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ON THE ANALYSIS OF INTERVAL DATA BASED GAME THEORETICAL PROBLEM UNDER VARIOUS SYMMETRIC FUZZY ENVIRONMENTS

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Abstract: In a real world environment, the payoffs of a matrix game are varying between two fixed values because of uncertainty. This type of payoffs may not be defined as a crisp numbers. In order to model such uncertainty, the payoffs can be represented as interval numbers. A matrix game with payoffs of such interval numbers are called interval data based matrix game. This work takes an initiative to solve the interval data based matrix game in an uncertain environment. In order to find the optimum solution of the interval matrix game without the knowledge of interval arithmetic, the fuzzy set theory is used in various symmetric environments. Finally the analysis is made on the solutions obtained through various symmetric fuzzy environments with the help of different shapes of fuzzy numbers.

Keywords: interval number; symmetric fuzzy number; matrix game; fuzzification; defuzzification.

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1. INTRODUCTION

The problems of optimization under uncertainty are appearing in many areas of application and present many interesting challenges in concept and computation. The key step of optimization is the exact formulation of real word problems. But in an uncertain environment, modelling of variables and their values of optimization problems are difficult in a precise manner. Stochastic and fuzzy valued variables are extensively used to model such uncertainties in optimization theories by the experts in their fields. However, the variables of interval number are also utilized for modelling the same. Although, the optimization models with random and fuzzy variables are widely used in many areas such as planning and scheduling of production, transportation etc. under uncertainty, the fuzzy mathematical formulation of the real world problems is familiar in recent years in an uncertain context. Moreover, on nowadays, many authors are involving themselves to solve such fuzzy optimization problems using fuzzy arithmetic and defuzzification techniques for finding its optimum solutions. As per the objective of this study, here, we have made a sample survey of recent works on fuzzy game theoretical problems, a familiar problem in optimization theory, to get the idea about the current trend in this field.

In 2012 [3], Deng-Feng Li and Feng-Xuan Hong developed and effective methodology for solving constrained matrix games with payoff of triangular fuzzy number. Mijanur et al. in 2015 [15] have proposed a ranking function to solve bi-matrix games with payoffs of triangular intuitionistic fuzzy numbers and also proved that generalization of fuzzy non-linear programming problem is non-linear intuitionistic fuzzy programming problem. Subsequently, Senthil Kumar & Kumaraghuru [20] and Stalin & Thirucheran [21] proposed a methodology for two person matrix games in which all elements of pay-off matrix are considered as fuzzy numbers and used a method of defuzzification to convert it into crisp game matrix in the same year. In 2016, hexagonal fuzzy number based Two Person Zero Sum Game was solved using ranking function proposed by Jon Arockiaraj and Sivasankari [9]. In 2017, Namarta and others introduced two person zero sum game with its payoff matrix are represented by dodecagonal fuzzy numbers and solve the same with the help of defuzzification technique. Subsequently in the same year, Arindam Chaudhuri solved the fuzzy game problem using minimax-maximin in which payoff matrices are represented by interval and LR trapezoidal fuzzy numbers, Iden Hassen and Zainab Saad [6] proposed new ranking function algorithm for solving Fuzzy Game problems to get the best gains, Monisha and Sangeetha [16] solved the fuzzy game problem in which all elements of payoffs are considered as pentagonal fuzzy numbers based on ranking function and discussed its solution with saddle point by using minimax – maximin, Thirucheran and others [23] has solved two person zero sum game with payoff matrices of triangular fuzzy numbers without converting it into crisp problem, Dinagar and Narayanan [22] solved fuzzy game problem with hexagonal fuzzy number as elements of its payoff matrix using minimax – maximini method, Konstantin [11] proposed an algorithm to formulate and construct a Nash equilibrium in a bimatrix game. In 2018, Dong Qiu [4] and others presented Bi-matrix game model with crisp payoffs is presented using ranking function method and hence its equilibrium solution is converted into optimal solution of non-linear programming problem. Subsequently in the same year Jishu Jana [8] proposed the method to solve fuzzy matrix game using linear ranking function to reduce its computational complexity, Krishnaveni and Ganesan [13] introduce an algorithm to find the optimal solution of fuzzy game without conversion of classical form using new ranking function, Iden [7] utilized three different ranking algorithms for solving fuzzy trapezoidal payoffs problem to get the best gains by its comparison. In 2019, Brikaa et al. [2] developed fuzzy multi-objective programming algorithm to solve fuzzy numbers based constrained matrix game and finally the comparison is made through numerical illustration between the results of proposed method and GAMS software. Successively Karthi and Selvakumari [10] introduced symmetric pentagonal and constant pentagonal fuzzy matrices in Game Theory to study the existence of Nash equilibrium in pure and mixed strategies; Namarta et al. [18] introduced a new procedure to convert the fuzzy game problems with payoff of octagonal fuzzy numbers into crisp game problems for finding its optimum solution using any one of the traditional method, Krishnapriya and Senbagam [12] also converted the fuzzy game problems with payoff of pentagonal and hexagonal fuzzy numbers into crisp game problems using ranking function to find its solution, Maheswari and Vijaya [14] introduced an algorithm to solve two person zero-sum matrix game problem in an uncertain environment using incenter of centroids based ranking function of generalized trapezoidal fuzzy numbers,

Recently in 2020, Yang and Song [24] proposed a method to solve matrix game in which payoffs are considered as triangular dual hesitant fuzzy numbers. Afterwards Seikh et al. [19] used lexicographic method to solve two non-linear programming problems with triangular fuzzy numbers in order to solve matrix games of with payoffs of triangular hesitant fuzzy elements by transforming two non-linear programming problems with triangular hesitant fuzzy elements formulated from the payoff matrix.

As per the above survey, most probably, the authors are used fuzzy set theory to model the game theoretical problems in the environment with uncertainty. But in 2011, Handan Akyar and Emrah Akyar used interval valued matrix to model such uncertainties [5]. Moreover, they used interval arithmetic to find its optimum solution using graphical method. In this work, we present an algorithm to solve the interval data based game theoretical problem using fuzzy set theory and classical methods of game theory. The speciality of the proposed algorithm is to analyse the optimum solution of interval matrix game based on various forms of symmetric fuzzy quantities under fuzzy environment.

The organization of this paper is given as follows: In section 2, we provide some basic ideas namely interval number, fuzzy set, various forms of fuzzy numbers and fuzzification of interval number. In Section 3, we introduce a new ranking function for ordering generalized trapezoidal, pentagonal and hexagonal fuzzy numbers based on its new centroids. Section 4 proposes an algorithm for solving interval data based Two Person Zero Sum Matrix Game under various symmetric fuzzy environments and also introduces its mathematical formulation. In Section 5, three numerical illustrations are given in order to analyse the optimum solutions of Interval Matrix Game under three symmetric fuzzy environments. The research work ends with conclusion in Section 6.

2. PRELIMINARIES

Definition 1 (Interval Number). A number is said to be an interval number if it is a subset of the real line R with the following form:

$$[a,b]=\left\{ x\in \mathrm{R} \,\middle|\, a\le x\le b \right\}$$

Definition 2 (Fuzzy Set). A set is said to be a fuzzy subset of the Universal Set X if the membership values of its objects are taken from the interval [0, 1]. The set and its membership function are defined as follows:

$$\tilde{\mathrm{A}}=\left\{ x \,\middle|\, x\in \mathrm{X} \ \& \ \mu_{\tilde{\mathrm{A}}}:\tilde{\mathrm{A}}\to [0,1] \right\}$$

Definition 3 (Fuzzy Number). A fuzzy number $\tilde{A}$ is a subset of real line R, with the membership function satisfying the following properties:

1. $\mu_{\tilde{A}}(x)$ is piecewise continuous in its domain.
2. $\tilde{A}$ is normal, i.e., there is a $x_0\in X$ such that $\mu_{\tilde{A}}(x_0)=1$.
3. $\tilde{A}$ is convex, i.e., $\mu_{\tilde{A}}\left(\lambda x_1+(1-\lambda)x_2\right)\ge \min(\left(\mu_{\tilde{A}}(x_1),\mu_{\tilde{A}}(x_2)\right),\ \forall\ x_1,x_2$ in X.

Definition 4 (Triangular Fuzzy Number). A finite fuzzy subset $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3;\omega)$ of real numbers is said to be generalized triangular fuzzy number if its membership function of the following form.

$$\mu_{\tilde{A}}(x)=\begin{cases} 0 & x\le a_1 \\ \omega\left(\dfrac{x-\varphi_1}{\varphi_2-\varphi_1}\right) & \varphi_1\le x\le \varphi_2 \\ \omega\left(\dfrac{\varphi_3-x}{\varphi_3-\varphi_2}\right) & \varphi_2\le x\le \varphi_3 \\ 0 & x>\varphi_3 \end{cases}$$

The triangular fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3;\omega)$ is said to be symmetric if $\varphi_2-\varphi_1=\varphi_3-\varphi_2$.

Definition 5 (Trapezoidal Fuzzy Number). A finite fuzzy subset $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4;\omega)$ of real numbers is said to be generalized trapezoidal fuzzy number if its membership function of the following form.

$$\mu_{\tilde{A}}(x)=\begin{cases}\omega\left(\dfrac{x-\varphi_1}{\varphi_2-\varphi_1}\right) & \varphi_1\le x\le\varphi_2\\ \omega & \varphi_2\le x\le\varphi_3\\ \omega\left(\dfrac{x-\varphi_4}{\varphi_3-\varphi_4}\right) & \varphi_3\le x\le\varphi_4\\ 0 & otherwise\end{cases}$$

The trapezoidal fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4;\omega)$ is said to be symmetric if $\varphi_2-\varphi_1=\varphi_4-\varphi_3$.

Definition 6 (Pentagonal Fuzzy Number). A finite fuzzy subset $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5;\omega)$ of real numbers is said to be generalized pentagonal fuzzy number if its membership function of the following form.

$$\mu_{\tilde{A}}(x)=\begin{cases}0, & x<a_1\\ \omega\left(\dfrac{x-\varphi_1}{\varphi_2-\varphi_1}\right) & \varphi_1\le x\le\varphi_2\\ \omega\left(\dfrac{x-\varphi_2}{\varphi_3-\varphi_1}\right) & \varphi_2\le x\le\varphi_3\\ \omega & x=\varphi_3\\ \omega\left(\dfrac{\varphi_4-x}{\varphi_4-\varphi_3}\right) & \varphi_3\le x\le\varphi_4\\ \omega\left(\dfrac{\varphi_5-x}{\varphi_5-\varphi_4}\right) & \varphi_4\le x\le\varphi_5\\ 0 & x>\varphi_5\end{cases}$$

The pentagonal fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5;\omega)$ is said to be symmetric if $\varphi_2-\varphi_1=\varphi_5-\varphi_4$ and $\varphi_3-\varphi_2=\varphi_4-\varphi_3$.

