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THE SOLUTIONS OF SOME CERTAIN NON-HOMOGENEOUS FRACTIONAL INTEGRAL EQUATIONS

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Abstract. In this paper, we propose the solutions of non-homogeneous fractional integral equations of the form

$$I_{0^+}^{2\sigma} y(t) + a \cdot I_{0^+}^{\sigma} y(t) + b \cdot y(t) = t^n,$$

and

$$I_{0^+}^{2\sigma} y(t) + a \cdot I_{0^+}^{\sigma} y(t) + b \cdot y(t) = t^n e^t,$$

where $I_{0^+}^{\sigma}$ is the Riemann-Liouville fractional integral of order $\sigma = 1/2, \sigma = 1, n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}^+$, and a, b are constants, by using the Laplace transform technique. We obtain the solutions of these equations are in the form of Mellin-Ross function and in the form of exponential function.

Keywords: Laplace transform; fractional differential equations; fractional integral equations; Riemann-Liouville fractional integral.

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1. Introduction

Fractional calculus is the theory of derivatives and integrals of arbitrary complex or real order. It began in 1695 when G. F. A. L'Hôpital asked G. W. Leibniz to give the meaning of $d^n y/dx^n$, where $n = 1/2$. In predictive answer, G. W. Leibniz expects the beginning of the area presently is named fractional calculus. Since that time, fractional calculus has interested many mathematicians such as L. Euler, H. Laurent, P. S. Laplace, J. B. J. Fourier, N. H. Abel, J. Liouville, and G. F. B. Riemann, etc. It has been shown that fractional calculus is very useful and active in mathematical areas.

Fractional derivative is a part of fractional calculus which has been of interest in recent years. It plays a key role in modeling phenomena with different branches of engineering and science in a real-world problem, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Many mathematical models of real problems appearing in various fields of engineering and science were established with the help of fractional calculus such as dielectric polarization, viscoelastic, electromagnetic waves, and electrode-electrolyte polarization, see [16, 17, 18, 19, 20, 21, 22, 23].

In addition, of course, the theory of fractional integral has been of interest in recent years, see [24, 25, 26, 27, 28, 29, 30, 31, 32]. In 1812, P. S. Laplace defined a fractional derivative through an integral. He developed it as a mere mathematical exercise generalizing from a case of integer order. Later, in 1832, J. Liouville recommended a definition based on the formula for differentiating the exponential function known as the first Liouville definition. Next, he presented the second definition formula in terms of an integral, called Liouville, to integrate noninteger order. After that J. Liouville and G. F. B. Riemann developed an approach to noninteger order derivatives in terms of convergent series, conversely to the Riemann-Liouville approach, that was given an integral. Many researchers focused on developing the theoretical aspects, methods of solution, and applications of fractional integral equations see [30, 31, 32, 33, 34, 35, 36, 37, 38].

In 2005, T. Morita [6] studied the initial value problem of fractional differential equations by using the Laplace transform. He obtained the solutions to the fractional differential equations with Riemann-Liouville fractional derivative and Caputo fractional derivative or its modification. In 2010, T. Morita and K. Sato [8] studied the initial value problem of fractional differential equations with constant coefficients of the form

$$ {}_0D_t^{\alpha}u(t)+c\cdot u(t)=f(t), $$

$$ {}_0D_t^{\alpha}u(t)+b\cdot {}_0D_t^{\beta}u(t)+c\cdot u(t)=f(t), $$

and

$$ {}_0D_t^{\sigma_m}u(t)+\sum_{l=0}^{m-1}c_l\cdot {}_0D_t^{\sigma_l}u(t)=f(t), $$

where ${}_0D_t^{\sigma_m}$ is the Riemann-Liouville fractional derivative, c_l are constants for $l=0,1,2,\dots,$ $m-1$, and $t\in\mathbb{R}^+$. They obtained solutions in terms of the Green's function and distribution theory. Next, they studied the solution of a fractional differential equation of the form

$$ (a_2t+b_2)_0D_t^{2\sigma}u(t)+(a_1t+b_1)_0D_t^{\sigma}u(t)+(a_0t+b_0)u(t)=f(t), $$

where $\sigma=1,\sigma=1/2,t\in\mathbb{R}^+$, and a_i,b_i are constants for $i=0,1,2$, see [9] for more details.

