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DYNAMICS OF PREY PREDATOR MODEL WITH PREY REFUGE USING A THRESHOLD PARAMETER

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Abstract. A lot of ecological studies deal with the coexistence of interacting species. The dynamics of this interaction helps in proper understanding of specie coexistence and resource conservation. In this work, we considered a prey-predator model with Holling type II functional response with threshold parameter incorporating a prey refuge. In order to understand the specie coexistence, we analyzed the long term dynamics of the model using a threshold parameter P_0 and a prey refuge parameter $m\varepsilon[0,1)$. We discussed the stability of the steady states and also carried out some numerical simulation in order to validate our analytical results.

Keywords: prey-predator; prey refuge; predation number; Holling Type II; functional response; coexistence.
2010 AMS Subject Classification: 93A30.

1. Introduction

Ecologists are often confronted with handling issues like the net effect of predation on the population dynamics of prey, variations in biomass of both prey and predator due to interaction, conditions for coexistence of interacting species using some parameter combinations and so on. The preservation of the ecological niche of species is salient to ecologists alongside the longevity of interacting species. Dawed et. al [2020] recommended that coexistence

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among predators and prey in the same environment is possible provided a good management of some factors (such as contacts between species, additional food supply, growth rate of species, etc).There are many ecological systems which describe the interaction between predator and prey and proper understanding of how the species coexist is a challenge in ecology (Holling, [1965]; Kar et al.,[2010]; Das et al, [2013]; Chauhan et al, [2017]; Sinha and Kumar, [2016]; Moustafa et al,[2018]). There are a lot of research works that have studied the dynamics of ecological systems with concentration on single predator and prey populations (Rosenzweig and MacArthur,[1963]). Kar [2006] looked at modelling and analysis of a harvested prey-predator system incorporating a prey refuge.

2. Prey-Predator Model for a Coexistence Ecosystem

We shall look at a single prey-predator system. In this work, we note that the ecological niche of the two species are not affected by external factors such as drought, fires, epidemics and so on. Besides the model description assumption and definition of variable and parameters, it is assumed that a constant proportion $m \in [0,1)$ of the prey can take refuge to avoid predation. This leaves $(1-m)x$ of the prey available for predation and hence the model under the above assumptions with Holling type II functional response is given by:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x\left(1-\frac{x}{k}\right) - \frac{\beta(1-m)xy}{1+a(1-m)x} \\ \frac{dy}{dt} &= -\gamma y + \frac{c\beta(1-m)xy}{1+a(1-m)x} \end{aligned} \tag{1}$$

A real life example is the interaction between the zebra (x) and the lion (y). The description of the variables and parameters that are used in the model (1) can be seen in Table 1.

2.1. Steady States of model (1).

We obtain the following equilibrium points from the system (1) as follows:

$$\begin{aligned} E_1(x,y) &= (0,0). \\ E_2(x,y) &= (k,0). \\ E_3(x^*,y^*) &= \left(\frac{\gamma}{(1-m)(c\beta-\gamma a)}, \frac{\alpha c\gamma(1+a(1-m)k}{k(1-m)^2(c\beta-\gamma a)^2}(P_0-1)\right) \end{aligned} \tag{2}$$

Variables/Parameters	Meaning	Unit
x	Population density of prey	g/m^2
y	Population density of predator	g/m^2
α	Growth rate of x	/year
k	Carrying capacity of x	g/m^2
β	x removal by y	/year
a	x when y is half	g/m^2
c	Conversion of x biomass into y biomass	Dimensionless
γ	Reduction of y due to other factors	/year

TABLE 1. Description of parameters and variables for model (1)

