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POWERFUL AND MAXIMAL RATIONAL METRIC DIMENSION OF A WHEEL

M. M. PADMA, B. SOORYANARAYANA*, M. JAYALAKSHMI

Department of Mathematics, Dr.Ambedkar Institute of Technology, Bengaluru- 560 056, Karnataka State, India

Abstract. The rational distance from the vertex u to the vertex v in a graph G, denoted by $d(v/u)$, is defined as the average distances from the vertex u to the closed neighbors of v if $u \neq v$, else it is 0. A subset S of vertices of G is called rational resolving set of G if for every pair u,v of distinct vertices in $V-S$, there is a $w \in S$ such that $d(u/w) \neq d(v/w)$ in G. In this paper powerful and maximal rational resolving sets are introduced and minimum cardinality of such sets are computed for the wheel graphs.

Keywords: resolving sets; rational resolving sets; rational metric dimension; wheel graphs.

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1. INTRODUCTION

Let $G(V,E)$ be a connected simple finite graph. A path $P_{a,b}$ from the vertex a to b is an alternating sequence of distinct vertices and edges starting with a and ending with b in such a way each edge lies between its end vertices. The number of edges in a path $P_{a,b}$ is called the length of the path and is denoted by $l(P_{a,b})$. The distance between two vertices a and b in G, denoted by $d_G(a,b)$ (or simply $d(a,b)$), is the minimum length of a path between a and b. That is, $d_G(a,b) = \min\{l(P_{a,b})\}$. The number of edges incident with a vertex v of G is the degree of

*Corresponding author
E-mail address: dr_bsnrao@yahoo.co.in
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the vertex v in G and is denoted by $\deg_G(v)$ or simply $\deg(v)$. Further, the closed neighborhood set of a vertex $v \in V$, denoted by $N[v]$, is defined as $N[v] = \{w : d(v,w) \leq 1\}$.

The notion of rational distance is introduced in [12]. The rational distance from the vertex u to the vertex $v \in V$, denoted by $d(v/u)$, is defined as

$$d(v/u) = \begin{cases} 0, & \text{if } v = u. \\ \sum\limits_{w \in N[v]} \frac{d_G(u,w)}{\deg_G(v)+1}, & \text{otherwise.} \end{cases}$$

Let $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ be an arbitrary ordered set. Then, for each vertex v of G, we can always associate a vector (called rational code of v) with respect to S, denoted by $\Gamma(v/S)$, as

$$\Gamma(v/S) = (d(v/s_1), d(v/s_2), \ldots, d(v/s_k)).$$

A subset $S \subseteq V$ is called a *rational resolving set* if $\Gamma(u/S) \neq \Gamma(v/S)$ for each pair u, v of distinct vertices in V. By the definition it follows that the rational resolving property is super hereditary. That is, if S is a rational resolving set of G then so as every super set of S. The minimum cardinality of a rational resolving set is called the *rational metric dimension* of G and is denoted by $rmd(G)$. Further, each rational resolving set with cardinality $rmd(G)$ is called an rmd-set of G.

The rational resolving sets are defined in [11] and studied by various authors in [6, 7, 8, 9, 10, 12]. We recall that a subset $S \subseteq V$ is a *resolving set* of G if for each pair $u, v \in V$ there exists a vertex $w \in S$ such that $d(v,w) \neq d(u,w)$. The *metric dimension* of G, denoted by $dim(G)$, is the minimum cardinality of a resolving set of G. A resolving set with minimum cardinality is called a *metric basis*. The concept of metric dimension was introduced by F. Harary and R. A. Melter [5] and independently by P. J. Slater [16] under the term locating set. For more works on metric dimension, we refer to [3, 4, 7, 13, 14, 15, 17, 18, 19, 21].

The complement of a minimum dominating set is also a dominating set. But, the complement of a rational resolving set of minimum cardinality need not be a rational resolving set. For example every rational resolving set of a triangle should include at least 2 vertices of it and hence its complement is not.

Recently, In 2021, an attempt is made by B. Sooryanarayana, Suma A. S and Chandrakala S. B in [22] to study some special classes of resolving sets as an extension to the earlier work of B. Sooryanarayana and Suma A. S [20]. In this paper, we obtain similar results on rational resolving sets of a wheel.

Throughout this paper, C_m denotes a cycle on m vertices with the vertex set $V = \{v_i : 0 \le i \le m-1\}$ and the edge set $E = \{v_i v_{i+1 \pmod m} : 0 \le i \le m-1\}$; K_m denotes the complete graph on m vertices with the vertex set $V = \{v_i : 1 \le i \le m\}$ and the edge set $E = \{v_i v_j : i \ne j, 1 \le i, j \le m\}$; and $W_{1,m}$ denotes wheel graph on $m+1$ vertices with vertex set $V = \{v_i : 0 \le i \le m-1\} \bigcup \{c_0\}$ and edge set $E = \{c_0 v_i, v_i v_{i+1 \pmod m} : 0 \le i \le m-1\}$. The vertex c_0 is called the central vertex and each v_i, $0 \le i \le m-1$, is called a rim vertex of $W_{1,m}$. The terms not defined here may be found in [1, 4].

2. Rational Distances in a Wheel

Since the diameter of the graph $W_{1,m}$ is 2, it follows that $d(u,v) \in \{0,1,2\}$ for all $u, v \in V(W_{1,m})$. Hence it is easy to see that each component of the rational vertex code $\Gamma(v_i/S) \in \left\{0, 1, \frac{3}{2}, \frac{7}{4}\right\}$ for $0 \le i \le m-1$, and each component of $\Gamma(C_0/S)$ is $\frac{2(m-3)+3}{m+1}$ (Note that $c_0 \notin S \subseteq V(W_{1,m})$) whenever $m \ge 5$. If $m = 3$, then the components of $\Gamma(c_0/S)$ and $\Gamma(v_i/S)$ are in $\left\{0, \frac{3}{4}\right\}$ for $0 \le i \le 2$. If $m = 4$, then $\Gamma(v_i/S) \in \left\{0, 1, \frac{5}{4}\right\}$ for $0 \le i \le 3$.

We recall the following results in [11] for immediate reference.

Theorem 2.1 ([11]). *For any integer $n \ge 3$,*

$$rmd(W_{1,m}) = \begin{cases} 3, & \text{if } n = 3. \\ 2, & \text{if } 4 \le m \le 9. \\ \lceil \frac{n}{4} \rceil - 1, & \text{if } n \ge 10 \text{ and } n \equiv 1 \pmod 8. \\ \lceil \frac{n}{4} \rceil, & \text{otherwise.} \end{cases}$$

Throughout this paper, let $\mathfrak{R}(G)$ be the collection of all rational resolving sets of the graph G. Then $\mathfrak{R}(G)$ is super hereditary, that is for every $S \in \mathfrak{R}(G)$, the set $T \in \mathfrak{R}(G)$ whenever $S \subseteq T$.