Definition 7 (Hexagonal Fuzzy Number). A finite fuzzy subset $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5,\varphi_6;\omega)$ of real numbers is said to be generalized hexagonal fuzzy number if its membership function of the following form.

$$\mu_{\tilde{A}}(x)=\begin{cases}0 & x\le\varphi_1\\ \frac{\omega}{2}\left(\frac{x-\varphi_1}{\varphi_2-\varphi_1}\right) & \varphi_1\le x\le\varphi_2\\ \frac{\omega}{2}+\frac{\omega}{2}\left(\frac{x-\varphi_2}{\varphi_3-\varphi_2}\right) & \varphi_2\le x\le\varphi_3\\ \omega & \varphi_3\le x\le\varphi_4\\ \omega-\frac{\omega}{2}\left(\frac{x-\varphi_4}{a_5-\varphi_4}\right) & \varphi_4\le x\le\varphi_5\\ \frac{\omega}{2}\left(\frac{a_6-x}{\varphi_6-\varphi_5}\right) & \varphi_5\le x\le\varphi_6\\ 0 & x\ge\varphi_6\end{cases}$$

The hexagonal fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5,\varphi_6;\omega)$ is said to be symmetric if $\varphi_2-\varphi_1=\varphi_6-\varphi_5$ and $\varphi_3-\varphi_2=\varphi_5-\varphi_4$.

Definition 8 (Fuzzification). In most of the real life situations, the observation of data is the numbers around some real value. In order to solve the problems of optimization, this type of data has to be represented in the standard form of numbers of mathematical theory. In 1965, Professor L.A. Zadeh [25] introduced fuzzy set theory to represent such vague information in some standard form of fuzzy numbers. In this section, we have introduced new approaches to fuzzify such vague information in the standard form of fuzzy numbers.

Let us consider the vague information as $[a_L, a_U]$. This is called interval number. The collection of such interval numbers is called interval data. The interval can be fuzzified different symmetric forms of standard fuzzy numbers in the following way.

The different symmetric forms of some standard fuzzy numbers for interval number are taken as follows:

Fuzzification of the interval number $[a_L, a_U]$ in the three different symmetric forms of trapezoidal fuzzy number as

(1) $$[a_L, a_U] \Rightarrow \begin{cases} (a_L, a_L + d, a_L + 2d, a_U) & \text{with} \quad d = (a_U - a_L)/3 \\ (a_L, a_L + d, a_L + 3d, a_U) & \text{with} \quad d = (a_U - a_L)/4 \\ (a_L, a_L + 2d, a_L + 3d, a_U) & \text{with} \quad d = (a_U - a_L)/5 \end{cases}$$

Fuzzification of the interval number $[a_L, a_U]$ in the three different symmetric forms of pentagonal fuzzy number as

(2) $$[a_L, a_U] \Rightarrow \begin{cases} (a_L, a_L + d, a_L + 2d, a_L + 3d, a_U) & \text{with} \quad d = (a_U - a_L)/4 \\ (a_L, a_L + d, a_L + 3d, a_L + 5d, a_U) & \text{with} \quad d = (a_U - a_L)/6 \\ (a_L, a_L + 2d, a_L + 3d, a_L + 4d, a_U) & \text{with} \quad d = (a_U - a_L)/6 \end{cases}$$

Fuzzification of the interval number $[a_L, a_U]$ in the three different symmetric forms of hexagonal fuzzy number as

(3) $$[a_L, a_U] \Rightarrow \begin{cases} (a_L, a_L + d, a_L + 2d, a_L + 3d, a_L + 4d, a_U) & \text{with} \quad d = (a_U - a_L)/5 \\ (a_L, a_L + d, a_L + 3d, a_L + 5d, a_L + 7d, a_U) & \text{with} \quad d = (a_U - a_L)/8 \\ (a_L, a_L + 2d, a_L + 3d, a_L + 4d, a_L + 5d, a_U) & \text{with} \quad d = (a_U - a_L)/7 \end{cases}$$

3. Ranking of Fuzzy Numbers Based on Centroid

In recent years many ranking functions have been introduced by the researchers based on centroid of fuzzy numbers. In this section, we introduce new real valued ranking function using new centroids of various shapes of fuzzy numbers in order to solve the optimization problems under fuzzy environment. First we introduce the new centroids of generalized trapezoidal, pentagonal and hexagonal fuzzy numbers in the following way.

In order to find these new centroids, three various centroids of generalized trapezoidal fuzzy number are found by splitting trapezoid into three plan figures in three different ways. First the trapezoid is divided into two triangles and one rectangle (Fig. 1a). Subsequently the trapezoid is divided into three triangles in two different ways (Fig. 1b & 1c). Finally we obtain the centroid of centroids of three plane figures which have been made in the above three ways.

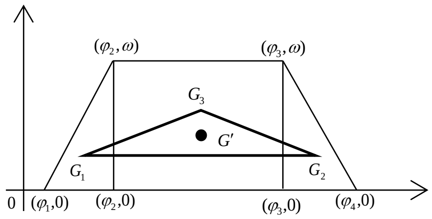


FIGURE 1a. Centroid G' of Trapezoid

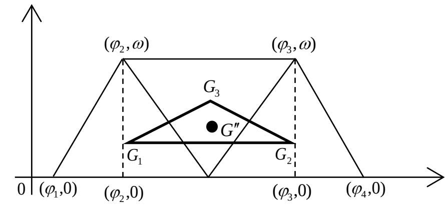


FIGURE 1b. Centroid G'' of Trapezoid

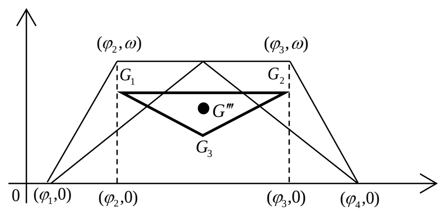


FIGURE 1c. Centroid G''' of Trapezoid

The three centroids of generalized trapezoidal fuzzy number are shown as follows:

$$G'=\left(\frac{4\varphi_1+5\varphi_2+5\varphi_3+4\varphi_4}{18},\frac{5\omega}{9}\right),G''=\left(\frac{3\varphi_1+\varphi_2+3\varphi_3+2\varphi_4}{9},\frac{4\omega}{9}\right),G'''=\left(\frac{5\varphi_1+4\varphi_2+4\varphi_3+5\varphi_4}{18},\frac{4\omega}{9}\right)$$

3.1. New Centroid of Generalized Trapezoidal Fuzzy Number

Since the three centroids $G'=\left(\frac{4\varphi_1+5\varphi_2+5\varphi_3+4\varphi_4}{18},\frac{5\omega}{9}\right)$, $G''=\left(\frac{3\varphi_1+\varphi_2+3\varphi_3+2\varphi_4}{9},\frac{4\omega}{9}\right)$ and $G'''=\left(\frac{5\varphi_1+4\varphi_2+4\varphi_3+5\varphi_4}{18},\frac{4\omega}{9}\right)$ of trapezoid are non – collinear, the triangle is made with these three vertices. These three centroids G', G'' and G''' are balancing point of the trapezoid of the generalized trapezoidal fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4;\omega)$. The centroid point G of these three balancing points (G', G'' and G''') will be much more balancing point of the trapezoid ADEF (See Fig. 2). So G can be considered as the new centroid point of the generalized trapezoidal fuzzy number. It is given as follows:

$$(x_c,y_c)=\left(\frac{15\varphi_1+11\varphi_2+15\varphi_3+13\varphi_4}{54},\frac{13\omega}{27}\right) \tag{4}$$

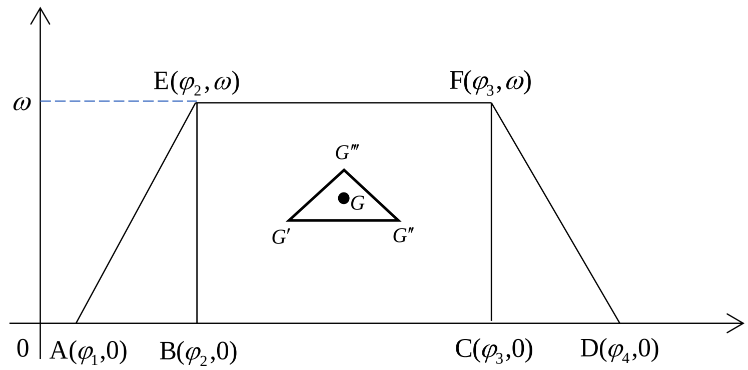


FIGURE 2. New Centroid of Generalized Trapezoidal Fuzzy Number

3.2. New Centroid of Generalized Pentagonal Fuzzy Number

First the pentagon of the generalized pentagonal fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5;\omega)$ is divided into two plane figures. The plane figures are triangle FGH and trapezoid AEGF. The centroids $G'=\left(\frac{\varphi_2+\varphi_3+\varphi_4}{3},\frac{2\omega}{3}\right)$ and $G''=\left(\frac{15\varphi_1+11\varphi_2+15\varphi_4+13\varphi_5}{54},\frac{13\omega}{54}\right)$ of these two plane figures are its balancing points (See Figure 3). Here G'' is the centroid of the triangle with the vertices $G_1'=\left(\frac{4\varphi_1+5\varphi_2+5\varphi_4+4\varphi_5}{18},\frac{5\omega}{18}\right)$, $G_2'=\left(\frac{5\varphi_1+4\varphi_2+4\varphi_4+5\varphi_5}{18},\frac{4\omega}{18}\right)$ and $G_3'=\left(\frac{6\varphi_1+2\varphi_2+6\varphi_4+4\varphi_5}{18},\frac{2\omega}{9}\right)$. The vertex points G_1',G_2',G_3' are made using the procedure introduced in the beginning of Section 3 and Section 3.1. The average point G of these two balancing points G' and G'' of the plane figures will be much more balancing point of the pentagon. So G can be considered as a new centroid of the generalized pentagonal fuzzy number $\tilde{A}=(\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5;\omega)$.