We remark that for each of the equilibrium points E_i (for $i = 1,2,3$) to exist, the inequalities $0 \leq x \leq k$ must be satisfied by each of them. Furthermore, for the equilibrium point E_3 to exist, it must each satisfy the inequality:

$$c\beta - \gamma a > 0.$$

We make use of a threshold quantity (predation number denoted by P_0) which gives a condition for the stability of the system around the equilibrium points. This quantity can also be likened to the basic reproduction number R_0 (Van Den Driessche and Watmough, [2002]) in epidemiological models. Its calculation is done in like manner by the approach of the next generation matrix (Collins and Duffy, [2016]):

$$P_0 = \frac{c\beta(1-m)k}{\gamma[1+a(1-m)k]}, \tag{3}$$

In ecological terms, we can describe P_0 as the combination of parameters which quantifies the prey consumption required for survival. So, $P_0 = 1$ implies that the predator uses prey biomass at a rate almost corresponding to their own biomass loss. For $P_0 < 1$ less prey is ingested per unit of predator biomass loss. For $P_0 > 1$ more prey is devoured per unit of predator biomass loss.

2.2. Stability analyses of model (1)**.** The investigation of the stability of this model is a tool that will assist us in describing the entire dynamics of the system. We will examine the stability using the threshold quantity P_0.

Theorem 1. *The equilibrium point E_1 is unstable no matter the value of P_0.*

Proof. It suffices to show that at least one of the eigenvalues of the Jacobian of the model (1), evaluated at the equilibrium point, has a positive real part.The eigenvalues of (1) at the trivial equilibrium point E_1 are $\lambda_1 = \alpha, \lambda_2 = -\gamma$. Hence, E_1 is unstable despite the value of P_0. □

Theorem 2.*The equilibrium point E_2 is stable if $P_0 \leq 1$ and unstable otherwise.*

Proof. When we evaluate the eigenvalues of (1) at E_2 we obtain

$$\lambda_1 = -\alpha \quad \text{and} \quad \lambda_2 = \gamma\left[\frac{c\beta(1-m)k}{\gamma[1+a(1-m)k]} - 1\right]$$

Similarly, in terms of the prey refuge parameter, the system will be stable if

$$m > \frac{(c\beta - \gamma a)k - \gamma}{(c\beta k - \gamma ak)}.$$

Theorem 3.*The equilibrium point E_3 is stable if $P_0 > 1$.*

Proof. We shall make use of the characteristic equation. This is given as:

$$\left[\alpha - \frac{2\alpha x^*}{k} - \frac{\beta(1-m)y^*}{[1+a(1-m)x^*]^2} - \lambda\right]\left[\frac{c\beta(1-m)x^*}{1+a(1-m)x^*} - \gamma - \lambda\right] + \left[\frac{\beta(1-m)x^*}{1+a(1-m)x^*}\right]\left[\frac{c\beta(1-m)y^*}{[1+a(1-m)x^*]^2}\right] = 0$$

If we set $\frac{\beta(1-m)}{1+a(1-m)x^*} = A$, then we will have that

$$\lambda^2 + \lambda\left[\frac{Ay^*}{1+a(1-m)x*} + \left(\frac{2\alpha}{k} - cA\right)x^* + (\gamma - \alpha)\right] + \left[cA\alpha x^*\left(1 - \frac{2x^*}{k}\right) + \alpha\gamma\left(\frac{2x^*}{k} - 1\right) + \frac{A\gamma y^*}{1+a(1-m)x^*}\right] = 0$$

The coexistence equilibrium point is stable if $\lambda_1 + \lambda_2$ is negative and $\lambda_1\lambda_2$ is positive.

$$\lambda_1 + \lambda_2 = (\alpha - \gamma + \left(cA - \frac{2\alpha}{k}\right)x^* - \frac{Ay^*}{1+a(1-m)x^*}.$$