In 1996, A. A. Kilbas and M. Saigo [33] introduced the connections of the Mittag-Leffler type function with the Riemann-Liouville fractional integrals and derivatives. Their applications are to solve the linear Abel-Volterra integral equations.

In 2015, R. Agarwal et al. [34] studied the solutions of fractional Volterra integral equation with Caputo fractional derivative using the integral transform of Pathway type. They discussed the solution of the non-homogeneous time-fractional heat equation in a spherical domain.

In 2017, C. Li et al. [32] studied a generalized Abel's integral equation and its variant in the distributional (Schwartz) sense based on fractional calculus of distributions. Next, in 2018, C. Li and K. Clarkson [35] studied Abel's integral equation of the second kind:

$$ y(t)+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-\lambda)^{\alpha-1}y(\tau)d\tau=f(t),\quad t>0, \tag{1} $$

where Γ is the gamma function, λ is a constant, and $\alpha\in\mathbb{R}$. Equation (1) can be written in the form

$$ \left(1+\lambda I_{0^+}^{\alpha}\right)y(t)=f(t), $$

where $I_{0^+}^{\alpha}$ is the Riemann-Liouville fractional integral. They applied Babenko's method and fractional integral for solving the above equation.

The linear fractional order integral equations with constant coefficients of the form

$$c_1 I_{a^+}^{\alpha_1} y(t) + c_2 I_{a^+}^{\alpha_2} y(t) + \cdots + c_n I_{a^+}^{\alpha_n} y(t) = f(t), \tag{2}$$

where $a \in \mathbb{R}, \alpha_i \in \mathbb{Q}^+, \alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0, c_i \in \mathbb{C}$, for $i \in \{1,2,\ldots,n\}$, and f is assumed to be a real valued function of real variable defined on an interval (a,b). The general solution of (2) can be found in [28] for $\alpha_i \in \mathbb{R}$ which is in the space S'_+ of tempered distributions with support in $[0,\infty)$.

In 2017, D. C. Labora and R. Rodriguez-Lopez [37] showed a new method by applying a suitable fractional integral operator for solving some fractional order integral equations with constant coefficients, and all the integration orders involving are rational. Next, they applied and extended ideas presented in [37] for solving fractional integral equations with Riemann-Liouville definition; see [31] for more details. Moreover, they studied the fractional integral equations with Caputo derivatives and non-rational orders by limiting fractional integral equations with rational orders.

As mentioned in the abstract, we propose the solutions of non-homogeneous fractional integral equations of the form

$$I_{0^+}^{2\sigma} y(t) + a \cdot I_{0^+}^{\sigma} y(t) + b \cdot y(t) = t^n,$$

and

$$I_{0^+}^{2\sigma} y(t) + a \cdot I_{0^+}^{\sigma} y(t) + b \cdot y(t) = t^n e^t,$$

where $I_{0^+}^{\sigma}$ is the Riemann-Liouville fractional integral of order $\sigma = 1/2, \sigma = 1, n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}^+$, and a,b are constants by using the Laplace transform technique and its variants in the classical sense. In Section 2, we introduce definitions of the Riemann-Liouville fractional integral and the Laplace transform which will help us to obtain our main results. In Section 3, we establish our main results and some examples as a consequently of our main results. Finally, we give the conclusions in Section 4.

2. PRELIMINARIES

Before we proceed to the main results, the following definitions, lemmas, and concepts are required.

Definition 2.1. [23] Let α be a constant, v a real number and t a positive real number. The Mellin-Ross function $E_t(v,\alpha)$ is defined by

$$E_t(v,\alpha) = t^v e^{\alpha t}\Gamma^*(v,\alpha t),$$

where Γ^* is the incomplete gamma function:

$$\Gamma^*(v,t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(v+k+1)},$$

in which Γ is the gamma function.

