(n,j)}{(k+1-j)!}\zeta^{(k+1-j)}(n), \tag{2.15}$$

and a generalized Euler reflection formula given as

$$\Gamma_k^*(z)\Gamma_k^*(1-z) = \frac{1}{s_k(z)}\exp\left(-\frac{1}{k+1}\ln^{k+1} z\right), \tag{2.16}$$

where

$$s_k(z) = \prod_{n=1}^{\infty}\exp\left(\frac{1}{k+1}\left(\ln^{k+1}(n+z)+\ln^{k+1}(n-z)-2\ln^{k+1} n\right)\right). \tag{2.17}$$

In particular, $s_0 = \frac{\sin(\pi z)}{\pi z}$.

A consequence of (2.16) yields

$$\Gamma_k^*(1+z)\Gamma_k^*(1-z) = \frac{1}{s_k(z)}. \tag{2.18}$$

Lemma 2.1. *(Dilcher, 1994)*

Let $z \in D$ be fixed and $k \in \mathbb{N}$. The identity

$$\ln^k(n+z) - \ln^k n = \frac{z}{n}k\ln^{k-1} n + O\left(\frac{1}{n^2}\ln^{k-1} n\right) \tag{2.19}$$

holds as $n \to \infty$.

3. Main Results

We begin this section by presenting a new generalized gamma function which is pivotal in achieving further results of this paper.

A new generalization of the gamma function is introduced as follows:

Definition 3.1. Let $z \in \mathbb{C} \setminus (\mathbb{Z}^- \cup 0)$ and $k \in \mathbb{N}_0$. Then

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^z\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)}. \tag{3.1}$$

Remark 3.2. For $k = 0$,

$$\Gamma_0(z) = \Gamma(z).$$

We check the validity of Definition 3.1 by showing that the conditions of the Bohr-Mullerup theorem are satisfied.

Theorem 3.3. *(Bohr-Mullerup)*

Let f(z) be a positive function on $(0,\infty)$. *Suppose that*

(a) $f(1) = 1$,

(b) $f(z+1) = \exp\left(\frac{1}{k+1}\ln^{k+1} z\right) f(z)$,

(c) $\ln f(z)$ *is convex.*

Then, $f(z) = \Gamma_k(z)$.

Proof.

(a) $\Gamma_k(1) = 1$.

(b) Replacing z by $z+1$ in (3.1), we have

$$\begin{aligned}
\Gamma_k(z+1) &= \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^{z+1}\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1}(z+1)\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z+1}{j}\right)\right)}, \\
&= \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^{z}\right)\exp\left(\frac{1}{k+1}\ln^{k+1}\left[1+\frac{1}{j}\right]\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1}(z+1)\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z+1}{j}\right)\right)}, \\
&= \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^{z}\right)}{\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}(z+j)\right)}, \\
&= \frac{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)} \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^{z}\right)}{\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}(z+j)\right)}, \\
&= \exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\Gamma_k(z). \qquad (3.2)
\end{aligned}$$

Remark 3.4. (3.2) was also established in [1] using Lemma 2.1.

(c) Taking logarithm on both sides of (3.1), we obtain

$$\ln\Gamma_k(z) = \sum_{j=1}^{\infty}\frac{z}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right) - \frac{1}{k+1}\ln^{k+1} z - \sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1}(z+j) - \sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1} j. \qquad (3.3)$$

Differentiating, we get

$$\ln\Gamma_k(z)' = \sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right) - \frac{\ln^k z}{z} - \sum_{j=1}^{\infty}\frac{\ln^k(z+j)}{(z+j)}. \qquad (3.4)$$

Thus,

$$\ln\Gamma_k(z)'' = \sum_{j=0}^{\infty}\left(\frac{k\ln^{k-1}(\frac{1}{z+j})}{(z+j)^2} + \frac{\ln^k(z+j)}{(z+j)^2}\right). \qquad (3.5)$$

For $z \in \mathbb{R}^+$, we have

$$\ln\Gamma_k(z)'' \geq 0, \qquad (3.6)$$

and the proof is complete. □

Now, we present the following theorem which establishes the equality of equations (2.12) and (3.1) for the same domain.