It is given as follows:

$$G=(x_0,y_0)=\left(\frac{15\varphi_1+29\varphi_2+18\varphi_3+33\varphi_4+13\varphi_5}{108},\frac{49\omega}{108}\right) \tag{5}$$

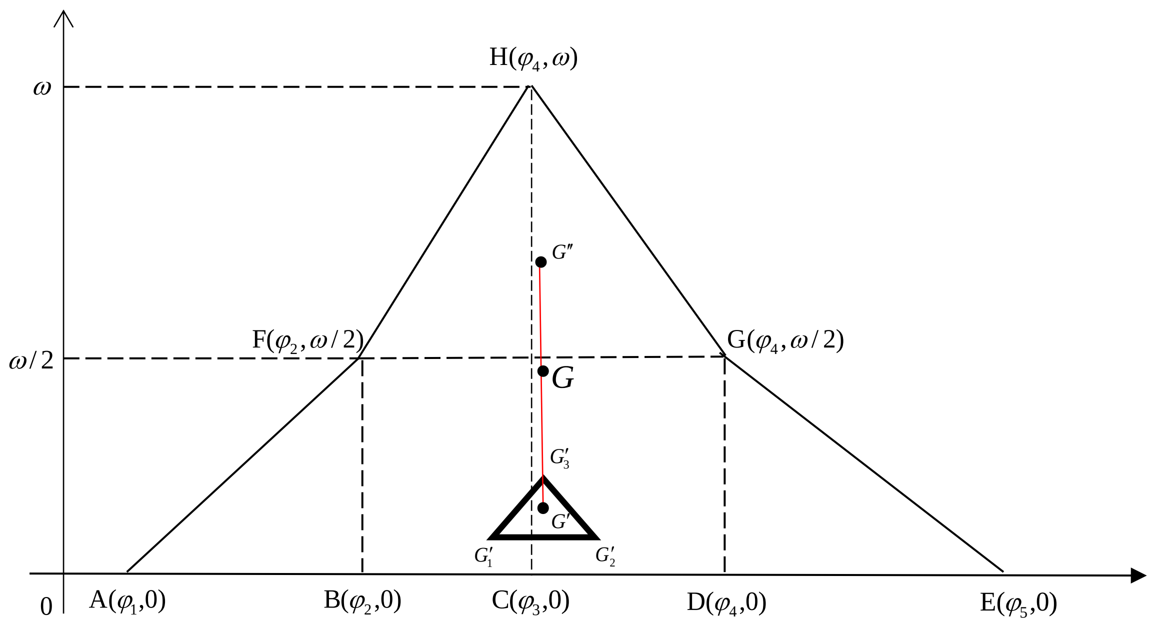


FIGURE 3. New Centroid of Generalized Pentagonal Fuzzy Number

3.3. New Centroid of Generalized Hexagonal Fuzzy Number

First the hexagon of the generalized hexagonal fuzzy number $\tilde{\mathrm{A}} = (\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5,\varphi_6;\omega)$ is divided into two trapezoids AFHG and GHJI. The centroids G' and G'' of these two trapezoids are its balancing point. Here $G' = \left(\frac{15\varphi_1+11\varphi_2+15\varphi_5+13\varphi_6}{54}, \frac{13\omega}{54}\right)$ is the centroid of the triangle with vertices $G_1' = \left(\frac{4\varphi_1+5\varphi_2+5\varphi_5+4\varphi_6}{18}, \frac{5\omega}{18}\right)$, $G_2' = \left(\frac{5\varphi_1+4\varphi_2+4\varphi_5+5\varphi_6}{18}, \frac{4\omega}{18}\right)$, $G_3' = \left(\frac{6\varphi_1+2\varphi_2+6\varphi_5+4\varphi_6}{18}, \frac{2\omega}{9}\right)$ and $G'' = \left(\frac{15\varphi_2+11\varphi_3+15\varphi_4+13\varphi_5}{54}, \frac{40\omega}{54}\right)$ is the centroid of the triangle with vertices $G_1'' = \left(\frac{4\varphi_2+5\varphi_3+5\varphi_4+4\varphi_5}{18}, \frac{7\omega}{9}\right)$, $G_2'' = \left(\frac{5\varphi_2+4\varphi_3+4\varphi_4+5\varphi_5}{18}, \frac{13\omega}{18}\right)$, $G_3'' = \left(\frac{6\varphi_2+2\varphi_3+6\varphi_4+4\varphi_5}{18}, \frac{26\omega}{36}\right)$. The two set of vertex points G_1', G_2', G_3' and G_1'', G_2'', G_3'' are made using the procedure introduced in the beginning of Section 3 and Section 3.1. Subsequently, the average G of these two balancing points G' and

G'' will be much more balancing point of the hexagon. So G can be considered as a new centroid of the generalized hexagonal fuzzy number $\tilde{A} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6; \omega)$. It is given as follows:

$$(x_0, y_0) = \left(\frac{15\varphi_1 + 26\varphi_2 + 11\varphi_3 + 15\varphi_4 + 28\varphi_5 + 13\varphi_6}{108}, \frac{53\omega}{108} \right) \tag{6}$$

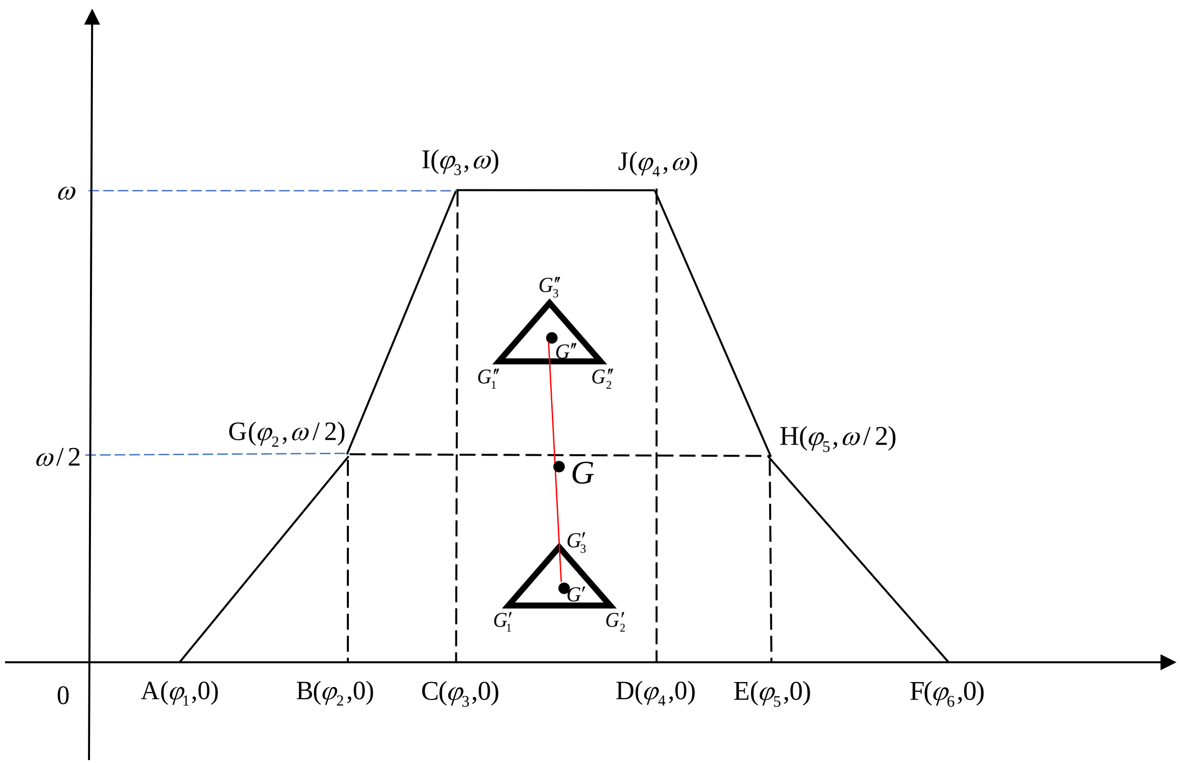


FIGURE 4. Centroid of Generalized Hexagonal Fuzzy Number

3.4. Ranking Function

Ranking function is a mapping which maps the set of all fuzzy numbers of various shapes into real numbers. Most probably, these types of ranking functions are based on centroids of fuzzy quantities. The set of real values arrived from Centroid based ranking function for various fuzzy quantities using Euclidean distance formula does not satisfy the total ordering. But in many fuzzy optimization problems, the fuzzy quantities have to be converted into real quantities and that should be satisfied by the total ordering properties. In order to overcome this problem we introduce the new centroid based ranking function using Euclidean distance formula.

$$R(\tilde{A}) = \begin{cases} S(\tilde{A}) & \text{if } x_c \text{ is positive} \\ -S(\tilde{A}) & \text{if } x_c \text{ is negative} \end{cases} \tag{7}$$

Where (x_c, y_c) is the centroid of fuzzy number $\tilde{A}$ and $S(\tilde{A}) = \sqrt{(x_c)^2 + (y_c)^2}$

4. Interval Data Based Game Theoretical Problem

4.1. Mathematical Formulation of Interval Data based Two-Person Zero-Sum Matrix Games

In game theoretical problems, the payoffs of matrix games are not observed in the form of crisp values in an uncertain context. But such values are mostly observed by the decision makers in the form of linguistic terms or interval values. The game theoretical problems with payoffs of interval values are called Interval Matrix Games or Interval data based Matrix Games. Here we introduce the mathematical form of interval data based two person zero sum matrix games in order to have an effective study under uncertainty.

Now we assume that $I_1, I_2 \ldots I_m$ are m strategies for Player I and $II_1, II_2 \ldots II_n$ are n strategies for Player II. In a Game Matrix with or without saddle point, that is, pure or mixed strategies, we assume that each player will have their own choices. Let $A = \left(\left[a_{ij}^L, a_{ij}^U\right]\right)_{m\times n}$ be an interval data based matrix game where $\left[a_{ij}^L, a_{ij}^U\right]$ is the payoff of interval number. Here player I gains from player II if strategy I_i and II_j are chosen by Players I and II respectively. The matrix with payoff of interval number is explicitly represented as follows:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} [a_{11}^l, a_{11}^u] & [a_{12}^l, a_{12}^u] & \cdots & \cdots & [a_{1n}^l, a_{1n}^u] \\ [a_{21}^l, a_{21}^u] & [a_{22}^l, a_{22}^u] & \cdots & \cdots & [a_{2n}^l, a_{2n}^u] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [a_{m1}^l, a_{m1}^u] & [a_{m2}^l, a_{m2}^u] & \cdots & \cdots & [a_{mn}^l, a_{mn}^u] \end{bmatrix} \end{array}$$

Remark

If the payoffs of the above matrix game are fuzzy numbers, then the game will become a fuzzy matrix game.

4.2. Proposed Method for solving Two-Person Zero-Sum Matrix Games

In this section, we propose to introduce an algorithm for solving interval data based two – person zero – sum matrix games under symmetric fuzzy environments.