3. Gap in a Wheel

We first define the gap between two vertices in G with respect to a set $S \in \mathfrak{R}(G)$.

Definition 3.1. *Let G be a graph and $S \in \mathfrak{R}(G)$. Let $x, y \in V$ and $\overline{S} = V(G) - S$. An $\overline{S}$ path between x and y in G is an xy-path of G containing all its internal vertices in $\overline{S}$. The gap between*

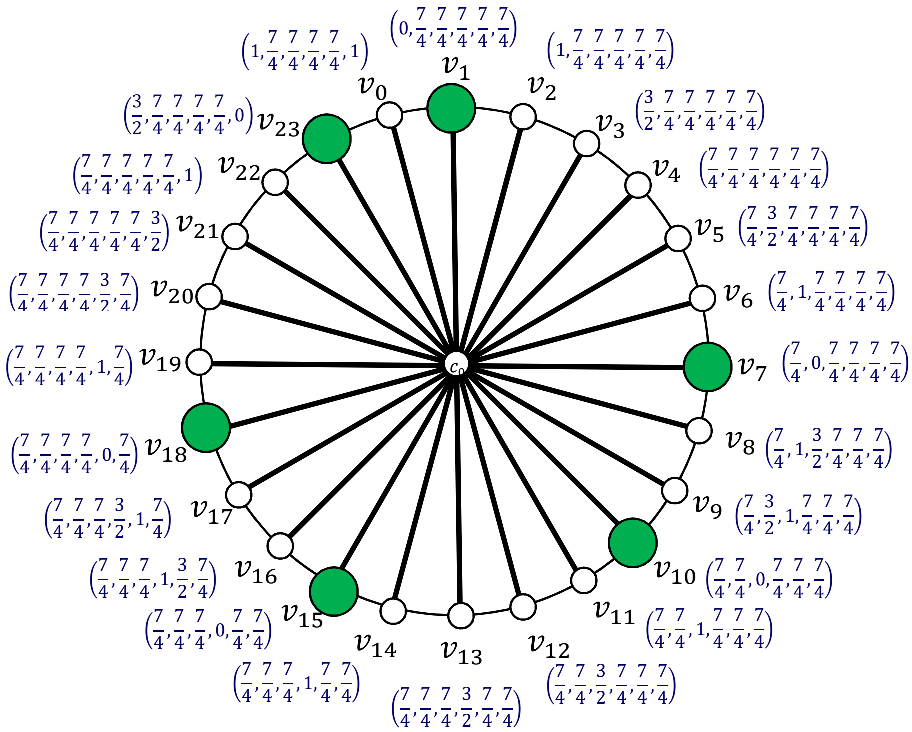

FIGURE 1. Rational code of each vertex of $W_{1,24}$ corresponding to the set $S = \{v_1, v_7, v_{10}, v_{15}, v_{18}, v_{23}\}$ with $\Gamma(c_0/S) = \left(\frac{9}{5}, \frac{9}{5}, \frac{9}{5}, \frac{9}{5}, \frac{9}{5}, \frac{9}{5}\right)$.

x and y with respect to S, denoted by $g_S(x,y)$, is defined as the minimum number of vertices of $\overline{S}$ in an $\overline{S}$ path (if it exists) between x and y, else it is 0.

Example: Consider the graph G of Figure 2. Let $S = \{c_0, v_1, v_8, v_{13}, v_{14}, v_{18}, v_{19}, v_{20}\}$. Then $\overline{S}$ path between v_1 and v_8 is $P : v_1 - v_2 - v_3 - \cdots - v_7 - v_8$. $g_S(v_1, v_8) = |V(P) \cap \overline{S}| = 6$. Also, taking $S_1 = \{v_1, v_8, v_{13}, v_{14}, v_{18}, v_{19}, v_{20}\}$, we see that $P : v_1 - c_0 - v_8$ is an $\overline{S}_1$ path between v_1 and v_8 with minimum length. Hence, $g_{S_1}(v_1, v_8) = 1$. Similarly, $g_S\{v_{18}, v_{19}\} = 0$, $g_S(v_{18}, v_{20}) = 0$.

We now begin with the following lemma which we often use in the proof of next theorems.

Lemma 3.2. *Let $m \in \mathbb{Z}^+$ and $m \geq 9$. Then a set S containing at least* 3 *rim vertices is in $\mathfrak{R}(W_{1,m})$ if and only if the following hold.*

i) $g_{S \cup \{c_0\}}(a,b) \leq 5$ for all $a, b \in S$ and $g_{S \cup \{c_0\}}(a,b) = 5$ for at most one pair $a, b \in S$.

ii) If $3 \leq g_{S \cup \{c_0\}}(a,b) \leq 5$ for any $a, b \in S$, then $g_{S \cup \{c_0\}}(b,c) \leq 2$ and $g_{S \cup \{c_0\}}(a,c) \leq 2$, for every $c \in S$.

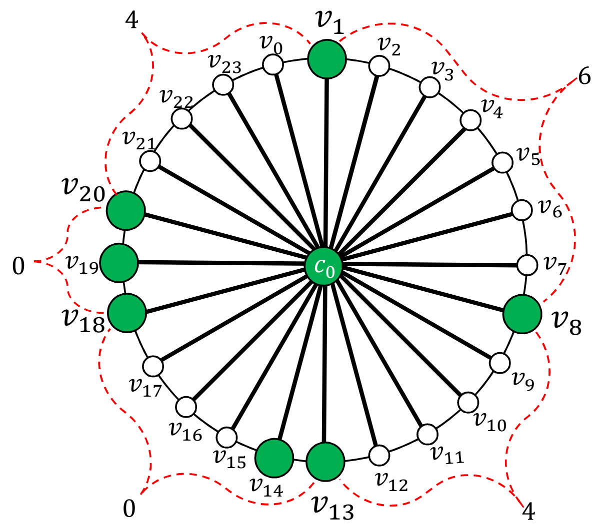

FIGURE 2. The graph between the vertices with respect to the set $S = \{c_0, v_1, v_8, v_{13}, v_{14}, v_{18}, v_{19}, v_{20}\}$.