Theorem 3.5. *Let $k \in \mathbb{N}_0$ and $z \in \mathbb{C} \setminus \mathbb{R}^-$. Then*

$$\Gamma_k(z) = \Gamma_k^*(z). \tag{3.7}$$

Proof. From (3.1), we have $\Gamma_k(z) =$

$$\lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^{z}\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}.$$

This simplifies to

$$\begin{aligned}\Gamma_k(z) = &= \lim_{n\to\infty} \frac{\exp\left(\frac{1}{k+1}\ln^{k+1}\left(\prod_{j=1}^{n}\left(1+\frac{1}{j}\right)\right)^{z}\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}(z+j)\right)},\\ &= \Gamma_k^*(z).\end{aligned}$$

Alternatively, from (2.12), we have,

$$\begin{aligned}\Gamma_k^*(z) &= \lim_{n\to\infty} \frac{\exp\left(\frac{1}{k+1}\ln^{k+1} n^{z}\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}{\prod_{j=0}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}(j+z)\right)},\\ &= \lim_{n\to\infty} \frac{\exp\left(\frac{1}{k+1}\ln^{k+1}\left(\prod_{j=1}^{n}\left(1+\frac{1}{j}\right)\right)^{z}\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}(z+j)\right)},\\ &= \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^{z}\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1} j\right)}.\end{aligned}$$

Hence,

$$\Gamma_k^*(z) = \Gamma_k(z).$$

□

Theorem 3.6. *Let $z \in D = \mathbb{C} \setminus (\mathbb{Z}^- \cup 0)$. Then the identity*

$$\frac{1}{\Gamma_k(z)} = e^{\gamma_k z}\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{\infty}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)\exp\left(\frac{-z}{j}\ln^{k} j\right). \tag{3.8}$$

holds, where γ_k is the generalized Euler-Mascheroni constant.

Proof. From (3.1) and using $j = \prod_{j=1}^{n} \left(1+\frac{1}{j}\right)$, we obtain

$$\frac{1}{\Gamma_k(z)} = \lim_{m\to\infty} \frac{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} j^z\right)}. \tag{3.9}$$

Taking logarithm on both sides of (3.9) and applying Lemma 2.1 gives

$$\ln\left(\frac{1}{\Gamma_k(z)}\right) = \frac{-z}{j}\ln^k j + \frac{1}{k+1}\ln^{k+1} z + \sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right). \tag{3.10}$$

$$\frac{1}{\Gamma_k(z)} = \exp\left(\frac{-z}{j}\ln^k j\right)\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\exp\left(\sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right). \tag{3.11}$$

By introducing convergence factors we obtain

$$\frac{1}{\Gamma_k(z)} = \exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\exp(\gamma_k z)\prod_{j=1}^{\infty}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)\exp\left(\frac{-z}{j}\ln^k j\right), \tag{3.12}$$

where γ_k is the generalized gamma function.

This completes the proof. □

Remark 3.7. Equation (3.8) is an improvement of the generalized reciprocal gamma function in [1].

Remark 3.8. By applying logarithm on both sides of (3.8), letting $k = 0$ and using (2.9), we obtain

$$\frac{1}{\Gamma(z)} = z\exp\left(\gamma z + \sum_{m=2}^{\infty}\frac{(-1)^{m+1}}{m}\zeta(m)z^m\right), \tag{3.13}$$

a result also found in [1], [2] and [5].

Remark 3.9. Substituting $z = 1$ into (3.13) yields

$$1 = \exp\left(\gamma + \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m}\zeta(m)\right). \tag{3.14}$$

By further taking logarithm on both sides of (3.14) gives

$$\gamma = \sum_{m=2}^{\infty} \frac{(-1)^{m}}{m}\zeta(m), \tag{3.15}$$

a result due to Euler.

Theorem 3.10. *The identities*

$$\Gamma_k(z)\Gamma_k(1-z) = \frac{1}{t_k(z)} \tag{3.16}$$

and

$$\Gamma_k(1+z)\Gamma_k(1-z) = \frac{1}{\prod_{j=1}^{\infty} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1-\frac{z^2}{j^2}\right)\right)} \tag{3.17}$$

hold for $z \in D$, where

$$t_k(z) = \exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{\infty}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1-\frac{z^2}{j^2}\right)\right). \tag{3.18}$$

Proof. Replacing z by $-z$ in (3.2) and multiplying the result by $\Gamma_k(z)$, we have

$$\Gamma_k(1-z)\Gamma_k(z) = \exp\left(\frac{1}{k+1}\ln^{k+1} -z\right)\Gamma_k(-z)\Gamma_k(z). \tag{3.19}$$

Using (3.1) we obtain

$$\Gamma_k(z)\Gamma(-z) = \lim_{n\to\infty} \frac{1}{\exp\left(\frac{1}{k+1}\ln^{k+1} -z^2\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1-\frac{z^2}{j^2}\right)\right)}. \tag{3.20}$$

Simplifying further yields

$$\exp\left(\frac{1}{k+1}\ln^{k+1} -z\right)\Gamma_k(z)\Gamma(-z) = \lim_{n\to\infty} \frac{1}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1-\frac{z^2}{j^2}\right)\right)}. \tag{3.21}$$

This completes the proof of the first part of the theorem.