Step 1: Given Interval data based two – person zero – sum matrix game is converted into fuzzy data based two – person zero – sum matrix games based on various symmetric forms of trapezoidal, pentagonal, hexagonal fuzzy numbers etc.

Step 2: The following fuzzy data based two – person zero – sum matrix games obtained from Step 1 are converted into crisp data based two – person zero – sum matrix games using ranking functions based on centroids of fuzzy numbers.

Case 1: **Fuzzy datum of trapezoidal shape**

Symmetric trapezoidal fuzzy data based two – person zero – sum matrix games are notated as $FGTP_{T1}$, $FGTP_{T2}$, $FGTP_{T3}$ …

Case 2: **Fuzzy datum of pentagonal shape**

Symmetric pentagonal fuzzy data based two – person zero – sum matrix games are notated as $FGTP_{P1}$, $FGTP_{P2}$, $FGTP_{P3}$ …

Case 3: **Fuzzy datum of hexagonal shape**

Symmetric trapezoidal fuzzy data based two – person zero – sum matrix games are notated as $FGTP_{H1}$, $FGTP_{H2}$, $FGTP_{H3}$ …

etc.

Step 3: The crisp data based two – person zero – sum matrix games obtained from Step 2 are notated as follows.

Case 1:

Crisp data based two – person zero – sum matrix games obtained from $FGTP_{T1}$, $FGTP_{T2}$, $FGTP_{T3}$,… using centroid based ranking function, are notated as $CGTP_{T1}$, $CGTP_{T2}$, $CGTP_{T3}$,…

Case 2:

Crisp data based two – person zero – sum matrix games obtained from $FGTP_{P1}$, $FGTP_{P2}$, $FGTP_{P3}$,… using centroid based ranking function, are notated as $CGTP_{P1}$, $CGTP_{P2}$, $CGTP_{P3}$,…

Case 3:

Crisp data based two – person zero – sum matrix games obtained from $FGTP_{H1}$, $FGTP_{H2}$, $FGTP_{H3}$,… using centroid based ranking function, are notated as $CGTP_{H1}$, $CGTP_{H2}$, $CGTP_{H3}$,…

etc.

Step 4: The crisp data based two – person zero – sum matrix games are solved based on existence and non-existence of saddle point in traditional ways as follows:

Game with Saddle Points – Pure Strategies

Here, the process for tracing the saddle point of a crisp data based payoff matrix is summarized as follows:

Step 1: The element of minimum value is selected in each row of the crisp payoff matrix and it is marked as (*)

Step 2: The element of maximum value is selected in each column of the crisp payoff matrix and it is marked as (+)

Step 3: The saddle point is an element marked both (*) and (+) in the crisp payoff matrix.

The above process is called *Maximin – Minimax Principle*. The strategies corresponding to the saddle point are the optimal strategies selected by the two players.

Remark

If the saddle point is zero, then the game is said to be a fair game. Otherwise it is said to be strictly determinable.

Game without Saddle Points – Mixed Strategies

If the saddle point does not exist, the size of the given crisp data based payoff matrix ($m \times n$) should be reduced and solved in the following way.

1. If it (payoff matrix of order $m \times n$) is reduced to 2×2 ordered matrix, then it has the following formulae for finding optimal strategies and the value of the game.

Any 2×2 two-person zero sum game without saddle point having the payoff matrix for player P

$$\begin{array}{cc} & \text{Player } Q \\ & \begin{array}{cc} Q_1 & Q_2 \end{array} \\ \text{Player } P \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \end{array}$$

the optimum mixed strategies

$$S_P = \begin{bmatrix} P_1 & P_2 \\ a_1 & a_2 \end{bmatrix} \quad \text{and} \quad S_Q = \begin{bmatrix} Q_1 & Q_2 \\ b_1 & b_2 \end{bmatrix}$$

are determined by

$$\frac{a_1}{a_2} = \frac{\alpha_{22} - \alpha_{21}}{\alpha_{11} - \alpha_{12}}, \quad \frac{b_1}{b_2} = \frac{\alpha_{22} - \alpha_{12}}{\alpha_{11} - \alpha_{21}}$$

(8) $$a_1 = \frac{\alpha_{22} - \alpha_{21}}{\alpha_{11} + \alpha_{22} - (\alpha_{12} + \alpha_{21})}, \quad b_1 = \frac{\alpha_{22} - \alpha_{12}}{\alpha_{11} + \alpha_{22} - (\alpha_{12} + \alpha_{21})}$$

Where $a_1 + a_2 = 1$ and $b_1 + b_2 = 1$. The value v of the game to Player P is given by

(9) $$v = \frac{\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}}{\alpha_{11} + \alpha_{22} - (\alpha_{12} + \alpha_{21})}$$

2. If the crisp data based payoff matrix $(m\times n)$ is reduced $2\times n$ or $m\times 2$ ordered matrix, then the graphical method shall be used for its further reduction to 2×2 ordered matrix. Finally 2×2 ordered matrix will be solved using the above formulae. The graphical method for solving $2\times n$ or $m\times 2$ crisp payoff matrix is given as follows:

In this graphical method, it is possible that $2\times n$ or $m\times 2$ ordered rectangular matrix may be reduced to 2×2 ordered matrix. The process for this reduction is given for $2\times n$ ordered matrix to the clarity of this method as follows:

$$\begin{array}{cc} & \text{Player } Q \\ & \begin{array}{ccccc} Q_1 & Q_2 & \cdots & \cdots & Q_n \end{array} \\ \text{Player } P \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \cdots & \alpha_{2n} \end{bmatrix} \end{array}$$

It is our assumption that the game will not have a saddle point. Let the optimum mixed strategy for the Player P be given by

$$S_P = \begin{bmatrix} P_1 & P_2 \\ a_1 & a_2 \end{bmatrix} \quad \text{where} \quad a_1 + a_2 = 1.$$

The expected payoff for P when he plays S_P against Q's pure moves Q_1, Q_2, …Q_n is given by

Q's pure move	P's expected payoff $E(a)$
Q_1	$E_1(a_1) = \alpha_{11}a_1 + \alpha_{12}(1 - a_1)$
Q_2	$E_2(a_1) = \alpha_{21}a_1 + \alpha_{22}(1 - a_1)$
….	…….
Q_n	$E_n(a_1) = \alpha_{1n}a_1 + \alpha_{2n}(1 - a_1)$

As per the maximin criterion, the values of a_1 and a_2 should be selected by player P to maximize his minimum expected payoffs. This is done by plotting the expected payoff lines:

$$E_j(b_1) = (\alpha_{1j} - \alpha_{2j})a_1 + \alpha_{2j}, \quad j = 1,2,....., n$$

The maximum of the minimum expected payoffs to player P will be given by the highest point on the lower envelope as also the maximum value of a_i.

The two lines passing through the maximin point identify the two critical moves of Q which combined the two of P, yield 2×2 matrix that can be used to determine the optimum strategies of the two players for the original game using the formulae (8) and (9).

Note

The $m \times 2$ game are also treated in the same way where the upper envelope of the straight lines corresponding to B's expected payoffs will give the maximum expected payoff to player Q and the lowest point on this then gives the maximum expected payoff and the optimum value of b_j.

Remark

If there is no saddle point to the payoff matrix of order m × n, then we may reduce it to m × 2 or 2 × n or 2 × 2 ordered payoff matrix with the help of dominance rule for finding its solution.

Step 5: Compare the optimum solutions obtained from various crisp assignment problems to find the minimum optimum assignment cost or time and its schedule for the given interval data based assignment problem.

$$\begin{aligned}&\{\text{Value of the Game for IBGTP}\}\\ &= \mathrm{Max}\{\mathrm{Max}\{VG_{CGTP_{T1}}, VG_{CGTP_{T2}}, \ldots\}, \mathrm{Max}\{VG_{CGTP_{P1}}, VG_{CGTP_{P2}}, \ldots\},\\ &\quad \mathrm{Max}\{VG_{CGTP_{H1}}, VG_{CGTP_{H2}}, \ldots\}, \ldots\ldots\}\end{aligned}$$

where VG denotes the value of the game.

5. Numerical Illustrations

Illustration 1

Solve the game with payoff matrix is given by

$$\begin{array}{cc} & \text{Player Q} \\ & \begin{array}{ccc} Q_1 & Q_2 & Q_3 \end{array} \\ \text{Player P} \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} & \begin{bmatrix} [0,2] & [2,4] & [0.5,1.5] \\ [-2,2] & [-6,-2] & [-4,-2] \\ [-2,4] & [1,9] & [-3,1] \end{bmatrix} \end{array}$$

Solution

As per the proposed algorithm, we wish to solve the given game with interval data based payoff matrix in the three different symmetric fuzzy environments with the help of their proposed fuzzification formulae.

First let us solve the game in symmetric trapezoidal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (1).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (0,0.67,1.33,2) & (2,2.67,3.33,4) & (0.5,0.83,1.17,1.5) \\ (-2,-0.67,0.67,2) & (-6,-4.67,-3.33,-2) & (-4,-3.33,-2.67,-2) \\ (-2,0,2,4) & (1,3.67,6.33,9) & (-3,-1.67,-0.33,1) \end{bmatrix}$$

$$\begin{bmatrix} (0,0.5,1.5,2) & (2,2.5,3.5,4) & (0.5,0.75,1.25,1.5) \\ (-2,-1,1,2) & (-6,-5,-3,-2) & (-4,-3.5,-2.5,-2) \\ (-2,-0.5,2.5,4) & (1,3,7,9) & (-3,-2,0,1) \end{bmatrix}$$

$$\begin{bmatrix} (0,0.8,1.2,2) & (2,2.8,3.2,4) & (0.5,0.9,1.1,1.5) \\ (-2,-0.4,0.4,2) & (-6,-4.4,-3.6,-2) & (-4,-3.2,-2.8,-2) \\ (-2,0.4,1.6,4) & (1,4.2,5.8,9) & (-3,-1.4,-0.6,1) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (4)

$$\begin{bmatrix} 1.0985 & 3.0260 & 1.0879 \\ -0.4821 & -4.0531 & -3.0508 \\ 1.0766 & 4.9737 & -1.1320 \end{bmatrix} \quad \begin{bmatrix} 1.1099 & 3.0384 & 1.1099 \\ -0.4815 & -4.0289 & -3.0384 \\ 1.1099 & 5.0231 & -1.1099 \end{bmatrix}$$

$$\begin{bmatrix} 1.0899 & 3.0165 & 1.0700 \\ -0.4835 & -4.0730 & -3.0603 \\ 1.0502 & 4.9347 & -1.1501 \end{bmatrix}$$

The maximin value and minimax values are coincided for the above three problems. So the problems have the saddle point.