Proof. Let $S \in \Re(W_{1,m})$ and $|S| = k$. Let $T = S \cup \{c_0\}$. Let us suppose to the contrary that the condition (i) fails. Then either $g_T(a,b) \geq 6$ for some $a,b \in S$ or there are vertices $a,b,c,d \in S$ with $|\{a,b\} \cup \{c,d\}| \geq 3$ such that $g_T(a,b) = 5$ and $g_T(c,d) = 5$.

Case 1: $g_T(a,b) \geq 6$.

In this case, there are at least six consecutive rim vertices, say $v_1, v_2, v_3, v_4, v_5, v_6$, in $\overline{S}$ with v_1 adjacent to a. But then, $\Gamma(v_3/S) = \Gamma(v_4/S) = (a_1, a_2, \ldots, a_k)$, where $a_i = 7/4$ for $1 \leq i \leq k$, a contradiction to the fact that $S \in \Re(W_{1,m})$.

Case 2: $g_T(a,b) = 5$ and $g_T(c,d) = 5$.

In this case, there are five consecutive rim vertices, say v_1, v_2, v_3, v_4, v_5, in $\overline{S}$ with v_1 adjacent to a, and there are five consecutive rim vertices u_1, u_2, u_3, u_4, u_5 in $\overline{S} - \{v_1, v_2, v_3, v_4, v_5\}$ with u_1 adjacent to c. But then, $\Gamma(u_3/S) = \Gamma(v_3/S) = (a_1, a_2, a_3, \ldots, a_k)$ where $a_i = 7/4$ for all $1 \leq i \leq k$, a contradiction to the fact that $S \in \Re(W_{1,m})$.

In case if the condition (ii) fails, then there are three vertices a,b,c in S such that $g_T(a,b) \in \{3,4,5\}$ and, $g_T(b,c) \geq 3$ or $g_T(a,c) \geq 3$. Without loss of generality, we take $g_T(b,c) \geq 3$. Then there are three consecutive rim vertices v_1, v_2, v_3 in $W_{1,m}$ with v_1 adjacent to b. Also there are three consecutive rim vertices u_1, u_2, u_3 with u_1 adjacent to b. But then, $\Gamma(v_1/S) = \Gamma(u_1/S) = (a_1, a_2, \ldots, a_k)$ where $a_i = 7/4$ for all $1 \leq i \leq k$, except for one i for which $a_i = 1$

(which corresponds to the vertex b), a contradiction to the fact that $S \in \Re(W_{1,m})$. Hence the conditions (i) and (ii) hold.

Now to prove the converse part, let S be a k-element subset of the vertex set of $W_{1,m}$ containing at least three rim vertices such that every pair of vertices in it satisfies the conditions (i) and (ii) of the theorem. We now show that $S \in \Re(W_{1,m})$ by the method of contradiction. If S does not belong to $\Re(W_{1,m})$, then there are two vertices $u, v \in V - S$ such that $d(u/w) = d(v/w)$ for every $w \in S$.

Case 1: $d(u/w) = d(v/w) = 1$ for some $w \in S$.

In this case u and v are the rim vertices of $W_{1,m}$ and w is adjacent to u and v ($\because m \neq 4$).

Subcase 1a: $d(u/w_1) = d(v/w_1) = 1$ for some $w_1 \in S$.

In this case, w_1 is also adjacent to u and v, and w_1 is the rim vertex, and hence $m = 4$, a contradiction.

Subcase 1b: $d(u/w_1) = d(v/w_1) = 3/2$ for some $w_1 \in S$.

This case is possible only if $m = 6$ ($\because$ $d(u,w) = d(v,w) = 1$ and $d(w_1,u) = d(w_1,v) = 2$ on the cycle C_m of G), a contradiction to the fact that $m \geq 9$.

From the above two sub cases we see that only possibility is $d(u/w_i) = d(v/w_i) = 7/4$ for every $w_i \in S$ other than w whenever $d(u/w) = d(v/w) = 1$. But then, for the vertices s_1, s_2 in S nearer to w in the cycle C_m of $W_{1,m}$ we see that $g_T(s_1,w) \geq 3$ and $g_T(w,s_2) \geq 3$, a contradiction to the assumption of the condition (ii).

Case 2: $d(u/w) = d(v/w) \neq 1$ for any $w \in S$.

In this case neither u nor v is non-adjacent to w, for every $w \in S$. We first show that $d(u/w) = d(v/w) = \frac{7}{4}$ for all $w \in S$. For this, let us assume to the contrary that $d(u/w') = d(v/w') = 3/2$ for some $w' \in S$.

Claim: $d(u/w) = d(v/w) \neq 3/2$ for any $w \in S - \{w'\}$.

If possible, suppose to contrary that $d(u/w_1) = d(v/w_1) = 3/2$ for some $w_1 \in S - \{w'\}$. Then w_1 can not be in a shortest uw'-path or vw'-path. Hence w_1 should be in a uv-path of C_m not containing w'. So, $d_{C_m}(u,w_1) = d_{C_m}(v,w_1) = d_{C_m}(u,w') = d_{C_m}(v,w') = 2$. This is possible only if $m = 8$, a contradiction to the fact that $m \geq 9$. Hence the claim.

By the above claim, $d(u/w) = d(v/w) = 7/4$ for all $w \in S - \{w'\}$. So, $d_{C_m}(u, w_i) \geq 3$ for all $w_i \in S - \{w'\}$. Since S contains at least 3 rim vertices, $|S - \{w'\}| \geq 3$. Let s_1 and s_2 be the two rim vertices in $S - \{w'\}$ which are nearer to w' in C_m. Then s_1 as well as s_2 can not be in a shortest $w'v$ path or $w'u$ path (else $d(u/s_1) \neq d(v/s_1)$), a contradiction to the assumption of u and v). Without loss of generality, let u be in the shortest $w's_1$ path and v be in the shortest $w's_2$ path. Then, as $d_{C_m}(s_1, u) \geq 3$ and $d_{C_m}(s_2, v) \geq 3$, we get $g_{S \cup \{c_0\}}(s_1, w') \geq 4$ and $g_{S \cup \{c_0\}}(w, s_2) \geq 4$, which is a contradiction to the assumption of condition (ii) of the lemma.

Thus, we have arrived at the conclusion that $d(u/w) = d(v/w) = 7/4$ for all $w \in S$. That is $\Gamma(u/S) = \Gamma(v/S) = (a_1, a_2, ...a_k)$ where $a_i = 7/4$, for all $1 \leq i \leq k$. This is possible only if one of the following hold.