Obtaining the second part of the theorem using (3.2), we have

$$\Gamma_k(z) = \frac{\Gamma_k(z+1)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)}. \tag{3.22}$$

Substituting (3.22) into (3.16) completes the proof. □

Remark 3.11. For $k=0$, (3.16) yields

$$\Gamma(1-z)\Gamma(z)=\frac{1}{t_0(z)}=\frac{1}{z\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right)}=\frac{\pi}{\sin(\pi z)}. \tag{3.23}$$

Lemma 3.12. *For* $|z|<1$,

(3.24)

$$\ln^{k+1}(j+z)-\ln^{k+1}j=\frac{z}{j}(k+1)\ln^k j+\sum_{m=2}^{\infty}\left(\frac{z}{j}\right)^m\frac{1}{m!}\sum_{n=1}^{m}\frac{(k+1)!}{(k+1-n)!}s(m,n)\ln^{k+1-n}j.$$

Proof.

$$\ln^{k+1}(j+z)-\ln^{k+1}j=\left(\ln\left(1+\frac{z}{j}\right)+\ln j\right)^{k+1}-\ln^{k+1}j, \tag{3.25}$$

$$=\sum_{n=1}^{k+1}\binom{k+1}{n}\ln^n\left(1+\frac{z}{j}\right)\ln^{k+1-n}j. \tag{3.26}$$

By (2.9) we obtain

$$\begin{aligned}\ln^{k+1}(j+z)-\ln^{k+1}j&=\sum_{n=1}^{k+1}\frac{(k+1)!}{(k+1-n)!}\ln^{k+1-n}j\sum_{m=n}^{\infty}\frac{s(m,n)}{m!}\left(\frac{z}{j}\right)^m,\\&=\sum_{m=1}^{\infty}\left(\frac{z}{j}\right)^m\frac{1}{m!}\sum_{n=1}^{m}\frac{(k+1)!}{(k+1-n)!}s(m,n)\ln^{k+1-n}j.\end{aligned} \tag{3.27}$$

The term belonging to $m=1$ is given by

$$\frac{z}{j}(k+1)s(1,1)\ln^k j=\frac{z}{j}(k+1)\ln^k j \tag{3.28}$$

Substituting (3.28) into (3.27) completes the proof. □

Theorem 3.13. *For* $-1<z\leq 1$, *the identity*

$$\ln\Gamma_k(z+1)=-\gamma_k-(-1)^k k!\sum_{m=2}^{\infty}\frac{z^m}{m!}\sum_{n=1}^{m}\frac{s(m,n)}{(k+1-n)!}\zeta^{(k+1-n)}(m) \tag{3.29}$$

holds.

Proof. Applying logarithm on both sides of (3.2) gives

$$\ln\Gamma_k(z+1)=\frac{1}{k+1}\ln^{k+1}z+\ln\Gamma_k(z). \tag{3.30}$$

Applying logarithm on (3.8), we obtain

$$\ln\Gamma_k(z) = -\gamma_k z - \frac{1}{k+1}\ln^{k+1} z - \sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right) + \sum_{j=1}^{\infty}\frac{z}{j}\ln^k j. \tag{3.31}$$

Substituting (3.31) into (3.30) yields

$$\ln\Gamma_k(z+1) = -\gamma_k z - \sum_{j=1}^{\infty}\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right) + \sum_{j=1}^{\infty}\frac{z}{j}\ln^k j. \tag{3.32}$$

By Lemma 3.11 , we get

$$\ln\Gamma_k(z+1) = -\gamma_k z - \sum_{m=2}^{\infty}\frac{z^m}{m!}\sum_{n=1}^{m}\frac{k!}{(k+1-n)!}s(m,n)\sum_{j=1}^{\infty}\frac{\ln^{k+1-n} j}{j^m}. \tag{3.33}$$

By (2.8) the proof of the theorem is complete. □

Remark 3.14. By letting $k=0$ and using (2.9), we obtain

$$\ln\Gamma(z+1) = -\gamma z + \sum_{m=2}^{\infty}\frac{(-1)^m}{m}z^m\zeta(m). \tag{3.34}$$

Equation (3.34) was also found in [1] and [5].

Remark 3.15. If $z=1$ is put in (3.34), (3.15) is obtained, a result due to Euler.