In addition, if $v>0$, then $E_t(v,\alpha)$ has an integral representation as

$$E_t(v,\alpha) = \frac{1}{\Gamma(v)}\int_0^t x^{v-1}e^{\alpha(t-x)}dx.$$

Example 2.1. Let α be a constant, v a real number and t a positive real number. Some special values and recursion relations of Mellin-Ross function needed for our calculations are as follows:

(i) $E_t(0,\alpha) = e^{\alpha t}$;

(ii) $E_t(v,0) = \dfrac{t^v}{\Gamma(v+1)}$;

(iii) $E_t(1,\alpha) = \dfrac{E_t(0,\alpha)-1}{\alpha}$;

(iv) $E_t\left(-\dfrac{1}{2},\alpha\right) = \alpha E_t\left(\dfrac{1}{2},\alpha\right) + \dfrac{t^{-1/2}}{\sqrt{\pi}}$;

(v) $E_t(v,\alpha) = \alpha E_t(v+1,\alpha) + \dfrac{t^v}{\Gamma(v+1)}$.

Definition 2.2. [23] Let $f(t)$ be piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval of $[0,\infty)$. Then the Riemann-Liouville fractional integral of $f(t)$ of order v is defined by

$$I_{0+}^v f(t) = \frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f(x)dx,$$

where $v\in\mathbb{R}^+$.

Example 2.2. Let α be a constant, μ a real number, v and t positive real numbers. Then the following Riemann-Liouville fractional integrals hold:

(i) $I_{0+}^v t^\mu = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)}t^{\mu+v}, \quad \mu > -1$;

(ii) $I_{0+}^{v} e^{\alpha t} = E_t(v, \alpha)$;

(iii) $I_{0+}^{v} \left[t e^{\alpha t} \right] = t E_t(v, \alpha) - v E_t(v+1, \alpha)$;

(iv) $I_{0+}^{v} \left[E_t(\mu, \alpha) \right] = E_t(\mu + v, \alpha), \quad \mu > -1$;

(v) $I_{0+}^{v} \left[t E_t(\mu, \alpha) \right] = t E_t(\mu + v, \alpha) - v E_t(\mu + v + 1, \alpha), \quad \mu > -2.$

Definition 2.3. [23] Let $f(t)$ be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$. Then Riemann-Liouville fractional derivative ${}_0D_t^{\beta} f(t)$ is defined by

$${}_0D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-x)^{n-\beta-1} f(x) dx,$$

where $\beta \in \mathbb{R}^+$ and n is an integer that satisfies $n - 1 \leq \beta < n$.

Definition 2.4. [23] Let $f(t)$ be a function satisfying the conditions in Definition 2.2 and of exponential order v where $v \in \mathbb{R}^+$. The Laplace transform of $f(t)$ is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt,$$

where *Re* $s > v$.

Example 2.3. Let α be a constant, n a real number, v and t positive real numbers. Then the following Laplace transforms hold:

(i) $\mathscr{L}\{1\} = \dfrac{1}{s}, \quad s > 0$;

(ii) $\mathscr{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0, n > -1$;

(iii) $\mathscr{L}\left\{e^{\alpha t}\right\} = \dfrac{1}{s-\alpha}, \quad s > \alpha$;

(iv) $\mathscr{L}\left\{t^n e^{\alpha t}\right\} = \dfrac{\Gamma(n+1)}{(s-\alpha)^{n+1}}, \quad s > \alpha, n > 0$;

(v) $\mathscr{L}\{E_t(v, \alpha)\} = \dfrac{1}{s^v (s-\alpha)}, \quad s > \alpha.$

Lemma 2.1. [23] Let $f(t)$ be a function satisfying the conditions in Definition 2.2 and of exponential order v where $v \in \mathbb{R}^+$. Then

$$\mathscr{L}\left[I_{0+}^{v} f(t)\right] = s^{-v} \mathscr{L}\left[f(t)\right].$$

Definition 2.5. Let $f(t)$ be a function satisfying the conditions in Definition 2.4 and $\mathscr{L}\{f(t)\} = F(s)$. The inverse Laplace transform of $F(s)$ is defined by

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{c-i\omega}^{c+i\omega} F(s)e^{st}ds,$$

where $Re(s) > \sigma_a, \sigma_a$ is an abscissa of absolute convergence for $\mathscr{L}\{f(t)\}$.