(1) Both u and v lie in a $\overline{S \cup \{c_0\}}$ path between some $w_1, w_2 \in S$ with $g_{S \cup \{c_0\}}(w_1, w_2) \geq 6$.
(2) The vertex u is in the center of the $\overline{S \cup \{c_0\}}$ path between w_1 and w_2 for some $w_1, w_2 \in S$, and, v is in the center of $\overline{S \cup \{c_0\}}$ path between w_3 and w_4 for some $w_3, w_4 \in S$ with $| \{w_1, w_2\} \bigcap \{w_3, w_4\} | \leq 1$. But then, $g_{S \cup \{c_0\}}(w_1, w_2) \geq 5$ and $g_{S \cup \{c_0\}}(w_2, w_3) \geq 5$.

In either of the above possibilities we arrive at a contradiction to the assumption of condition (i). Hence the lemma. □

Remark 3.3. *The conditions in the above lemma holds for all $a, b \in V$.*

4. Powerful Rational Metric Dimension

A rational resolving set $S \in \mathfrak{R}(G)$ is called *powerful* if $\overline{S} \in \mathfrak{R}(G)$. The least cardinality of a powerful rational resolving set (if it exists) of G is called powerful rational metric dimension of G and is denoted by $rmd_p(G)$.

In this section we determine powerful rational metric dimension of a Wheel.

Lemma 4.1. *If $S \in \mathfrak{R}(W_{1,m})$ and $|S| = \min\{|T| : T \in \mathfrak{R}(W_{1,m})\} \geq 3$, then S has no three consecutive rim vertices whenever $m \neq 3$.*

Proof. Let $S \in \mathfrak{R}(W_{1,m})$ be of minimum cardinality and $m \neq 3$. If possible, let a_1, a_2, a_3 be the three consecutive rim vertices in S and $|S| \geq 3$. Then, by Theorem 2.1, $m \geq 10$. Let $S' = S - \{a_2\}$. Let a_i be the rim vertex at a distance $i - 1$ from a_1, along the circle C_m of $W_{1,m}$, in a shortest path containing the vertex a_2. Let a_{-i} be the rim vertex at a distance $i+1$ from a_1, along the circle C_m of $W_{1,m}$, in a shortest path not containing the vertex a_2

Claim: $S' \in \mathfrak{R}(w_{1,m})$.

We prove the claim by contradiction. Suppose that $S' \notin \mathfrak{R}(W_{1,m})$. Then there are two vertices u and $v \in V(W_{1.m})$ such that

$$d(u/a_i) = d(v/a_i), \text{for } i = 1,3 \tag{1}$$

and

$$d(u/a_2) \neq d(v/a_2). \tag{2}$$

Let $d(u/a_i) = l_i$, $i = 1,3$. Then we have the following possibilities.

Case 1: $l_1 = l_3 = 1$.

In this case, $u, v \in \{a_2, v_0\}$ and hence $n = 4$, a contradiction to the fact that $n \geq 10$.

Case 2: $l_1 = l_3 = 3/2$.

Since $l_1 = 3/2$, either $u = a_3$ or $v = a_3$. If $u = a_3$ then $v = a_{-1}$ and hence, $d(u/a_3) = 0$ and $d(v/a_3) = 7/4$ ($\because m \geq 10$). Similarly, if $v = a_3$ then $u = a_{-1}$ and hence $d(u/a_3) = 7/4$ and $d(v/a_3) = 0$. In either of the cases $l_3 = d(u/a_3) \neq 3/2$, a contradiction.

Case 3: $l_1 = l_3 = 7/4$.

Since $l_1 = 7/4$, $u = a_i$ for some $i \geq 4$ or $i \leq -2$. Since $l_3 = 7/4$, $u = a_i$ for some $i \geq 6$ or $i \leq 0$. These two together imply $u = a_i$ for some $i \geq 6$ or $i \leq -2$. In either of the cases, $d(u/a_2) = d(v/a_2) = 7/4$, a contradiction to equation (2).

Case 4: $l_1 = 1$, $l_3 \in \{3/2, 7/4\}$

Since $l_1 = 1$, we have $u, v \in \{a_2, a_0\}$ but then $d(u/a_3) = 1$ if $u = a_2$, or, $d(v/a_3) = 1$ if $u = a_0$. In either of the cases $l_3 = d(u/a_3) = d(v/a_3) = 1 \notin \{3/2, 7/4\}$, a contradiction.

Case 5: $l_1 = 3/2$ and $l_3 = 7/4$.

Since $l_1 = 3/2$, $a_3 \in \{u, v\}$. If $a_3 = u$, then $l_3 = d(u/a_3) = 0$. Else if $a_3 = v$ then $d(v/a_3) = 0$. Therefore, in either of the cases, $l_3 \neq 7/4$, a contradiction.

Other cases follows by symmetry. Hence the Claim.

Therefore, by the above Claim, $S' \in \mathfrak{R}(W_{1,m})$, a contradiction to the fact that S is of minimum cardinality in $\mathfrak{R}(W_{1,m})$. Hence the lemma. □

Theorem 4.2. *For each integer $m \geq 4$, every rmd-set $S \in \mathfrak{R}(W_{1,m})$ is powerful.*

Proof. For $4 \leq m \leq 7$, it is easy to see that a set S containing exactly two adjacent rim vertices of $W_{1,m}$ is in $\mathfrak{R}(W_{1,m})$ and is an *rmd*-set of $W_{1,m}$. Further, $\overline{S}$ also contains two adjacent rim vertices and hence by the super hereditary property of $\mathfrak{R}(G)$ it follows immediately that $\overline{S} \in \mathfrak{R}(W_{1,m})$. When $m = 8$, every *rmd*-set S of $W_{1,m}$ is a 2-element set of its rim vertices v_i, v_j such that $2 \leq d_{C_m}(v_i, v_j) \leq 3$. For each of such sets, $\overline{S}$ contains a 2-element subset S' of rim vertices v_{i-1}, v_{i+1} which is again an *rmd*-set of $W_{1,8}$ in $\mathfrak{R}(W_{1,m})$ implies that $\overline{S} \in \mathfrak{R}(W_{1,m})$ (by super hereditary property). Finally, when $n \geq 9$, by Lemma 4.1, for each *rmd*-set $S \in \mathfrak{R}(W_{1,m})$ we have $g_{\overline{S} \cup \{c_0\}}(a,b) \leq 2$ for every pair of vertices $a, b \in \overline{S}$ and hence $\overline{S} \in \mathfrak{R}(W_{1,m})$ (by Lemma 3.2). Hence the theorem. □

Corollary 4.3. *For every integer $m \geq 4$, $rmd_p(W_{1,m}) = rmd(W_{1,m})$.*

5. Maximal and Foul Rational Metric Dimension

A rational resolving set $S \in \mathfrak{R}(G)$ is called *maximal* if $\overline{S} \notin \mathfrak{R}(G)$. The least cardinality of a maximal rational resolving set of G is called maximal rational metric dimension of G and is denoted by $rmd_m(G)$.