Remark 3.16. For $k=1$ and applying (2.9), we obtain

$$\ln\Gamma_1(z+1) = -\gamma_1 z + \sum_{m=2}^{\infty}\frac{(-1)^m z^m}{m}\left(\zeta'(m) - \zeta(m)\sum_{i=1}^{m-1}\frac{1}{i}\right). \tag{3.35}$$

Remark 3.17. By letting $k=0$ and substituting $z=\frac{1}{2}$ into (3.33) yields

$$2\ln\left(\frac{\sqrt{\pi}}{2}\right) = -\gamma + 2\sum_{m=2}^{\infty}\frac{(-1)^m}{m2^m}\zeta(m), \tag{3.36}$$

a result that is also established in [5].

Lemma 3.18. *Let* $|z|<1$ *and* $k\in\mathbb{N}_0$. *Then*

$$\ln^{k+1}\left(1+\frac{1}{j}\right) = \frac{(k+1)}{j}\ln^k j + \sum_{m=2}^{\infty}\left(\frac{1}{j}\right)^m\frac{1}{m!}\sum_{n=1}^{m}\frac{(k+1)!}{(k+1-n)!}\ln^{k+1-n} j. \tag{3.37}$$

The proof follows the same procedure as that of Lemma 3.11.

Theorem 3.19. *Let $z=-\frac{1}{2}$ and $k=0$. Then*

$$\Gamma\left(-\frac{1}{2}\right)=-2\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^m\zeta(m)}{m}+\sum_{m=2}^{\infty}\frac{\zeta(m)}{m*2^m}\right). \tag{3.38}$$

Proof. By Lemma 3.11 and Lemma 3.17, (3.1) becomes

$$\Gamma_k(z)=\frac{\exp\left(\sum_{m=2}^{\infty}\left(\frac{z}{m!}\right)\sum_{n=1}^{m}\frac{k!s(m,n)}{(k+1-n)!}\zeta^{(k+1-n)}(m)\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1}z\right)*\exp\left(\sum_{m=2}^{\infty}\left(\frac{z^m}{m!}\right)\sum_{n=1}^{m}\frac{k!s(m,n)}{(k+1-n)!}\zeta^{(k+1-n)}(m)\right)}. \tag{3.39}$$

By letting $z=-\frac{1}{2}$ and $k=0$, we obtain

$$\Gamma\left(-\frac{1}{2}\right)=\frac{\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\left(\frac{1}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}{\exp\left(\ln\left(\frac{-1}{2}\right)\right)\exp\left(\sum_{m=2}^{\infty}\left(\frac{(-1)^m\left(\frac{1}{2}\right)^m}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}, \tag{3.40}$$

$$=\frac{\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^{m+1}(m-1)!}{m(m-1)!}\zeta(m)\right)}{-\frac{1}{2}\exp\left(\sum_{m=2}^{\infty}\frac{(-1)^m*(-1)^{m+1}(m-1)!}{2^m*m(m-1)!}\zeta(m)\right)}. \tag{3.41}$$

This completes the proof. □

Theorem 3.20.

$$\Gamma(i)=\exp\left(\sum_{m=2}^{\infty}\frac{(i)^{(m)}(-1)^m\zeta(m)}{m}-i\gamma-\ln(i)\right). \tag{3.42}$$

Proof. By substituting $z=i$ and $k=0$ into (3.39), we get

$$\Gamma(i)=\frac{\exp\left(\sum_{m=2}^{\infty}\left(\frac{i}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}{\exp(\ln i)*\exp\left(\sum_{m=2}^{\infty}\left(\frac{i^m}{m!}\right)\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m)\right)}, \tag{3.43}$$

$$=\exp\left(\sum_{m=2}^{\infty}\frac{(i)^{(m)}(-1)^m\zeta(m)}{m}-i\gamma-\ln(i)\right). \tag{3.44}$$

This completes the proof. □

Remark 3.21. Using the Wolfram Infinite series analyzer to analyze $\sum_{m=2}^{\infty} \frac{(i)^{(m)}(-1)^m \zeta(m)}{m}$ yields

$$\Gamma(i) = \exp\left(\ln\left(\Gamma(1+i) - \ln(i)\right)\right), \tag{3.45}$$

$$= \exp\left(\ln\left(-\Gamma(1+i)\right) - i\pi - \ln(i)\right), \tag{3.46}$$

$$= \exp\left(\ln\left(\left(-\frac{1}{2}+\frac{i}{2}\right)\right)(1+i)! - i\pi - \ln(i)\right). \tag{3.47}$$

Theorem 3.22. *Let $k = 1$ and $z = -\frac{1}{2}$. Then*

$$\Gamma_1\left(-\frac{1}{2}\right) = \exp\left(\frac{1}{2}\left(\sum_{m=2}^{\infty} \frac{(-1)^m \zeta'(m)}{m} - \sum_{m=2}^{\infty} \frac{(-1)^m H_{m-1}\zeta(m)}{m}\right)\right) \times \tag{3.48}$$

$$\exp\left(-\frac{1}{2}\ln^2\left(\frac{-1}{2}\right) + \sum_{m=2}^{\infty} \frac{\zeta'(m)}{2^m m} - \sum_{m=2}^{\infty} \frac{H_{m-1}\zeta(m)}{2^m m}\right), \tag{3.49}$$

where H_{m-1} is the (m-1)th harmonic number.