The maximin and minimax values are marked by (*) and (+) respectively as shown below:

$$\begin{bmatrix} 1.0985^{*+} & 3.0260 & 1.1043^{+} \\ -0.4821 & -4.0531^{*} & -3.0508 \\ 1.0766 & 4.9737^{+} & -1.1320^{*} \end{bmatrix} \quad \begin{bmatrix} 1.1099^{*+} & 3.0384 & 1.1099^{*+} \\ -0.4815 & -4.0289^{*} & -3.0384 \\ 1.1099^{+} & 5.0231^{+} & -1.1099^{*} \end{bmatrix}$$

$$\begin{bmatrix} 1.0899^{*+} & 3.0165 & 1.0999^{+} \\ -0.4835 & -4.0730^{*} & -3.0603 \\ 1.0502 & 4.9347^{+} & -1.1501^{*} \end{bmatrix}$$

We observed the following things from the above three problems.

In problem I, that there exist one saddle point at the position (1, 3). In problem II, there exist two saddle points at positions (1, 1) and (1, 3). Finally in problem III, that there exist one saddle point at the position (1, 1).

Thus the solutions to the crisp data based above problems are given by

Problem solved under Trapezoidal Fuzzy Environment				
Problems	Optimum Strategy		Value of the Game	
	Player P	Player Q	Player P	Player Q
I	P_1	Q_1	1.0985	-1.0985
II	P_1	Q_1 & Q_3	1.1099	-1.1099
III	P_1	Q_1	1.0899	-1.0899

Now the given game is solved in symmetric pentagonal fuzzy environment.

Similarly, the payoff matrix for the given game is converted in to three fuzzy payoff matrices using eq. (2)

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (0,0.5,1,1.5,2) & (2,2.5,3,3.5,4) & (0.5,0.75,1,1.25,1.5) \\ (-2,-1,0,1,2) & (-6,-5,-4,-3,-2) & (-4,-3.5,-3,-2.5,-2) \\ (-2,-0.5,1,2.5,4) & (1,3,5,7,9) & (-3,-2,-1,0,1) \end{bmatrix}$$

$$\begin{bmatrix} (0,0.33,1,1.67,2) & (2,2.33,3,3.67,4) & (0.5,0.67,1,1.33,1.5) \\ (-2,-1.33,0,1.33,2) & (-6,-5.33,-4,-2.67,-2) & (-4,-3.67,-3,-2.33,-2) \\ (-2,-1,1,3,4) & (1,2.33,5,7.67,9) & (-3,-2.33,-1,0.33,1) \end{bmatrix}$$

$$\begin{bmatrix} (0,0.67,1,1.33,2) & (2,2.67,3,3.33,4) & (0.5,0.83,1,1.17,1.5) \\ (-2,-0.67,0,0.67,2) & (-6,-4.67,-4,-3.33,-2) & (-4,-3.33,-3,-2.67,-2) \\ (-2,0,1,2,4) & (1,3.67,5,6.33,9) & (-3,-1.67,-1,-0.33,1) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (5)

$$\begin{bmatrix} 1.0981 & 3.0341 & 1.0981 \\ -0.4537 & -4.0256 & -3.0341 \\ 1.0981 & 5.0205 & -1.0981 \end{bmatrix} \quad \begin{bmatrix} 1.1037 & 3.0402 & 1.1009 \\ -0.4539 & -4.0134 & -3.0280 \\ 1.1150 & 5.0451 & -1.0869 \end{bmatrix}$$

$$\begin{bmatrix} 1.0925 & 3.0280 & 1.0953 \\ -0.4539 & -4.0379 & -3.0402 \\ 1.0813 & 4.9960 & -1.1094 \end{bmatrix}$$

The maximin value and minimax values are coincided for the above three problems. So the problems have the saddle point.

The maximin and minimax values are marked by (*) and (+) respectively as shown below:

$$\begin{bmatrix} 1.0981^{*+} & 3.0341 & 1.0981^{*+} \\ -0.4537 & -4.0256^{*} & -3.0341 \\ 1.0981^{+} & 5.0205^{+} & -1.0981^{*} \end{bmatrix} \begin{bmatrix} 1.1037 & 3.0402 & 1.1009^{*+} \\ -0.4539 & -4.0134^{*} & -3.0280 \\ 1.1150^{+} & 5.0451^{+} & -1.0869^{*} \end{bmatrix}$$

$$\begin{bmatrix} 1.0925^{*+} & 3.0280 & 1.0953^{+} \\ -0.4539 & -4.0379^{*} & -3.0402 \\ 1.0813 & 4.9960^{+} & -1.1094^{*} \end{bmatrix}$$

We observed the following things from the above three problems.

In problem I, that there exist two saddle points at the positions (1, 1) and (1, 3). In problem II, there exists one saddle point at the position (1, 3). Finally in problem III, that there exist one saddle point at the position (1, 1).

Thus the solutions to the crisp data based above problems are given by

Problem solved under Pentagonal Fuzzy Environment				
Problems	Optimum Strategy		Value of the Game	
	Player P	Player Q	Player P	Player Q
I	P_1	Q_1 & Q_3	1.0981	-1.0981
II	P_1	Q_3	1.1009	-1.1009
III	P_1	Q_1	1.0925	-1.0925

Now the given game is solved in symmetric hexagonal fuzzy environment.

Similarly, the payoff matrix for the give game is converted in to three fuzzy payoff matrices using eq. (3).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (0,0.4,0.8,1.2,1.6,2) & (2,2.4,2.8,3.2,3.6,4) & (0.5,0.7,0.9,1.1,1.3,1.5) \\ (-2,-1.2,-0.4,0.4,1.2,2) & (-6,-5.2,-4.4,-3.6,-2.8,-2) & (-4,-3.6,-3.2,-2.8,-2.4,-2) \\ (-2,-0.8,0.4,1.6,2.8,4) & (1,2.6,4.2,5.8,7.4,9) & (-3,-2.2,-1.4,-0.6,0.2,1) \end{bmatrix}$$

$$\begin{bmatrix} (0,0.25,0.75,1.25,1.75,2) & (2,2.25,2.75,3.25,,3.75,4) & (0.5,0.625,0.875,1.125,1.375,1.5) \\ (-2,-1.5,-0.5,0.5,1.5,2) & (-6,-5.5,-4.5,-3.5,-2.5,-2) & (-4,-3.75,-3.25,-2.75.-2.25,-2) \\ (-2,-1.25,0.25,1.75,3.25,4) & (1,2,4,6,8,9) & (-3,-2.5,-1.5,-0.5,0.5,1) \end{bmatrix}$$

$$\begin{bmatrix} (0,0.57,0.86,1.14,1.43,2) & (2,2.57,2.86,3.14,3.43,4) & (0.5,0.79,0.93,1.07,1.21,1.5) \\ (-2,-0.86,-0.29,0.29,0.86,2) & (-6,-4.86,-4.29,-3.71,-3.14,-2) & (-4,-3.43,-3.14,-2.86,-2.57,-2) \\ (-2,-0.29,0.57,1.43,2.29,4) & (1,3.29,4.43,5.57,6.71,9) & (-3,-1.86,-1.29,-0.71,-0.14,1) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (6)

$$\begin{bmatrix} 1.1139 & 3.0399 & 1.1139 \\ -0.4907 & -4.0300 & -3.0399 \\ 1.1139 & 5.0240 & -1.1139 \end{bmatrix} \quad \begin{bmatrix} 1.1181 & 3.0444 & 1.1160 \\ -0.4908 & -4.0208 & -3.0353 \\ 1.1264 & 5.0425 & -1.1056 \end{bmatrix}$$

$$\begin{bmatrix} 1.1092 & 3.0347 & 1.1115 \\ -0.4909 & -4.0405 & -3.0451 \\ 1.0997 & 5.0030 & -1.1234 \end{bmatrix}$$

The maximin value and minimax values are coincided for the above three problems. So the problems have the saddle point.

The maximin and minimax values are marked by (*) and (+) respectively as shown below:

$$\begin{bmatrix} 1.1139^{*+} & 3.0399 & 1.1139^{*+} \\ -0.4907 & -4.0300^{*} & -3.0399 \\ 1.1139^{+} & 5.0240^{+} & -1.1139^{*} \end{bmatrix} \quad \begin{bmatrix} 1.1181 & 3.0444 & 1.1160^{*+} \\ -0.4908 & -4.0208^{*} & -3.0353 \\ 1.1264^{+} & 5.0425^{+} & -1.1056^{*} \end{bmatrix}$$

$$\begin{bmatrix} 1.1092^{*+} & 3.0347 & 1.1115^{+} \\ -0.4909 & -4.0405^{*} & -3.0451 \\ 1.0997 & 5.0030^{+} & -1.1234^{*} \end{bmatrix}$$

We observed the following things from the above three problems.

In problem I, that there exist one saddle point at the position (1, 3). In problem II, there exist two saddle points at positions (1, 1) and (1, 3). Finally in problem III, that there exist one saddle point at the position (1, 1).

Thus the solutions to the crisp data based above problems are given by

Problem solved under Hexagonal Fuzzy Environment				
Problems	Optimum Strategy		Value of the Game	
	Player P	Player Q	Player P	Player Q
I	P_1	Q_1 & Q_3	1.1139	-1.1139
II	P_1	Q_3	1.1160	-1.1160
III	P_1	Q_1	1.1092	-1.1092

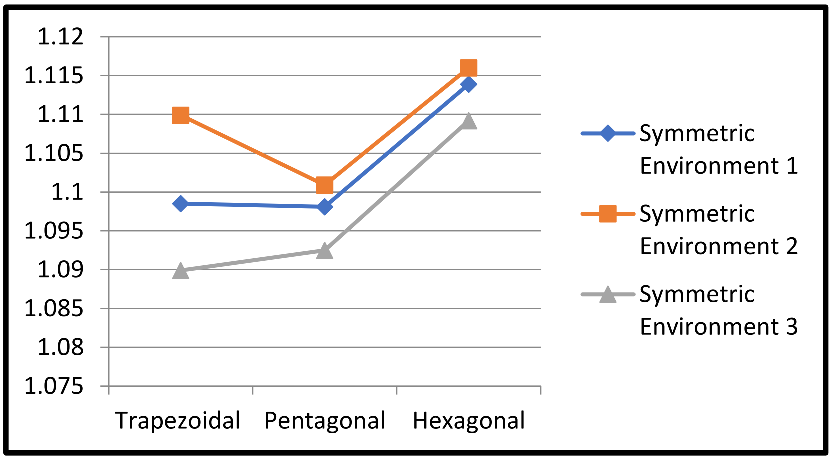


FIGURE 5. Values of the Game for IDBGTP

Therefore, the Value of the Game for given IDGTP = Max { Max {1.0985, 1.1099, 1.0899}, Max {1.0981, 1.1009, 1.0925}, Max {1.1139, 1.1160, 1.1092} } = Max {1.1099, 1.1009, 1.1160} = 1.1160.