A subset $S \subseteq V(G)$ is called *foul rational resolving set* if $S \notin \mathfrak{R}(G)$ and $\overline{S} \notin \mathfrak{R}(G)$. The least cardinality of a foul rational resolving set of G is called foul rational metric dimension of G and is denoted by $rmd_f(G)$.

Let $\hat{\mathfrak{R}}(G)$ and $\neg\mathfrak{R}(G)$ be the set of all maximal and foul rational resolving sets of G, respectively. In this section we determine maximal and foul rational metric dimension of a Wheel.

Lemma 5.1. *For any integer $4 \leq m \leq 8$, a 2-element set $S = \{v_i, v_j\}$ of rim vertices is in $\mathfrak{R}(W_{1,m})$ if and only if one of the following hold.*

- (i) *m is odd.*
- (ii) *$m = 4, 6$ and $j \neq i + \frac{m}{2}$.*
- (iii) *$m = 8$ and $j \notin \{i+1, i+4\}$.*

Proof. When m is odd and $4 \leq m \leq 8$, at most one vertex in $V - S$ is at a distance at least 3 from both the vertices v_i and v_j in C_m. Hence if $\Gamma(u/S) = \Gamma(v/S)$ for any $u, v \in V - S$, then both u and v can not be in a common v_iv_j-path of C_m. But then, $d_{C_m}(v_i,u) = d_{C_m}(v_i,v)$ and $d_{C_m}(v_j,u) = d_{C_m}(v_j,v)$ implies that $m = 2(d(v_i,u)+d(u,v_j)) =$ even, a contradiction. Further, when m is even, $\Gamma(v_{i+1 \pmod m}/S) = \Gamma(v_{i-1 \pmod m}/S) = (1,a)$ where $a = 1, \frac{3}{4}, \frac{7}{4}$ if $m = 4,6,8$, respectively. So, $j \neq i + \frac{m}{2}$ whenever m is even. Also, when $m = 8$, $\Gamma(v_{i+4 \pmod 8}/S) = \Gamma(v_{i-4 \pmod 8}/S) = (\frac{7}{4}, \frac{7}{4})$ whenever $j = i+1$ and hence, $j \neq i+1$.

On the other hand, suppose that all the conditions in the lemma hold. If $S \notin \mathfrak{R}(W_{1,m})$ for any $1 \leq m \leq 8$, then there are two vertices $u,v \in V-S$, such that $\Gamma(u/S) = \Gamma(v/S)$. If $\Gamma(u/S) = \Gamma(v/S) = (1,1)$, then $u,v \in \{v_{i-1 \pmod m}, v_{i+1 \pmod m}\}$, $j = i+2 \equiv i-2 \pmod m$ and hence, $m = 4$ and $j = i + \frac{m}{2}$, a contradiction to condition (ii). If $\Gamma(u/S) = \Gamma(v/S) = \left(\frac{3}{4}, \frac{3}{4}\right)$, then $u,v \in \{v_{i-2 \pmod m}, v_{i+2 \pmod m}\}$, $j = i+4 \equiv i-4 \pmod m$ and hence $m = 8$, a contradiction to condition (iii). If $\Gamma(u/S) = \Gamma(v/S) = \left(\frac{7}{4}, \frac{7}{4}\right)$, then length of a longest path on C_m between v_i and v_j to be at least 7 (so $m \geq 9$ by condition (iii) as $j \neq i+1$), or there are two longest paths of length 6 between v_i and v_j in C_m (so $m = 12$). In either of the cases $m \geq 9$, a contradiction to $m \leq 8$. If $\Gamma(u/S) = \Gamma(v/S) = \left(1, \frac{3}{4}\right)$, then $u,v \in \{v_{i-1 \pmod m}, v_{i+1 \pmod m}\}$, $j = i+3 \equiv i-3 \pmod m$ and hence $m = 6$ and $j = i + \frac{m}{2}$, a contradiction to condition (ii). If $\Gamma(u/S) = \Gamma(v/S) = \left(1, \frac{7}{4}\right)$, then $u,v \in \{v_{i-1 \pmod m}, v_{i+1 \pmod m}\}$, $j = i+k \equiv i-l \pmod m$ for some $k,l \geq 4$. So, $m = 8$ and $j = i+4$, or $m \geq 9$, a contradiction to condition (iii) or $m \leq 8$, respectively. If $\Gamma(u/S) = \Gamma(v/S) = \left(\frac{3}{4}, \frac{7}{4}\right)$, then $u,v \in \{v_{i-2 \pmod m}, v_{i+2 \pmod m}\}$, $j = i+k \equiv i-l \pmod m$ for some $k,l \geq 5$, and hence $m \geq 10$, again a contradiction. □

Lemma 5.2. *For any $m \geq 13$, let $S \in \mathfrak{R}(W_{1,m})$ be such that $|S \cap \{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+6}\}| \geq 3$ for some $v_i \in S$. Then $rmd_m(W_{1,m}) \leq |S|+3$.*

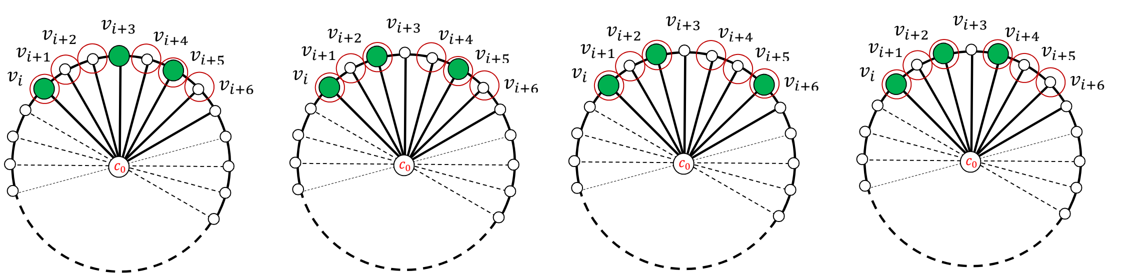

FIGURE 3. Possible *rmd*-sets and the corresponding rmd_m-set as in the proof of Lemma 5.2.