Example 2.4. Let α be a constant, v a real number, n and t positive real numbers. Then the following inverse Laplace transforms hold:

(i) $\mathscr{L}^{-1}\left\{\frac{1}{s^{v+1}}\right\} = \frac{t^v}{\Gamma(v+1)}, \quad v > 0;$

(ii) $\mathscr{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = E_t(0,\alpha) = e^{\alpha t};$

(iii) $\mathscr{L}^{-1}\left\{\frac{\Gamma(n+1)}{(s-\alpha)^{n+1}}\right\} = t^n e^{\alpha t};$

(iv) $\mathscr{L}^{-1}\left\{\frac{1}{s^{1/2}-\alpha}\right\} = E_t\left(-\frac{1}{2},\alpha^2\right) + \alpha E_t\left(0,\alpha^2\right);$

(v) $\mathscr{L}^{-1}\left\{\frac{1}{s^v(s-\alpha)^2}\right\} = tE_t(v,\alpha) - vE_t(v+1,\alpha), \quad v > -2.$

Lemma 2.2. [23] Let n be a positive integer, α be a constant, v be a real number, and t be a positive real number. Then

$$\mathscr{L}^{-1}\left\{\frac{1}{s^v(s-\alpha)^n}\right\} = \frac{1}{(n-1)!\Gamma(v)}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\Gamma(v+i)t^{n-1-i}E_t(v+i,\alpha),$$

where $v > -n$.

3. Main Results

In this section, we will state our main results and give their proofs.

Theorem 3.1. Consider the non-homogeneous fractional integral equation of the form

$$I_{0^+}^{2\sigma}y(t) + a\cdot I_{0^+}^{\sigma}y(t) + b\cdot y(t) = t^n, \tag{3}$$

where $I_{0^+}^{\sigma}$ is the Riemann-Liouville fractional integral of order $\sigma = 1/2, \sigma = 1, n \in \mathbb{N}\cup\{0\}, a, b$ are constants and $t \in \mathbb{R}^+$. Then the solutions of (3) are as the follows:

(i) If $\sigma = 1/2$, and $j,k \in \mathbb{R}\setminus\{0\}$ with $j \neq k$ such that $a = j+k$ and $b = jk$, then the solution of (3) is of the form

$$\begin{aligned} y(t) = {} & \frac{n!}{j-k}\sum_{i=0}^{2n}(-1)^i\left[\frac{j^{2n-i+1}-k^{2n-i+1}}{\Gamma(i/2)}\right]t^{(i-2)/2}+\frac{n!\,j^{2n}}{j-k}\left[E_t\left(-\frac{1}{2},\frac{1}{j^2}\right)-\frac{1}{j}E_t\left(0,\frac{1}{j^2}\right)\right] \\ & -\frac{n!\,k^{2n}}{j-k}\left[E_t\left(-\frac{1}{2},\frac{1}{k^2}\right)-\frac{1}{k}E_t\left(0,\frac{1}{k^2}\right)\right]. \end{aligned} \tag{4}$$

(ii) If $\sigma = 1$, and $j,k \in \mathbb{R}\setminus\{0\}$ with $j \neq k$ such that $a = j+k$ and $b = jk$ then the solution of (3) is of the form

$$y(t) = \frac{n!}{j-k}\sum_{i=0}^{n}(-1)^{n-i+1}\left[\frac{j^{n-i}-k^{n-i}}{\Gamma(i)}\right]t^{i-1}+\frac{(-1)^{n+1}n!}{j-k}\left[j^{n-1}e^{-t/j}-k^{n-1}e^{-t/k}\right]. \tag{5}$$

Proof. Applying the Laplace transform to both sides of (3), we have

$$\mathscr{L}\{I_{0^+}^{2\sigma}y(t)\}+a\mathscr{L}\{I_{0^+}^{\sigma}y(t)\}+b\mathscr{L}\{y(t)\}=\mathscr{L}\{t^n\}. \tag{6}$$

Using Lemma 2.1, Example 2.3 (ii), and denoting the Laplace transform $\mathscr{L}\{y(t)\} = Y(s)$ to (6), we obtain

$$Y(s) = \frac{n!\,s^{2\sigma}}{s^{n+1}\left(bs^{2\sigma}+as^{\sigma}+1\right)}. \tag{7}$$