Illustration 2

There is a competition in business between two firms F_1 and F_2 such that the gain of one firm is loss in the other. The interval data based payoff matrix for firm F1 is given as follows:

$$
\begin{array}{lll}
 & & \text{Firm } F_2 \\
 & & \begin{array}{ccc} \text{No Advt} & \text{Medium Advt} & \text{High Advt} \end{array} \\
 & \begin{array}{r} \text{No advertising} \\ \text{Medium advertising} \\ \text{High advertising} \end{array} & \begin{bmatrix} [9,11] & [3,7] & [-4,0] \\ [10,16] & [11,13] & [13,17] \\ [14,18] & [11,17] & [8,12] \end{bmatrix} \\
\text{Firm } F_1 & &
\end{array}
$$

Find the optimum strategies for F_1 and F_2 and also calculate the net outcome.

Solution

As per the proposed algorithm, we wish to solve the given game with interval data based payoff matrix in the three different symmetric fuzzy environments with the help of their proposed fuzzification formulae.

First let us solve the game in symmetric trapezoidal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (1).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (9,9.67,10.33,11) & (3,4.33,5.67,7) & (-4,-2.67,-1.33,0) \\ (10,12,14,16) & (11,11.67,12.33,13) & (13,14.33,15.67,17) \\ (14,15.33,16.67,18) & (11,13,15,17) & (8,9.33,10.67,12) \end{bmatrix}$$

$$\begin{bmatrix} (9,9.5,10.5,11) & (3,4,6,7) & (-4,-3,-1,0) \\ (10,11.5,14.5,16) & (11,11.5,12.5,13) & (13,14,16,17) \\ (14,15,17,18) & (11,12.5,15.5,17) & (8,9,11,12) \end{bmatrix}$$

$$\begin{bmatrix} (9,9.8,10.2,11) & (3,4.6,5.4,7) & (-4,-2.4,-1.6,0) \\ (10,12.4,13.6,16) & (11,11.8,12.2,13) & (13,14.6,15.4,17) \\ (14,15.6,16.4,18) & (11,13.4,14.6,17) & (8,9.6,10.4,12) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (4)

$$\begin{bmatrix} 9.9993 & 4.9986 & -2.0812 \\ 12.9719 & 11.9973 & 14.9830 \\ 15.9826 & 13.9713 & 9.9869 \end{bmatrix} \quad \begin{bmatrix} 10.0116 & 5.0231 & -2.0571 \\ 13.0089 & 12.0097 & 15.0077 \\ 16.0072 & 14.0083 & 10.0116 \end{bmatrix}$$

$$\begin{bmatrix} 9.9894 & 4.9789 & -2.1004 \\ 12.9423 & 11.9875 & 14.9633 \\ 15.9628 & 13.9416 & 9.9672 \end{bmatrix}$$

In the above three problems, the second column dominates the first column. Thus the first column is eliminated as per the dominance property.

$$\begin{bmatrix} 4.9986 & -2.0812 \\ 11.9973 & 14.9830 \\ 13.9713 & 9.9869 \end{bmatrix} \quad \begin{bmatrix} 5.0231 & -2.0571 \\ 12.0097 & 15.0077 \\ 14.0083 & 10.0116 \end{bmatrix} \quad \begin{bmatrix} 4.9789 & -2.1004 \\ 11.9875 & 14.9633 \\ 13.9416 & 9.9672 \end{bmatrix}$$

The second and third rows dominate the first row again. So the first row can be eliminated as per the dominance property.

$$\begin{bmatrix} 11.9973 & 14.9830 \\ 13.9713 & 9.9869 \end{bmatrix} \quad \begin{bmatrix} 12.0097 & 15.0077 \\ 14.0083 & 10.0116 \end{bmatrix} \quad \begin{bmatrix} 11.9875 & 14.9633 \\ 13.9416 & 9.9672 \end{bmatrix}$$

The above payoff matrices do not have any saddle point. So the strategies are mixed strategies. Using the traditional method for payoff matrices of order 2 × 2, the solutions of the above three problems are obtained as follows:

Mixed Strategies		**Value of the Game**
Strategy for Firm F1	**Strategy for Firm F2**	
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5716 & 0.4284 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7168 & 0.2832 \end{bmatrix}$	$v = 12.8429$
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5714 & 0.4286 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7143 & 0.2857 \end{bmatrix}$	$v = 12.8663$
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5718 & 0.4282 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7188 & 0.2812 \end{bmatrix}$	$v = 12.8242$

Now let us solve the game in symmetric pentagonal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (2).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (9,9.5,10,10.5,11) & (3,4,5,6,7) & (-4,-3,-2,-1,0) \\ (10,11.5,13,14.5,16) & (11,11.5,12,12.5,13) & (13,14,15,16,17) \\ (14,15,16,17,18) & (11,12.5,14,15.5,17) & (8,9,10,11,12) \end{bmatrix}$$

$$\begin{bmatrix} (9,9.33,10,10.67,11) & (3,3.67,5,6.33,7) & (-4,-3.33,-2,-0.67,0) \\ (10,11,13,15,16) & (11,11.33,12,12.67,13) & (13,13.67,15,16.33,17) \\ (14,14.67,16,17.33,18) & (11,12,14,16,17) & (8,8.67,10,11.33,12) \end{bmatrix}$$

$$\begin{bmatrix} (9,9.67,10,10.33,11) & (3,4.33,5,5.67,7) & (-4,-2.67,-2,-1.33,0) \\ (10,12,13,14,16) & (11,11.67,12,12.33,13) & (13,14.33,15,15.67,17) \\ (14,15.33,16,16.67,18) & (11,13,14,15,17) & (8,9.33,10,10.67,12) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (5)

$$\begin{bmatrix} 10.0103 & 5.0205 & -2.0508 \\ 13.0079 & 12.0086 & 15.0069 \\ 16.0064 & 14.0073 & 10.0103 \end{bmatrix} \begin{bmatrix} 10.0165 & 5.0328 & -2.0388 \\ 13.0264 & 12.0147 & 15.0192 \\ 16.0188 & 14.0259 & 10.0226 \end{bmatrix}$$

$$\begin{bmatrix} 10.0041 & 5.0082 & -2.0629 \\ 12.9894 & 12.0024 & 14.9945 \\ 15.9941 & 13.9888 & 9.9980 \end{bmatrix}$$

Similarly, the above matrices are reduced to the payoff matrices of order 2 × 2 using dominance property.

$$\begin{bmatrix} 12.0086 & 15.0069 \\ 14.0073 & 10.0103 \end{bmatrix} \quad \begin{bmatrix} 12.0147 & 15.0192 \\ 14.0259 & 10.0226 \end{bmatrix} \quad \begin{bmatrix} 12.0024 & 14.9945 \\ 13.9888 & 9.9980 \end{bmatrix}$$

The above payoff matrices do not have any saddle point. So the strategies are mixed strategies. Using the traditional method for payoff matrices of order 2 × 2, the solutions of the above three problems are obtained as follows:

Mixed Strategies		**Value of the Game**
Strategy for Firm F1	**Strategy for Firm F2**	
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5714 & 0.4286 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7143 & 0.2857 \end{bmatrix}$	$v = 12.8653$
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5713 & 0.4287 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7130 & 0.2870 \end{bmatrix}$	$v = 12.8770$
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5715 & 0.4285 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7155 & 0.2845 \end{bmatrix}$	$v = 12.8536$

Now let us solve the game in symmetric hexagonal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (3).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (9,9.4,9.8,10.2,10.6,11) & (3,3.8,4.6,5.4,6.2,7) & (-4,-3.2,-2.4,-1.6,-0.8,0) \\ (10,11.2,12.4,13.6,14.8,16) & (11,11.4,11.8,12.2,12.6,13) & (13,13.8,14.6,15.4,16.2,17) \\ (14,14.8,15.6,16.4,17.2,18) & (11,12.2,13.4,14.6,15.8,17) & (8,8.8,9.6,10.4,11.2,12) \end{bmatrix}$$

$$\begin{bmatrix} (9,9.25,9.75,10.25,10.75,11) & (3,3.5,4.5,5.5,6.5,7) & (-4,-3.5,-2.5,-1.5,-0.5,0) \\ (10,10.75,12.25,13.75,15.25,16) & (11,11.25,11.75,12.25,12.75,13) & (13,13.5,14.5,15.5,16.5,17) \\ (14,14.5,15.5,16.5,17.5,18) & (11,11.75,13.25,14.75,16.25,17) & (8,8.5,9.5,10.5,11.5,12) \end{bmatrix}$$

$$\begin{bmatrix} (9,9.57,9.86,10.14,11) & (3,4.14,4.71,5.29,5.86,7) & (-4,-2.86,-2.29,-1.71,-1.14,0) \\ (10,11.71,12.57,13.43,14.29,16) & (11,11.57,11.86,12.14,12.43,13) & (13,14.14,14.71,15.29,15.86,17) \\ (14,15.14,15.71,16.29,16.86,18) & (11,12.71,13.57,14.43,15.29,17) & (8,9.14,9.71,10.29,10.86,12) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (6)

$$\begin{bmatrix} 10.0120 & 5.0240 & -2.0593 \\ 13.0093 & 12.0100 & 15.0080 \\ 16.0075 & 14.0086 & 10.0120 \end{bmatrix} \begin{bmatrix} 10.0167 & 5.0332 & -2.0503 \\ 13.0231 & 12.0147 & 15.0173 \\ 16.0168 & 14.0225 & 10.0213 \end{bmatrix}$$

$$\begin{bmatrix} 10.0067 & 5.0135 & -2.0696 \\ 12.9934 & 12.0047 & 14.9974 \\ 15.9969 & 13.9927 & 10.0015 \end{bmatrix}$$

Similarly, the above matrices are reduced to the payoff matrices of order 2 * 2 using dominance property.

$$\begin{bmatrix} 12.0100 & 15.0080 \\ 14.0086 & 10.0120 \end{bmatrix} \quad \begin{bmatrix} 12.0147 & 15.0173 \\ 14.0225 & 10.0213 \end{bmatrix} \quad \begin{bmatrix} 12.0047 & 14.9974 \\ 13.9927 & 10.0015 \end{bmatrix}$$

The above payoff matrices do not have any saddle point. So the strategies are mixed strategies. Using the traditional method for payoff matrices of order 2×2, the solutions of the above three problems are obtained as follows:

Mixed Strategies		**Value of the Game**
Strategy for Firm F_1	**Strategy for Firm F_2**	
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5714 & 0.4286 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7143 & 0.2857 \end{bmatrix}$	$v = 12.8666$
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5713 & 0.4287 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7133 & 0.2867 \end{bmatrix}$	$v = 12.8755$
$S_{F_1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ 0 & 0.5715 & 0.4285 \end{bmatrix}$	$S_{F_2} = \begin{bmatrix} F_{21} & F_{22} & F_{23} \\ 0 & 0.7153 & 0.2847 \end{bmatrix}$	$v = 12.8566$

Therefore, the Value of the Game for given IDGTP = Max { Max {12.8429, 12.8663, 12.8242}, Max {12.8653, 12.8770, 12.8536}, Max {12.8666, 12.8755, 12.8566} } = Max {12.8663, 12.8770, 12.8755} = 12.8770.