Proof. Let $a = \min\{j : j > i, v_j \in S\}$ and $b = \min\{k : k > a, v_k \in S\}$. If $i+3 \notin \{a,b\}$, then the set $S' = S \cup \{v_{i+1}, v_{i+2}\} \cup \{v_{i+4}, v_{i+5}, v_{i+6}\} \in \hat{\mathfrak{R}}(W_{1,m})$ (by Lemma 3.2, as $v_i \in S'$) and $v_a, v_b \in S'$. Hence, $rmd_m(W_{1,m}) \leq |S'| \leq |S|+3$. If $i+3 \in \{a,b\}$, then the set $S' = (S - \{v_{i+3}\}) \cup \{v_{i+1}, v_{i+2}\} \cup \{v_{i+4}, v_{i+5}, v_{i+6}\} \in \hat{\mathfrak{R}}(W_{1,m})$ (by Lemma 3.2, as $v_i \in S'$) and $|S' \cap \{v_a, v_b\}| = 1$. Hence, $rmd_m(W_{1,m}) \leq |S'| \leq (|S|-1)+4 = |S|+3$. □

Lemma 5.3. *If S is an rmd-set of a wheel $W_{1,m}$ and $m \geq 13$, then there exist integers $0 \leq a < b$ such that $v_a, v_b \in S$ and $g_{S\cup\{c_0\}}(v_a, v_b) \geq 3$.*

Proof. If not, then $g_{S\cup\{c_0\}}(v_a, v_b) \leq 2$. But then, as $rmd(W_{1,m}) \geq 3$ (since $m \geq 13$), we get an integer $c > b$ such that $g_{S\cup\{c_0\}}(v_b, v_c) \leq 2$. Further, by assumption, $g_{S\cup\{c_0\}}(v_x, v_a) \leq 2$ and $g_{S\cup\{c_0\}}(v_c, v_y) \leq 2$ for every $v_x, v_y \in S$ with $x < a$ and $y > c$. Hence, the set $S' = S - \{v_b\}$ satisfies the conditions of Lemma 3.2. So, $S' \in \mathfrak{R}(W_{1,m})$ with $|S'| < |S| = rmd(W_{1,m})$, a contradiction. □

Corollary 5.4. *For every integer $m \geq 13$, $rmd_m(W_{1,m}) \leq rmd(W_{1,m}) + 4$.*

Proof. Let S be an rmd-set of $W_{1,m}$ for $m \geq 13$. Then, Theorem 2.1 and Lemma 5.3, there are suffixes $a < b < c$ such that $v_a, v_b, v_c \in S$ with $g_{S\cup\{c_0\}}(v_a, v_b) \geq 3$. But then, $g_{S\cup\{c_0\}}(v_b, v_c) \leq 2$ and hence, $v_c \in \{v_{b+1}, v_{b+2}, v_{b+3}\}$. Let $S' = (S - \{v_b\}) \cup \{v_{b-3}, v_{b-2}, v_{b-1}, v_{b+1}, v_{b+2}, v_{b+3}\}$. Then, S' satisfies all the conditions of Lemma 3.2 and hence $S \in \mathfrak{R}(W_{1,m})$. Further, $g_{\overline{S}\cup\{c_0\}}(v_{a-1}, v_b) \geq 3$, $g_{\overline{S}\cup\{c_0\}}(v_b, v_{b+4}) = 3$ and $|S'| \leq |S|+4$. Therefore, by Lemma 3.2, $S' \notin \mathfrak{R}(W_{1,m})$. Hence $S' \in \hat{\mathfrak{R}}(W_{1,m})$. This shows that $rmd_m(W_{1,m}) \leq |S'| = |S|+4$ (since $v_c \in S$)=$rmd(W_{1,m})+4$. □

Theorem 5.5. *For any integer $m \geq 3$,*

$$rmd_m(W_{1,m}) = \begin{cases} 3, & \textit{if } m = 3,4. \\ 4, & \textit{if } m = 5,6. \\ 6, & \textit{if } 7 \leq m \leq 11. \\ \lceil \frac{m}{4} \rceil + 4, & \textit{if } m \geq 12 \textit{ and } m \equiv 0 \pmod 8. \\ \lceil \frac{m}{4} \rceil + 3, & \textit{if } m \geq 12 \textit{ and } m \not\equiv 0 \pmod 8. \end{cases}$$

Proof. Let S be an element of minimum cardinality in $\hat{\mathfrak{R}}(W_{1,m})$. Then $S \in \mathfrak{R}(W_{1,m})$. For $m = 3$, as $3 = rmd(W_{1,3}) = \min\{|T| : T \in \mathfrak{R}(W_{1,m})\}$, $\overline{T} \notin \mathfrak{R}(W_{1,m})$ for every $T \in \mathfrak{R}(W_{1,4})$. Hence $|S| =$

$rmd(W_{1,m}) = 3$. For $m = 4$, $|\overline{S}| \leq 1$ (else by condition (ii) of Lemma 5.1 and Theorem 2.1, $S \notin \mathfrak{R}(W_{1,4})$). Also, the set $S_1 = \{v_0, v_1, v_2\}$ is in $\mathfrak{R}(W_{1,4})$ and $\overline{S}_1 \notin \mathfrak{R}(W_{1,4})$. Hence $rmd_m(W_{1,4}) = 3$.

When $m = 5,7$, by Lemma 5.1, $|\overline{S}| \leq 1$ (else $\overline{S} \in \mathfrak{R}(W_{1,m})$) and hence $|S| \geq m-1$. Also, for each odd $m, 4 \leq m \leq 8$, the set $S' = \{v_0, v_1, \ldots, v_{m-2}\} \in \mathfrak{R}(W_{1,m})$ being the super set of $\{v_0, v_1\}$, which is in $\mathfrak{R}(W_{1,m})$ (by Lemma 5.1) and $\overline{S'} \notin \mathfrak{R}(W_{1,m})$. Hence $|S| = m-1$ for $m = 5,7$.

For $m = 6,8$, by Lemma 5.1, $|\overline{S}| \leq 2$ (else $\overline{S} \in \mathfrak{R}(W_{1,m})$) and hence $|S| \geq m-2$. Also, for $m = 6,8$ the set $\{v_0, v_2, v_4, v_5, \ldots, v_{m-1}\} \in \mathfrak{R}(W_{1,m})$ being the super set of $\{v_0, v_2\}$, which is in $\mathfrak{R}(W_{1,m})$ (by Lemma 5.1). Hence $|S| = m-2$ for $m = 6,8$.