For $\sigma = 1/2$, equation (7) becomes

$$Y(s) = \frac{n!}{s^n\left(bs+as^{1/2}+1\right)},$$

and turns into

$$Y(s) = \frac{n!}{u^{2n}\left(bu^2+au+1\right)},$$

with a substitution of $u = s^{1/2}$. Using partial fractions with explicit values of a,b, we can rewrite it as

$$Y(s) = \frac{n!}{j-k}\sum_{i=1}^{2n}(-1)^i\left[j^{2n-i+1}-k^{2n-i+1}\right]\frac{1}{u^i}+\frac{n!\,j^{2n}}{j-k}\left(\frac{1}{u+1/j}\right)-\frac{n!\,k^{2n}}{j-k}\left(\frac{1}{u+1/k}\right). \tag{8}$$

Finally, resubstituting $u = s^{1/2}$ and taking the inverse Laplace transform to (8) with the help of Example 2.4 (i), (iv), we obtain a solution of (3) in the form of (4).

For $\sigma = 1$, equation (7) becomes

$$Y(s) = \frac{n!}{s^{n-1}\left(bs^2+as+1\right)}.$$

Using partial fractions with explicit values of a,b, we can rewrite the above equation as

(9)
$$Y(s)=\frac{n!}{j-k}\sum_{i=1}^{n}(-1)^{n-i+1}\left[j^{n-i}-k^{n-i}\right]\frac{1}{s^i}+\frac{(-1)^{n+1}n!}{j-k}\left[j^{n-1}\left(\frac{1}{s+1/j}\right)-k^{n-1}\left(\frac{1}{s+1/k}\right)\right].$$

Applying the inverse Laplace transform to (9) and using Example 2.4 (i), and (ii), yield a solution of (3) in the form of (5). In order to include the case $n=0$ into the solution formulas of both cases, we adopt the notation $1/\Gamma(0)=0$. The proof is completed. □

Remark 3.1. Let n be a non-negative integer and a,b satisfy condition in Theorem 3.1. Then (5) is a solution of

$$b\cdot y''(t)+a\cdot y'(t)+y(t)=n(n-1)t^{n-2}$$

see [37] for more details.

Example 3.1. Letting $a=\frac{5}{2}, b=1$, and $\sigma=1/2$, equation (3) changes to

$$I_{0^+}y(t)+\frac{5}{2}\cdot I_{0^+}^{1/2}y(t)+y(t)=t^n. \tag{10}$$

From Theorem 3.1, equation (10) has a solution

$$\begin{aligned} y(t)=&\frac{n!}{3}\sum_{i=0}^{2n}(-1)^i\left[\frac{2^{2n-i+2}-(1/2)^{2n-i}}{\Gamma(i/2)}\right]t^{(i-2)/2}+\frac{n!\,2^{2n+1}}{3}\left[E_t\left(-\frac{1}{2},\frac{1}{4}\right)-\frac{1}{2}e^{t/4}\right]\\ &-\frac{n!\,(1/2)^{2n-1}}{3}\left[E_t\left(-\frac{1}{2},4\right)-2e^{4t}\right]. \end{aligned} \tag{11}$$

By applying Example 2.2 (i), (ii), and (iv), it is not difficult to verify that (11) satisfies (10).

Moreover, if $n=1$, then equation (10) becomes

$$I_{0^+}y(t)+\frac{5}{2}\cdot I_{0^+}^{1/2}y(t)+y(t)=t. \tag{12}$$

From (11), it follows that (12) has a solution

$$y(t)=\frac{8}{3}E_t\left(-\frac{1}{2},\frac{1}{4}\right)-\frac{1}{6}E_t\left(-\frac{1}{2},4\right)-\frac{4e^{t/4}}{3}+\frac{e^{4t}}{3}-\frac{5t^{-1/2}}{2\sqrt{\pi}}+1. \tag{13}$$

It is not difficult to verify that (13) satisfies (12).