For $x^* = \frac{\beta(1-m)}{1+a(1-m)x^*}$, $y^* = \frac{\alpha c\gamma[a(1-m)k+1](C_0-1)}{k[(c\beta-\gamma a)(1-m)}$, $A = \frac{\beta(1-m)}{1+a(1-m)x^*}$, then we have that

$$\lambda_1+\lambda_2 = (\alpha-\gamma)+\left[\frac{c\beta(1-m}{1+a(1-m)x^*}+\frac{2\alpha}{k}\right]x^*-\frac{\beta(1-m)y^*}{1+a(1-m)x^*}$$

$$= (\alpha-\gamma)+\left[\frac{c\beta(1-m)(c\beta-\gamma}{k}\right]\left[\frac{\gamma}{(1-m)(c\beta-\gamma a}\right]-\left[\frac{\alpha c\gamma[a(1-m)k+1](P_0-1}{ck(c\beta-\gamma a)(1-m)}\right]$$

$$= (\alpha-\gamma)+\gamma-\frac{2\alpha\gamma}{k[(1-m)(c\beta}-\frac{\alpha\gamma[a(1-m)k+1](P_0-1)}{k[(c\beta-\gamma a(1-m)}$$

$$= \frac{\alpha}{k}\left[\frac{kc\beta-kcm\beta+\gamma akmP_0-\gamma akP_0+\gamma^2-\gamma^2P_0-2\gamma}{(1-m)(c\beta-\gamma a)}\right]$$

$$= \frac{\alpha}{k}\left[\frac{kc\beta(1-m)+\gamma[akP_0(m-1)-2]+\gamma^2(1-P_0)}{(1-m)(c\beta-\gamma a)}\right]$$

We see that $\lambda_1+\lambda_2$ is negative for $P_0>1$.

Similarly, using the product of roots:

$$\lambda_1\lambda_2 = \left[cA\alpha x^*\left(1-\frac{2x^*}{k}\right)+\alpha\gamma\left(\frac{2x^*}{k}-1\right)+\frac{A\gamma y^*}{1+a(1-m)x^*}\right]$$

where $A = \frac{\beta(1-m)}{1+a(1-m)x^*}$, $x^* = \frac{\gamma}{(1-m)(c\beta-\gamma a)}$, $y^* = \frac{\alpha c\gamma[a(1-m)k+1]P_0-1}{k[(c\beta-\gamma a)(1-m)]^2}$

We then have

$$\left[\frac{\alpha\gamma^2aP_0-\alpha\gamma^2akmP_0+\alpha\gamma^2aP_0-\alpha\gamma^2ak+\alpha\gamma^2akm-\alpha\gamma^2a}{kc\beta}\right]$$

$$= \frac{\alpha\gamma^2ak(P_0-1)+\alpha\gamma^2akm(1-P_0)+\alpha\gamma^2a(P_0-1)}{kc\beta}$$

$$= \frac{\alpha\gamma^2a}{kc\beta}[k(P_0-1)+(P_0-1)mk+(P_0-1)]$$

$$= \frac{\alpha\gamma^2ak(P_0-1)+\alpha\gamma^2akm(1-P_0)+\alpha\gamma^2a(P_0-1)}{kc\beta}$$

$$= \frac{\alpha\gamma^2a}{kc\beta}[k(P_0-1)+mk(P_0-1)+(P_0-1)]$$

$$= \frac{\alpha\gamma^2ak(P_0-1)+\alpha\gamma^2akm(1-P_0)+\alpha\gamma^2a(P_0-1)}{kc\beta}$$

$$= \frac{\alpha\gamma^2a}{kc\beta}[k(P_0-1)+(P_0-1)-mk(P_0-1)]$$

$$= \frac{\alpha\gamma^2ak(P_0-1)+\alpha\gamma^2akm(1-P_0)+\alpha\gamma^2a(P_0-1)}{kc\beta}$$

$$= \frac{\alpha\gamma^2 a}{kc\beta}[(P_0 - 1)(1-m)k+1] > 0, \text{ if } P_0 > 1 \text{ since } m \in [0,1).$$

3. Model Simulation and Discussion of Results

In order to validate our analytical results, we examine the long-term dynamics of the model by carrying out numerical simulations using the parameter values given in Table 2.