When $9 \leq m \leq 12$, by Lemma 3.2, at least one of the conditions (i) or (ii) must fail with respect to the set $\overline{S}$ ($\because \overline{S} \notin \mathfrak{R}(W_{1,m})$). Hence, $\overline{S}$ should contain at least 6 vertices. Therefore, $|S| \geq 6$. On the other hand it is easy to see, by Lemma 3.2, that the set $S = \{v_0, v_1, v_2, v_4, v_5, v_6\} \in \hat{\mathfrak{R}}(W_{1,m})$ for $9 \leq m \leq 12$ ($\because g_{\overline{S} \cup \{c_0\}}(v_1, v_3) = g_{\overline{S} \cup \{c_0\}}(v_3, v_6) = 3; g_{S \cup \{c_0\}}(v_1, v_4) = 1, g_{S \cup \{c_0\}}(v_0, v_6) \leq 5$ and for all other pairs $a, b \in V - S$ we get $g_{S \cup \{c_0\}}(a, b) = 0$). Therefore, $|S| = 6$ for $9 \leq m \leq 12$.

Let us now consider the cases $m \geq 13$. In these cases, as $\overline{S} \notin \mathfrak{R}(W_{1,m})$, by Lemma 3.2 we see that S shall contain a 6-element proper subset T which is of the form $T = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ or $T = \{v_i, v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5}, v_{i+6}\}$ or a 10-element subset $T = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+k}, v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, v_{i+k+4}\}$ for some $k \geq 6$. For the minimality of $|S|$, we consider the second option (which selects 6 vertices out of 7). Without loss of generality, we take $T = \{v_0, v_1, v_2, v_4, v_5, v_6\} \subseteq S$ and $v_3 \notin S$.

Let $S' = S \cup \{v_3\} - \{v_1, v_2, v_4, v_5\}$. Then S' satisfies all the conditions of Lemma 3.2 ($\because S$ satisfies all the conditions of Lemma 3.2 and by the inclusion of v_3) and hence, $S' \in \mathfrak{R}(W_{1,m})$. But then, $|S'| \geq rmd(W_{1,m})$ and $|S'| = |S| - 3$. Thus, $|S| \geq rmd(W_{1,m}) + 3$. We now show that the equality can not be achieved in the cases $m \equiv 0, 1 \pmod 8$.

Claim: $|S| \geq rmd(W_{1,m}) + 4$, whenever $m \equiv 0, 1 \pmod 8$.

Let a and b be the least and greatest indices (a may be b) such that $v_a, v_b \in S - \{v_1, v_2, v_4, v_5\}$ (such a vertex exists because $m \geq 13$ and by Lemma 3.2, otherwise $g_{S \cup C_0}(v_0, v_6) \geq 6$). Then, by

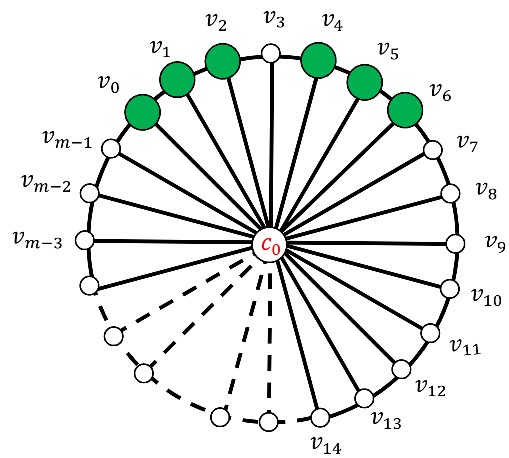

FIGURE 4. The subset T of $S \in \hat{\mathfrak{R}}(W_{1,m})$.

Lemma 3.2, either $g_{S\cup\{c_0\}}(v_6, v_a) \leq 4$ or $g_{S\cup\{c_0\}}(v_0, v_b) \leq 4$. Without loss of generality, due to symmetry we take $g_{S\cup\{c_0\}}(v_6, v_a) \leq 4$. Let $l = g_{S\cup\{c_0\}}(v_6, v_a)$ and $G = W_{1,m}$.

Since $m \geq 13$ and $m \equiv 0, 1 \pmod 8$, we have $m = 8k$ or $m = 8k+1$ for some integer $k \geq 7$. When $m = 8k$, by Theorem 2.1, $rmd(G) = rmd(W_{1,8k}) = \lceil \frac{8k}{4} \rceil = 2k$. When $m = 8k+1$, by Theorem 2.1, $rmd(G) = rmd(W_{1,8k+1}) = \lceil \frac{8k+1}{4} \rceil - 1 = 2k+1-1 = 2k$. Thus, in either of the cases, it suffices to show that $|S| \geq 2k+4$ whenever $m = 8k$ or $8k+1$.

Case 1: $l = 3, 4$.

In this case, $a = 10$ or 11 if $l = 3$ or 4 respectively, and $g_{S\cup\{c_0\}}(v_a, v_{14}) \leq 2$. Let $G' = (G - \{v_1, v_2, \dots, v_a\}) + v_0v_{a+1}$. Then, $G' \equiv W_{1,m-a}$ and $S' = S - \{v_1, v_2, v_4, v_5, v_6, v_a\} \in \mathfrak{R}(G')$ (since S' satisfies all the conditions of Lemma 3.2 as S fulfilled the conditions and by the construction of G'). Therefore, $|S'| \geq rmd(G')$. But $rmd(G') = \lceil \frac{m-11}{4} \rceil = 2k-2$ (by Theorem 2.1 as $m - a \not\equiv 1 \pmod 8$ for $m = 8k$ or $8k+1$, $10 \leq a \leq 11$) and $|S| = |S'| + 6$. Hence, $|S| = |S'| + 6 \geq 2k - 2 + 6 = 2k + 4$.

Case 2: $l = 2$.

In this case, $a = 19$. Let $G' = (G - \{v_1, v_2, \dots, v_6\}) + v_0v_7$. Then, $G' \equiv W_{1,m-6}$ and $S' = S - \{v_1, v_2, v_4, v_5, v_6\} \in \mathfrak{R}(G')$. Therefore, $|S'| \geq rmd(G') = \lceil \frac{m-6}{4} \rceil = 2k-1$ (by Theorem 2.1 as $m - 6 \not\equiv 1 \pmod 8$ for $m = 8k$ or $8k+1$) and hence $|S| = |S'| + 5 \geq 2k - 1 + 5 = 2k + 4$.

Case 3: $l = 0, 1$.

In this case, $a \in \{7,8\}$. Let $G' = (G - \{v_1, v_2, \ldots, v_5\}) + v_0v_6$. Then, $G' \equiv W_{1,m-5}$ and $S' = S - \{v_1, v_2, v_4, v_5, v_6\} \in \Re(G')$. Therefore, $|S'| \geq rmd(G') = \lceil \frac{m-5}{4} \rceil = 2k-1$ (by Theorem 2.1 as $m-5 \not\equiv 1 \pmod 8$ for $m = 8k$ or $8k+1$) and hence $|S| = |S'| + 5 \geq 2k-1+5 = 2k+4$.