Example 3.2. Letting $a=\frac{5}{2}, b=1$, and $\sigma=1$, equation (3) changes to

$$I_{0^+}^2y(t)+\frac{5}{2}\cdot I_{0^+}y(t)+y(t)=t^n. \tag{14}$$

From Theorem 1, equation (14) has a solution

(15)
$$y(t) = \frac{n!}{3}\sum_{i=0}^{n}(-1)^{n-i+1}\left[\frac{2^{n-i+1}-(1/2)^{n-i-1}}{\Gamma(i)}\right]t^{i-1} + \frac{(-1)^{n+1}n!}{3}\left[2^n e^{-t/2} - (1/2)^{n-2}e^{-2t}\right].$$

By applying Example 2.2 (ii), it is not difficult to verify that (15) satisfies (14).

Moreover, if $n = 2$, then equation (14) becomes

$$I_{0^+}^2 y(t) + \frac{5}{2}\cdot I_{0^+}y(t) + y(t) = t^2. \tag{16}$$

From (15), it follows that (16) has a solution

$$y(t) = 2 - \frac{8}{3}e^{-t/2} + \frac{2}{3}e^{-2t}. \tag{17}$$

It is not difficult to verify that (17) satisfies (16).

According to Remark 3.1, function (17) is a solution of $y''(t) + \frac{5}{2}y'(t) + y(t) = 2$.

Theorem 3.2. Consider the non-homogeneous fractional integral equation of the form

$$I_{0^+}^{2\sigma} y(t) + a\cdot I_{0^+}^{\sigma} y(t) + b\cdot y(t) = t^n e^t, \tag{18}$$

where $I_{0^+}^{\sigma}$ is the Riemann-Liouville fractional integral of order $\sigma = 1/2, \sigma = 1, n \in \mathbb{N}\cup\{0\}, a, b$ are constants and $t \in \mathbb{R}^+$. Then the solutions of (18) are as the follows:

(i) If $\sigma = 1/2$, and $j,k \in \mathbb{R}\setminus\{-1,0,1\}$ with $j \neq k$ such that $a = j+k$ and $b = jk$, then the solution of (18) is of the form

$$\begin{aligned} y(t) = {} & \frac{n!}{j-k}\sum_{i=0}^{n+1}(-1)^{n-i}\left[\frac{k^{2n-2i+2}}{(k^2-1)^{n-i+2}} - \frac{j^{2n-2i+2}}{(j^2-1)^{n-i+2}}\right]\times \\ & \frac{1}{\Gamma(-1/2)}\sum_{l=0}^{i}(-1)^{l-1}\frac{\Gamma(-3/2+l)}{\Gamma(i-l+1)\Gamma(l)}t^{i-l}E_t\left(-\frac{3}{2}+l,1\right) \\ & + \frac{n!e^t}{j-k}\sum_{i=0}^{n}\frac{(-1)^{n-i+1}}{\Gamma(i)}\left[\frac{k^{2n-2i+1}}{(k^2-1)^{n-i+2}} - \frac{j^{2n-2i+1}}{(j^2-1)^{n-i+2}}\right]t^{i-1} \\ & + \frac{(jk+1)t^n e^t}{(j^2-1)(k^2-1)} + \frac{(-1)^{n-1}n!\, j^{2n}}{(j^2-1)^{n+1}(j-k)}\left[E_t\left(-\frac{1}{2},\frac{1}{j^2}\right) - \frac{1}{j}E_t\left(0,\frac{1}{j^2}\right)\right] \\ & + \frac{(-1)^n n!\, k^{2n}}{(k^2-1)^{n+1}(j-k)}\left[E_t\left(-\frac{1}{2},\frac{1}{k^2}\right) - \frac{1}{k}E_t\left(0,\frac{1}{k^2}\right)\right]. \end{aligned} \tag{19}$$

(ii) If $\sigma = 1$, and $j,k \in \mathbb{R}\setminus\{-1,0\}$ with $j \neq k$ such that $a = j+k$ and $b = jk$, then the solution of (18) is of the form

$$y(t) = \frac{n!e^t}{j-k}\sum_{i=0}^{n}\frac{(-1)^{n-i}}{\Gamma(i)}\left[\frac{k^{n-i}}{(k+1)^{n-i+2}} - \frac{j^{n-i}}{(j+1)^{n-i+2}}\right]t^{i-1} + \frac{t^n e^t}{(j+1)(k+1)}$$
$$+\frac{(-1)^{n+1}n!}{j-k}\left[\frac{j^{n-1}e^{-t/j}}{(j+1)^{n+1}} - \frac{k^{n-1}e^{-t/k}}{(k+1)^{n+1}}\right]. \tag{20}$$