Parameters	**Parameter values**	**Source**
α	0.18	Estimated
β	0.6	Sinha and Kumar, [2016]
a	0.02	Sinha and Kumar, [2016]
m	$0-0.6$	Estimated
c	0.02	Sinha and Kumar, [2016]
γ	$0.3-0.45$	Estimated

TABLE 2. Parameter values for numerical simulations

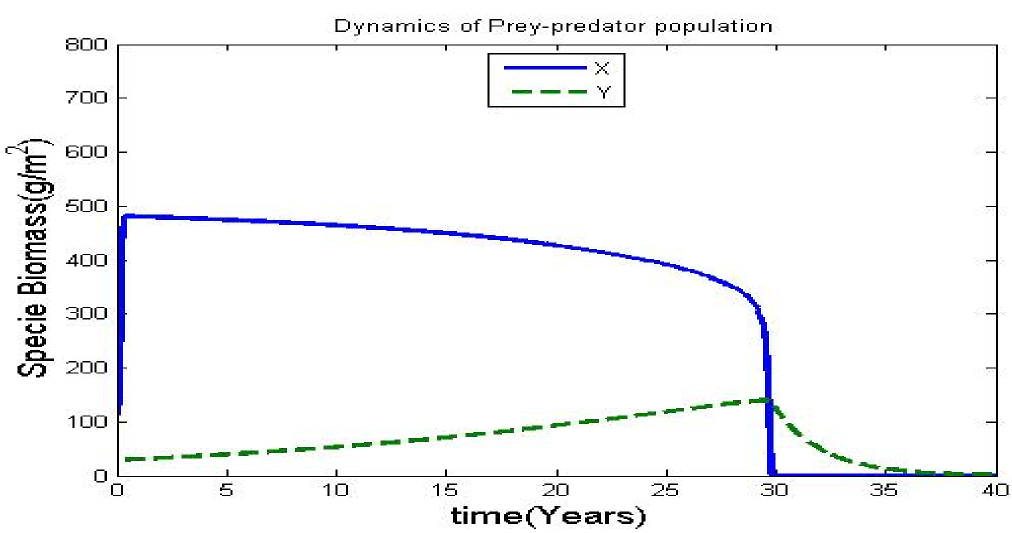

FIGURE 1. Dynamics of prey-predator model for P_0<1

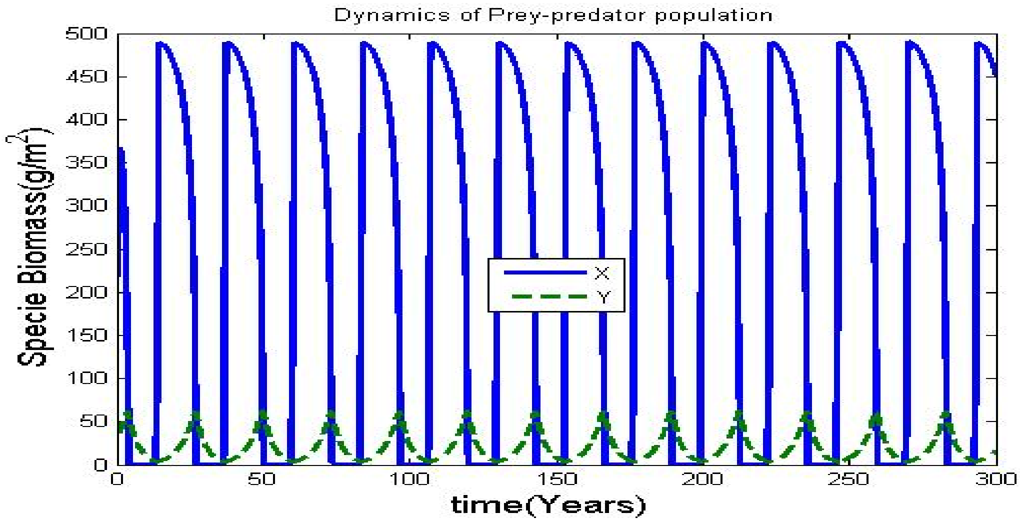

FIGURE 2. Dynamics of prey-predator model for $P_0 < 1$

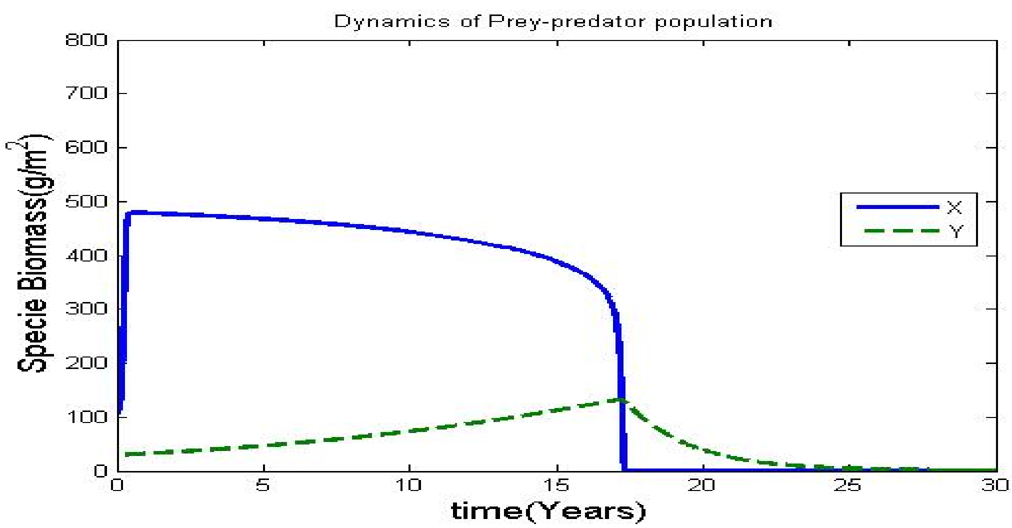

FIGURE 3. Dynamics of prey-predator model for $m = 0.4$

3.1. Discussion of Results.

To validate our analytical results, we examine the long-term dynamics of the model by carrying out numerical simulations using the parameter values given in Table 2 and varying γ (Figures 1-2) and m (Figures 3-5). γ is varied because its alteration easily affects the predation number. For Figure 5, $\gamma = 0.45$ results in $P_0 = 0.889$. For Figure **??**, $\gamma = 0.3$ results in $P_0 = 1.333$. From Figure 5, the trajectory of the predator biomass increases and then decreases due as the prey biomass decreases. So, less prey is devoured per unit of predator biomass loss. From Figure **??**, when the predation number is greater than one, the