Hence the Claim.

By the above Claim, Theorem 2.1, and above explanation, we now conclude that

$$rmd_m(W_{1,m}) \geq \begin{cases} \lceil \frac{m}{4} \rceil + 3 & \text{if } m \not\equiv 0 \pmod 8. \\ \lceil \frac{m}{4} \rceil + 4 & \text{if } m \equiv 0 \pmod 8. \end{cases}$$

To prove the reverse inequality we execute rmd_m-sets of $W_{1,m}$ in different cases as follows:

Let $m = 8k+q$, where $0 \leq q \leq 7$. Let $S_1 = \{v_{8j} : 0 \leq j \leq k\}$ and $S_2 = \{v_{5+8j} : 0 \leq j \leq k-1\}$. Then $|S_1 \cup S_2| = 2k+1$ if $m \not\equiv 0 \pmod 8$ and $S_1 \cap S_2 = \emptyset$. Let $S = S_1 \cup S_2$.

Case 1: $q = 0, 1$.

In this case, by Corollary 5.4, $rmd_m(W_{1,m}) \leq rmd(W_{1,m}) + 4$. Hence, by Theorem 2.1, $rmd(W_{1,m}) \leq \lceil \frac{m}{4} \rceil + 4$ if $q = 0$, and $rmd(W_{1,m}) \leq \lceil \frac{m}{4} \rceil - 1 + 4 = \lceil \frac{m}{4} \rceil + 3$ if $q = 1$.

Case 2: $q = 2$.

By Lemma 3.2, the set $S \in \Re(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_0\} \subseteq S$ and hence $|S \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_0, v_1\}| \geq 3$. Therefore, by Lemma 5.2, $rmd_m(W_{1,m}) \leq |S| + 3 = 2k + 1 + 3 = \lceil \frac{m}{4} \rceil + 3$.

Case 3: $q = 3$.

By Lemma 3.2, the set $S \in \Re(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_0\} \subseteq S$ and hence $|S \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_0\}| \geq 3$. Therefore, by Lemma 5.2, $rmd_m(W_{1,m}) \leq |S| + 3 = 2k + 1 + 3 = \lceil \frac{m}{4} \rceil + 3$.

Case 4: $q = 4$.

By Lemma 4.1, the set $S' = (S - \{v_0\}) \cup \{v_{8k+3}\} \in \Re(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_{8k+3}\} \subseteq S'$ and hence $|S' \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_{8k+3}\}| \geq 3$. Therefore, by Lemma 5.2, $rmd_m(W_{1,m}) \leq |S'| + 3 = [(|S|-1)+1] + 3 = |S| + 3 = 2k+1+3 = \lceil \frac{m}{4} \rceil + 3$.

Case 5: $q = 5, 6$.

By Lemma 3.2, the set $S' = S \cup \{v_{8k+3}\} \in \Re(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_{8k+3}\} \subseteq S'$ and hence $|S' \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_{8k+3}\}| \geq 3$. Therefore, by Lemma 5.2, $rmd_m(W_{1,m}) \leq |S'| + 3 = [|S|+1] + 3 = [2k+2] + 3 = \lceil \frac{m}{4} \rceil + 3$.

Case 6: $q = 7$.

By Lemma 3.2, the set $S' = (S - \{v_0\}) \cup \{v_{8k+6}, v_{8k+3}\} \in \mathfrak{R}(W_{1,m})$. Also, $\{v_{5+8(k-1)}, v_{8k}, v_{8k+3}\} \subseteq S'$ and hence $|S' \cap \{v_{8k-3}, v_{8k-2}, v_{8k-1}, v_{8k}, v_{8k+1}, v_{8k+2}, v_{8k+3}\}| \geq 3$. Therefore, by Lemma 5.2, $rmd_m(W_{1,m}) \leq |S'| + 3 = [|S| + 1] + 3 = [2k + 2] + 3 = \lceil \frac{m}{4} \rceil + 3$. □

Theorem 5.6. *For any integer $m \geq 3$,*

$$rmd_f(W_{1,m}) = \begin{cases} 2, & \text{if} \quad m = 3, 4. \\ 6, & \text{if} \quad m \geq 12. \end{cases}$$

Further, for $5 \leq m \leq 12$, $\neg\mathfrak{R}(W_{1,m}) = \emptyset$.

Proof. When $m = 3$, by Theorem 2.1, $rmd(W_{1,3}) = 3$ and hence the set $S = \{v_0, v_1\} \in \neg\mathfrak{R}(W_{1,3})$ and is of minimum cardinality. So, $rmd_f(W_{1,3}) = 2$. When $m = 4$, by Lemma 5.1, the sets $S = \{v_0, v_2\} \notin \mathfrak{R}(W_{1,4})$ and $\overline{S} = \{c_0, v_1, v_3\} \notin \mathfrak{R}(W_{1,4})$. Further, by Lemma 5.1, every 4-element subset of vertices of $W_{1,4}$ is a super set of an *rmd*, and hence it is in $\mathfrak{R}(W_{1,4})$ implies that $rmd_f(W_{1,4}) = |S| = 2$. When $5 \leq m \leq 11$, each subset S of vertices of $W_{1,m}$ shall contain at least 6 vertices in S to make $\overline{S} \notin \mathfrak{R}(W_{1,m})$, but then $S \in \mathfrak{R}(W_{1,m})$. Hence $\neg\mathfrak{R}(W_{1,m}) = \emptyset$ for $5 \leq m \leq 11$.

For $m \geq 12$, by Lemma 3.2, S shall contain at least 6 rim vertices to make $\overline{S} \notin \mathfrak{R}(W_{1,m})$. Thus, $|S| \geq 6$ for every $S \in \neg\mathfrak{R}(W_{1,m})$. To prove the reverse inequality, let $S = \{v_1, v_2, \ldots, v_6\}$. Then $g_{S \cup \{c_0\}}(v_1, v_6) \geq 6$ and $g_{\overline{S} \cup \{c_0\}}(v_0, v_7) \geq 6$. So, by Lemma 3.2, $S \notin \mathfrak{R}(W_{1,m})$ and $\overline{S} \notin \mathfrak{R}(W_{1,m})$ implies that $S \in \neg\mathfrak{R}(W_{1,m})$ with $|S| = 6$. Hence the theorem. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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