Proof. Performing the Laplace transform to both sides of (18), we have

$$\mathscr{L}\{I_{0^+}^{2\sigma}y(t)\} + a\cdot\mathscr{L}\{I_{0^+}^{\sigma}y(t)\} + b\cdot\mathscr{L}\{y(t)\} = \mathscr{L}\{t^n e^t\}. \tag{21}$$

Using Lemma 2.1, Example 2.3 (iv), and denoting the Laplace transform $\mathscr{L}\{y(t)\} = Y(s)$ to (21), we obtain

$$Y(s) = \frac{n!\,s^{2\sigma}}{(s-1)^{n+1}\,(bs^{2\sigma}+as^{\sigma}+1)}. \tag{22}$$

For $\sigma = 1/2$, equation (22) becomes

$$Y(s) = \frac{n!\,s}{(s-1)^{n+1}(bs+as^{1/2}+1)},$$

and turns into

$$Y(s) = \frac{n!\,u^2}{(u^2-1)^{n+1}\,(bu^2+au+1)}, \tag{23}$$

with a substitution of $u = s^{1/2}$. Using partial fractions with explicit values of a,b, we can rewrite it as

$$Y(s) = \frac{n!}{j-k}\sum_{i=1}^{n+1}(-1)^{n-i}\left[\frac{k^{2n-2i+2}}{(k^2-1)^{n-i+2}} - \frac{j^{2n-2i+2}}{(j^2-1)^{n-i+2}}\right]\frac{u}{(u^2-1)^i}$$
$$+\frac{n!}{j-k}\sum_{i=1}^{n}(-1)^{n-i+1}\left[\frac{k^{2n-2i+1}}{(k^2-1)^{n-i+2}} - \frac{j^{2n-2i+1}}{(j^2-1)^{n-i+2}}\right]\frac{1}{(u^2-1)^i}$$
$$+\left[\frac{n!}{(j+1)(k+1)}\right]\frac{1}{(u^2-1)^{n+1}} + \left[\frac{(-1)^{n-1}n!\,j^{2n}}{(j^2-1)^{n+1}(j-k)}\right]\frac{1}{u+1/j}$$
$$+\left[\frac{(-1)^n n!\,k^{2n}}{(k^2-1)^{n+1}(j-k)}\right]\frac{1}{u+1/k}. \tag{24}$$

Finally, resubstituting $u = s^{1/2}$ and taking the inverse Laplace transform to (24) with the help of Lemma 2.2, Example 2.4 (iii), and (iv), we obtain a solution of (18) in the form of (19).

For $\sigma = 1$, equation (22) becomes

$$Y(s) = \frac{n!\, s^2}{(s-1)^{n+1}(bs^2+as+1)}.$$

Using partial fractions with explicit values of a,b, we can rewrite the above equation as

$$Y(s) = \frac{n!}{j-k}\sum_{i=1}^{n}(-1)^{n-i}\left[\frac{k^{n-i}}{(k+1)^{n-i+2}} - \frac{j^{n-i}}{(j+1)^{n-i+2}}\right]\frac{1}{(s-1)^i} + \left[\frac{n!}{(j+1)(k+1)}\right]\frac{1}{(s-1)^{n+1}}$$

$$(25) \qquad + \frac{(-1)^{n+1}n!}{j-k}\left[\left[\frac{j^{n-1}}{(j+1)^{n+1}}\right]\frac{1}{s+1/j} - \left[\frac{k^{n-1}}{(k+1)^{n+1}}\right]\frac{1}{s+1/k}\right].$$

Applying the inverse Laplace transform to (25) with the help of Example 2.4 (iii), and (iv), yield a solution of (18) in the form of (20). In order to include the case $n = 0$ into the solution formulas of both cases, we adopt the notation $1/\Gamma(0) = 0$. The proof is completed. □