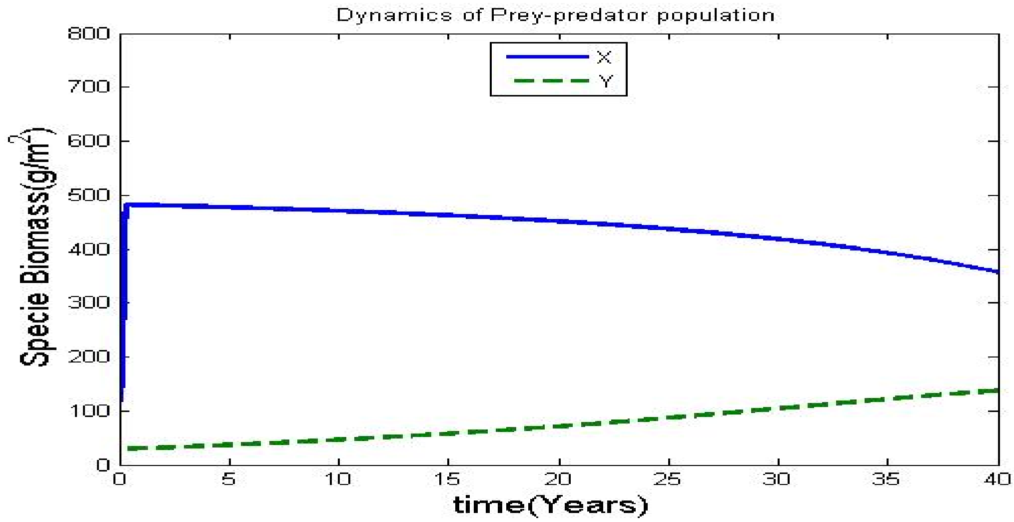

FIGURE 4. Dynamics of prey-predator model for $m = 0.5$

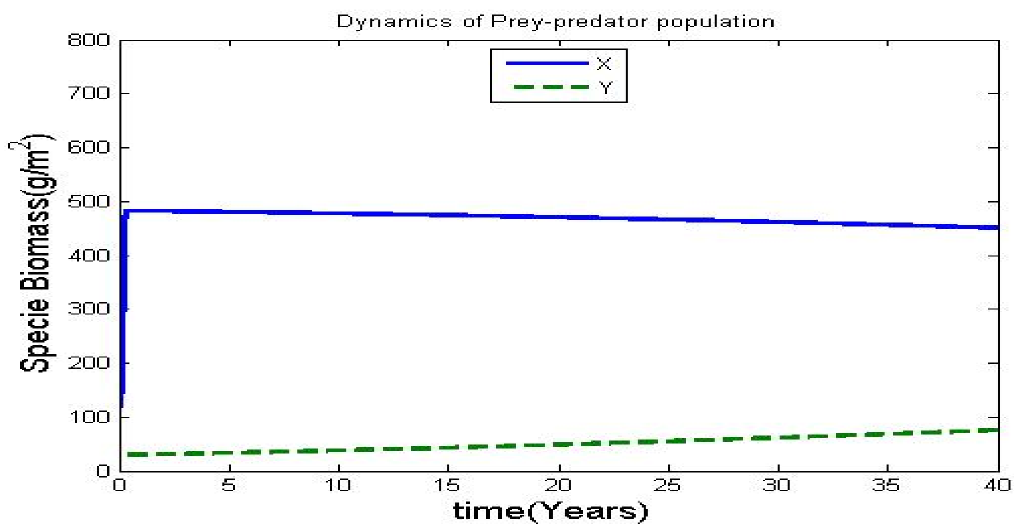

FIGURE 5. Dynamics of prey-predator model for $m = 0.6$

trajectories for both prey and predator become periodic or cyclic. So, more prey is eaten up per unit of predator biomass loss. From Figures 3-5, we see the degree of decline in the prey biomass for different increased values of m.

When we understand the dynamics of certain ecosystem, it helps in the preservation of the co-existing species. An insight into a homogeneous community of a single predator and prey helps to study the dynamics of their interaction and useful in analysis. We illustrated this using x as the prey with y as the associated predator. In order to identify system stability, we computed a threshold parameter, P_0 (Predation number) which summarizes most of the parameters. Also, we see that the prey refuge parameter also contributes to investigating the survival of both prey and predator. A possibility to introduce heterogeneity in the population will change the entire dynamics and generate more interesting result for specie coexistence. In general, this work can be extended to a heterogeneous population in order to account for multiple species coexistence. We can also use a different functional response apart from Holling type II which will give us a widespread view in our analysis.

3.2. Authors' Contribution. This present work discussed the coexistence of both species using both a threshold parameter and a prey refuge parameter. A threshold quantity, the predation number P_0 was introduced to evaluate prey consumption per equivalent of predator biomass (Collins and Duffy,[2016]). In addition to this, we investigated how to promote long term coexistence of both species in relation to the prey refuge parameter $m\varepsilon[0,1)$. The first and corresponding author originated the work, supervised and performed the simulations. The co-author contributed also by handling the analysis.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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