Proof. By substituting $k = 1$ and $z = -\frac{1}{2}$ into (3.39), we get

$$\Gamma_1\left(-\frac{1}{2}\right) = \frac{\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\left(\frac{1}{m!}\right)\sum_{n=1}^{m} \frac{1!s(m,n)}{(2-n)!}\zeta^{(2-n)}(m)\right)}{\exp\left(\frac{1}{2}\ln^2\left(\frac{-1}{2}\right)\right)\exp\left(\sum_{m=2}^{\infty}\left(\frac{(-1)^m\left(\frac{1}{2}\right)^m}{m!}\right)\sum_{n=1}^{m}\frac{1!s(m,n)}{(2-n)!}\zeta^{(2-n)}(m)\right)}, \tag{3.50}$$

$$= \frac{\exp\left(-\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m!}\left(\frac{s(m,1)\zeta'(m)}{1!} + \frac{s(m,2)\zeta(m)}{0!}\right)\right)}{\exp\left(\frac{1}{2}\ln^2\left(\frac{-1}{2}\right)\right) * \exp\left(\sum_{m=2}^{\infty}\frac{(-1)^m}{m!*2^m}\left(\frac{s(m,1)\zeta'(m)}{1!} + \frac{s(m,2)\zeta(m)}{0!}\right)\right)}. \tag{3.51}$$

By (2.9), we obtain

$$\Gamma_1\left(-\frac{1}{2}\right) = \frac{\exp\left(\frac{1}{2}\left(\sum_{m=2}^{\infty}\frac{(-1)^m\zeta'(m)}{m} - \sum_{m=2}^{\infty}\frac{(-1)^m H_m\zeta(m)}{m}\right)\right)}{\exp\left(\frac{1}{2}\ln^2\left(\frac{-1}{2}\right)\right) * \exp\left(\sum_{m=2}^{\infty}\frac{-\zeta'(m)}{m*2^m} + \sum_{m=2}^{\infty}\frac{H_{m-1}\zeta(m)}{m*2^m}\right)}. \tag{3.52}$$

This completes the proof. □

Remark 3.23. For $k = 1$, $z = -\frac{1}{2}$ and using (3.2), we obtain

$$\Gamma_1\left(\frac{1}{2}\right) = \exp\left(\frac{1}{2}\left(\sum_{m=2}^{\infty}\frac{(-1)^m\zeta'(m)}{m} - \sum_{m=2}^{\infty}\frac{(-1)^m H_m\zeta(m)}{m}\right)\right) \times$$

$$\exp\left(\sum_{m=2}^{\infty}\frac{\zeta'(m)}{2^m m} - \sum_{m=2}^{\infty}\frac{H_{m-1}\zeta(m)}{2^m m}\right),. \tag{3.53}$$

Remark 3.24. Substituting $z=-\frac{1}{2}$ in (3.35) and using (3.22) yields

$$\gamma_1=\sum_{m=2}^{\infty}\frac{(-1)^m\zeta'(m)}{m}-\sum_{m=2}^{\infty}\frac{(-1)^m H_{m-1}\zeta(m)}{m}. \tag{3.54}$$

Remark 3.25. For $k=1$ and $z=2$, we obtain

$$\Gamma_1(2)=\exp\left(-2\gamma_1-\frac{1}{2}\ln^2 2+\sum_{m=2}^{\infty}\frac{(-1)^m}{m2^m}\left(\zeta'(m)-H_{m-1}\zeta(m)\right)\right). \tag{3.55}$$

Remark 3.26. For $k=1$, $z=3$ and using (3.22), we get

$$\Gamma_1(3)=\exp\left(-2\gamma_1+\sum_{m=2}^{\infty}\frac{(-1)^m}{m2^m}\left(\zeta'(m)-H_{m-1}\zeta(m)\right)\right). \tag{3.56}$$

4. Conclusions

A generalized gamma function has been presented. Some properties of this generalized gamma function have been established and a generalized Euler-Mascheroni constant obtained for $k=1$. For some values of k and z, some identities of the generalized gamma function were established.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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