Remark 3.2. Let n be a non-negative integer and a,b satisfy condition in Theorem 3.2. Then (20) is a solution of

$$b\cdot y''(t) + a\cdot y'(t) + y(t) = t^n e^t + 2nt^{n-1}e^t + n(n-1)t^{n-2}e^t.$$

Example 3.3. For $\sigma = 1/2, a = \frac{5}{6}$, and $b = \frac{1}{6}$, equation (18) changes to

$$(26) \qquad I_{0^+}y(t) + \frac{5}{6}\cdot I_{0^+}^{1/2}y(t) + \frac{1}{6}\cdot y(t) = t^n e^t.$$

From Theorem 3.2, equation (26) has a solution

$$y(t) = n!\sum_{i=0}^{n+1}\left[\frac{27}{2^{3n-3i+5}} - \frac{8}{3^{n-i+1}}\right]\frac{1}{\Gamma(-1/2)}\sum_{l=0}^{i}(-1)^{l-1}\frac{\Gamma(-3/2+l)}{\Gamma(i-l+1)\Gamma(i)}t^{i-l}E_t\left(-\frac{3}{2}+l,1\right)$$
$$+ n!e^t\sum_{i=0}^{n}\left[\frac{16}{3^{n-i+1}} - \frac{81}{2^{3n-3i+5}}\right]\frac{t^{i-1}}{\Gamma(i)} + \frac{7t^n e^t}{4} + \frac{8n!}{3^n}\left[E_t\left(-\frac{1}{2},4\right) - 2e^{4t}\right]$$
$$(27) \qquad - \frac{27n!}{2^{3n+2}}\left[E_t\left(-\frac{1}{2},9\right) - 3e^{9t}\right].$$

For a fixed $n = 1$, equation (26) becomes

$$(28) \qquad I_{0^+}y(t) + \frac{5}{6}\cdot I_{0^+}^{1/2}y(t) + \frac{1}{6}\cdot y(t) = te^t.$$

From (27), it follows that (28) has a solution

$$\begin{aligned} y(t) =& \frac{269}{96}e^t + \frac{7}{4}te^t - \frac{175}{96}E_t\left(-\frac{1}{2},1\right) - \frac{5}{4}\left[tE_t\left(-\frac{1}{2},1\right) + \frac{1}{2}E_t\left(\frac{1}{2},1\right)\right] \\ &+ \frac{8}{3}\left[E_t\left(-\frac{1}{2},4\right) - 2e^{4t}\right] - \frac{27}{32}\left[E_t\left(-\frac{1}{2},9\right) - 3e^{9t}\right]. \end{aligned} \tag{29}$$

By applying Example 2.2 (ii), (iii), (iv), and (v), it is not difficult to verify that (29) satisfies (28).

Example 3.4. Letting $a = 3, b = 2$, and $\sigma = 1$, equation (18) changes to

$$I_{0+}^2 y(t) + 3 \cdot I_{0+} y(t) + 2 \cdot y(t) = t^n e^t. \tag{30}$$

From Theorem 3.2, equation (30) has a solution

(31)

$$y(t) = n!e^t \sum_{i=0}^{n} \frac{(-1)^{n-i}}{\Gamma(i)}\left[\frac{1}{2^{n-i+2}} - \frac{2^{n-i}}{3^{n-i+2}}\right]t^{i-1} + \frac{t^n e^t}{6} + (-1)^{n+1}n!\left[\frac{2^{n-1}e^{-t/2}}{3^{n+1}} - \frac{e^{-t}}{2^{n+1}}\right].$$

By applying Example 2.2 (ii), and (iii), it is not difficult to verify that (31) satisfies (30).

Moreover, if $n = 1$, then equation (30) becomes

$$I_{0+}^2 y(t) + 3 \cdot I_{0+} y(t) + 2 \cdot y(t) = te^t. \tag{32}$$

From (31), it follows that (32) has a solution

$$y(t) = \frac{5e^t}{36} + \frac{te^t}{6} + \frac{e^{-t/2}}{9} - \frac{e^{-t}}{4}. \tag{33}$$

It is not difficult to verify that (33) satisfies (32).

According to Remark 3.2, function (33) is a solution of $2y''(t) + 3y'(t) + y(t) = te^t + 2e^t$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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