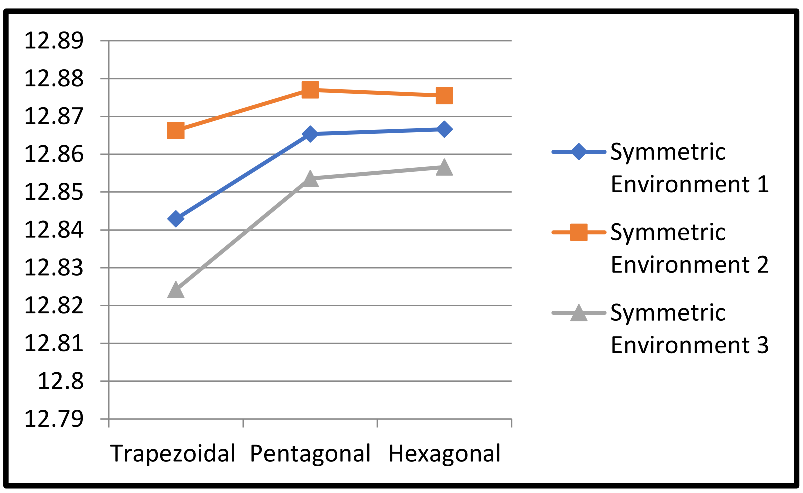


FIGURE 6. Values of the Game for IDBGTP

Illustration 3

Use graphical method to solve the following game:

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} [1,3] & [0,2] & [-2,2] & [-3,-1] \\ [0,2] & [-1,1] & [2,4] & [0,4] \end{bmatrix} \end{array}$$

Solution

As per the proposed algorithm, we wish to solve the given game with interval data based payoff matrix in the three different symmetric fuzzy environments with the help of their proposed fuzzification formulae.

First let us solve the game in symmetric trapezoidal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (1).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (1,1.67,2.33,3) & (0,0.67,1.33,2) & (-2,-0.67,0.67,2) & (-3,-2.33,-1.67,-1) \\ (0,0.67,1.33,2) & (-1,-0.33,0.33,1) & (2,2.67,3.33,4) & (0,1.33,2.67,4) \end{bmatrix}$$

$$\begin{bmatrix} (1,1.5,2.5,3) & (0,0.5,1.5,2) & (-2,-1,1,2) & (-3,-2.5,-1.5,-1) \\ (0,0.5,1.5,2) & (-1,-0.5,0.5,2) & (2,2.5,3.5,4) & (0,1,3,4) \end{bmatrix}$$

$$\begin{bmatrix} (1,1.8,2.2,3) & (0,0.8,1.2,2) & (-2,-0.4,0.4,2) & (-3,-2.2,-1.8,-1) \\ (0,0.8,1.2,2) & (-1,-0.2,0.2,1) & (2,2.8,3.2,4) & (0,1.6,2.4,4) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (4)

$$\begin{bmatrix} 2.0451 & 1.0988 & -0.4821 & -2.0691 \\ 1.0988 & -0.4816 & 3.0262 & 2.0331 \end{bmatrix}$$

$$\begin{bmatrix} 2.0571 & 1.1099 & 0.4815 & -2.0571 \\ 1.1099 & 0.4815 & 3.0384 & 2.0571 \end{bmatrix}$$

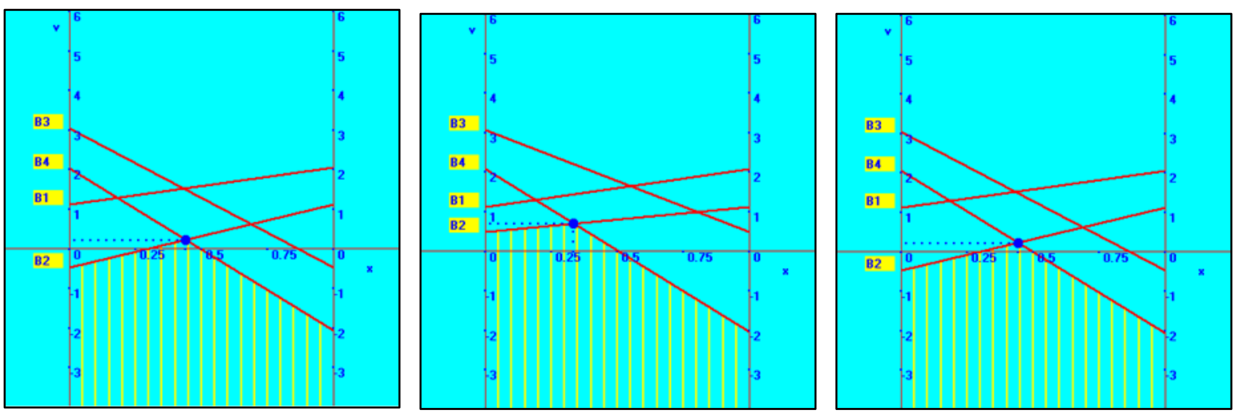
$$\begin{bmatrix} 2.0355 & 1.0899 & -0.4835 & -2.0788 \\ 1.0899 & -0.4820 & 3.0165 & 2.0140 \end{bmatrix}$$

Clearly, the above crisp data based problems does not possess a saddle point.

Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ a_1 & a_2 \end{bmatrix}$ and $a_1 + a_2 = 1.$ against B. Then A's expected payoffs against B's pure moves are given by

The expected payoff for A when he plays S_A against B's pure moves B_1, B_2, … B_n is given by

B's pure move	A's expected payoff E(a)		
	Problem 1	Problem 2	Problem 3
B1	$E_1(a_1) = 0.9463a_1 + 1.0988$	$E_1(a_1) = 0.9472a_1 + 1.1099$	$E_1(a_1) = 0.9456a_1 + 1.0899$
B2	$E_2(a_1) = 1.5804a_1 - 0.4816$	$E_2(a_1) = 0.6284a_1 + 0.4815$	$E_2(a_1) = 1.5719a_1 - 0.4820$
B3	$E_3(a_1) = -3.5083a_1 + 3.0262$	$E_3(a_1) = -2.5569a_1 + 3.0384$	$E_3(a_1) = -3.5a_1 + 3.0165$
B4	$E_4(a_1) = -4.1022a_1 + 2.0331$	$E_4(a_1) = -4.1142a_1 + 2.0571$	$E_4(a_1) = -4.0928a_1 + 2.0140$



FIGURES 7a, 7b, 7c. The Maximin Values

In the above three games with crisp payoff matrices, the player A wishes to maximize his minimum expected payoff. As per the graphical method, the highest point of intersection H on the lower envelop of the A's expected payoff equations is considered. At this point the lines B_2 and B_4 are intersected. The above three 2×4 games therefore reduce to the games with 2×2 following payoff matrices:

$$\begin{bmatrix} 1.0988 & -2.0691 \\ -0.4816 & 2.0331 \end{bmatrix} \qquad \begin{bmatrix} 1.1099 & -2.0571 \\ 0.4815 & 2.0571 \end{bmatrix} \qquad \begin{bmatrix} 1.0899 & -2.0788 \\ -0.4820 & 2.0140 \end{bmatrix}$$

Using eq. (8) and (9), the solutions of the above three problems are obtained as follows:

Mixed Strategies		**Value of the Game**
Strategy for Firm A	**Strategy for Firm B**	
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.4425 & 0.5575 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.7219 & 0 & 0.2781 \end{bmatrix}$	$v = 0.2178$
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.3322 & 0.6678 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.8675 & 0 & 0.1325 \end{bmatrix}$	$v = 0.6903$
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.4406 & 0.5594 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.7225 & 0 & 0.2775 \end{bmatrix}$	$v = 0.2106$

Now let us solve the game in symmetric pentagonal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (2).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (1,1.5,2,2.5,3) & (0,0.5,1,1.5,2) & (-2,-1,0,1,2) & (-3,-2.5,-2,-1.5,-1) \\ (0,0.5,1,1.5,2) & (-1,-0.5,0,0.5,1) & (2,2.5,3,3.5,4) & (0,1,2,3,4) \end{bmatrix}$$

$$\begin{bmatrix} (1,1.33,2,2.67,3) & (0,0.33,1,1.67,2) & (-2,-1.33,0,1.33,2) & (-3,-2.67,-2,-1.33,-1) \\ (0,0.33,1,1.67,2) & (-1,-0.67,0,0.67,2) & (2,2.33,3,3.67,4) & (0,0.67,2,3.33,4) \end{bmatrix}$$

$$\begin{bmatrix} (1,1.67,2,2.33,3) & (0,0.67,1,1.33,2) & (-2,-0.67,0,0.67,2) & (-3,-2.33,-2,-1.67,-1) \\ (0,0.67,1,1.33,2) & (-1,-0.33,0,0.33,1) & (2,2.67,3,3.33,4) & (0,1.33,2,2.67,4) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (5)

$$\begin{bmatrix} 2.0508 & 1.0981 & 0.4537 & -2.0508 \\ 1.0981 & 0.4537 & 3.0341 & 2.0508 \end{bmatrix}$$

$$\begin{bmatrix} 2.0568 & 1.1037 & 0.4539 & -2.0448 \\ 1.1037 & 0.4537 & 3.0402 & 2.0629 \end{bmatrix}$$

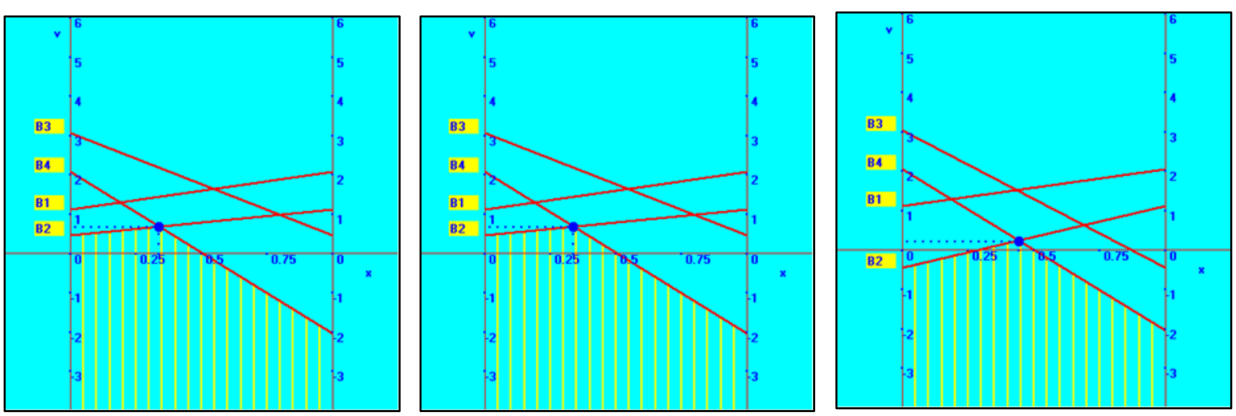
$$\begin{bmatrix} 2.0448 & 1.0925 & -0.4539 & -2.0568 \\ 1.0925 & -0.4537 & 3.0280 & 2.0388 \end{bmatrix}$$

Clearly, the above crisp data based problems does not possess a saddle point.

Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ a_1 & a_2 \end{bmatrix}$ and $a_1 + a_2 = 1.$ against B. Then A's expected payoffs against B's pure moves are given by

The expected payoff for A when he plays S_A against B's pure moves B_1, B_2, …B_n is given by

B's pure move	A's expected payoff E(a)		
	Problem 1	Problem 2	Problem 3
B1	$E_1(a_1) = 0.9527a_1 + 1.0981$	$E_1(a_1) = 0.9531a_1 + 1.1037$	$E_1(a_1) = 0.9523a_1 + 1.0925$
B2	$E_2(a_1) = 1.5518a_1 - 0.4537$	$E_2(a_1) = 0.65a_1 + 0.4537$	$E_2(a_1) = 1.5462a_1 - 0.4537$
B3	$E_3(a_1) = -3.4878a_1 + 3.0341$	$E_3(a_1) = -2.5863a_1 + 3.0402$	$E_3(a_1) = -3.4819a_1 + 3.0280$
B4	$E_4(a_1) = -4.1016a_1 + 2.0508$	$E_4(a_1) = -4.1077a_1 + 2.0629$	$E_4(a_1) = -4.0956a_1 + 2.0388$



FIGURES 8a, 8b, 8c. The Maximin Values

In the above three games with crisp payoff matrices, the player A wishes to maximize his minimum expected payoff. As per the graphical method, the highest point of intersection H on the lower envelop of the A's expected payoff equations is considered. At this point the lines B_2 and B_4 are intersected. The above three 2×4 games therefore reduce to the games with 2×2 following payoff matrices:

$$\begin{bmatrix} 1.0981 & -2.0508 \\ 0.4537 & 2.0508 \end{bmatrix} \quad \begin{bmatrix} 1.1037 & -2.0448 \\ 0.4537 & 2.0629 \end{bmatrix} \quad \begin{bmatrix} 1.0925 & -2.0568 \\ -0.4537 & 2.0388 \end{bmatrix}$$

Using eq. (8) and (9), the solutions of the above three problems are obtained as follows:

Mixed Strategies		**Value of the Game**
Strategy for Firm A	**Strategy for Firm B**	
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.3365 & 0.6635 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.8642 & 0 & 0.1358 \end{bmatrix}$	$v = 0.6706$
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.3382 & 0.6618 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.8634 & 0 & 0.1366 \end{bmatrix}$	$v = 0.6736$
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.4418 & 0.5582 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.7259 & 0 & 0.2741 \end{bmatrix}$	$v = 0.2294$

Now let us solve the game in symmetric hexagonal fuzzy environment.

In order to find its solution, the interval data based payoff matrix is converted into three fuzzy data based payoff matrices using eq. (3).

The fuzzy data based payoff matrices are

$$\begin{bmatrix} (1,1.4,1.8,2.2,2.6,3) & (0,0.4,0.8,1.2,1.6,2) & (-2,-1.2,-0.4,0.4,1.2,2) & (-3,-2.6,-2.2,-1.8,-1.4,-1) \\ (0,0.4,0.8,1.2,1.6,2) & (-1,-0.6,-0.2,0.2,0.6,1) & (2,2.4,2.8,3.2,3.6,4) & (0,0.8,1.6,2.4,3.2,4) \end{bmatrix}$$

$$\begin{bmatrix} (1,1.25,1.75,2.25,2.75,3) & (0,0.25,0.75,1.25,1.75,2) & (-2,-1.5,-0.5,0.5,1.5,2) & (-3,-2.75,-2.25,-1.75,-1.25,-1) \\ (0,0.25,0.75,1.25,1.75,2) & (-1,-0.75,-0.25,0.25,0.75,2) & (2,2.25,2.75,3.25,3.75,4) & (0,0.5,1.5,2.5,3.5,4) \end{bmatrix}$$

$$\begin{bmatrix} (1,1.57,1.86,2.14,2.43,3) & (0,0.57,0.86,1.14,1.43,2) & (-2,-0.86,-0.29,0.29,0.86,2) & (-3,-2.43,-2.14,-1.86,-1.57,-1) \\ (0,0.57,0.86,1.14,1.43,2) & (-1,-0.43,-0.14,0.14,0.43,1) & (2,2.57,2.86,3.14,3.43,4) & (0,1.14,1.71,2.29,2.86,4) \end{bmatrix}$$

The above fuzzy payoff matrices are converted into crisp payoff matrices using the ranking function (7) of proposed centroid (6)

$$\begin{bmatrix} 2.0593 & 1.1139 & 0.4907 & -2.0593 \\ 1.1139 & 0.4907 & 3.0399 & 2.0593 \end{bmatrix}$$

$$\begin{bmatrix} 2.0639 & 1.1181 & 0.4908 & -2.0548 \\ 1.1181 & 0.4908 & 3.0444 & 2.0683 \end{bmatrix}$$

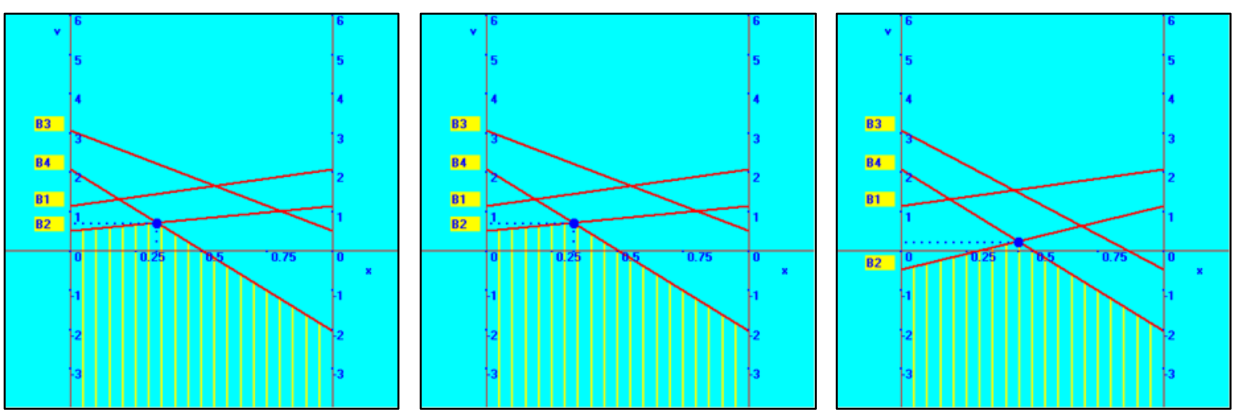
$$\begin{bmatrix} 2.0593 & 1.1139 & -0.4907 & -2.0593 \\ 1.1139 & -0.4907 & 3.0399 & 2.0593 \end{bmatrix}$$

Clearly, the above crisp data based problems does not possess a saddle point.

Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ a_1 & a_2 \end{bmatrix}$ and $a_1 + a_2 = 1.$ against B. Then A's expected payoffs against B's pure moves are given by

The expected payoff for A when he plays S_A against B's pure moves $B_1, B_2, \ldots B_n$ is given by

B's pure move	A's expected payoff E(a)		
	Problem 1	Problem 2	Problem 3
B1	$E_1(a_1) = 0.9454a_1 + 1.1139$	$E_1(a_1) = 0.9458a_1 + 1.1181$	$E_1(a_1) = 0.9454a_1 + 1.1139$
B2	$E_2(a_1) = 1.6046a_1 - 0.4907$	$E_2(a_1) = 0.6273a_1 + 0.4908$	$E_2(a_1) = 1.6046a_1 - 0.4907$
B3	$E_3(a_1) = -3.5306a_1 + 3.0399$	$E_3(a_1) = -2.5536a_1 + 3.0444$	$E_3(a_1) = -3.5306a_1 + 3.0399$
B4	$E_4(a_1) = -4.1186a_1 + 2.0593$	$E_4(a_1) = -4.1231a_1 + 2.0683$	$E_4(a_1) = -4.1186a_1 + 2.0593$



FIGURES 9a, 9b, 9c. The Maximin Values

In the above three games with crisp payoff matrices, the player A wishes to maximize his minimum expected payoff. As per the graphical method, the highest point of intersection H on the lower envelop of the A's expected payoff equations is considered. At this point the lines B_2 and B_4 are intersected. The above three 2 × 4 games therefore reduce to the games with 2 × 2 following payoff matrices:

$$\begin{bmatrix} 1.1139 & -2.0593 \\ 0.4907 & 2.0593 \end{bmatrix} \quad \begin{bmatrix} 1.1181 & -2.0548 \\ 0.4908 & 2.0683 \end{bmatrix} \quad \begin{bmatrix} 1.1139 & -2.0593 \\ -0.4907 & 2.0593 \end{bmatrix}$$

Using eq. (8) and (9), the solutions of the above three problems are obtained as follows:

Mixed Strategies		**Value of the Game**
Strategy for Firm A	**Strategy for Firm B**	
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.3308 & 0.6692 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.8686 & 0 & 0.1314 \end{bmatrix}$	$v = 0.6969$
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.3321 & 0.6679 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.8680 & 0 & 0.1320 \end{bmatrix}$	$v = 0.6991$
$S_A = \begin{bmatrix} A_1 & A_2 \\ 0.4456 & 0.5546 \end{bmatrix}$	$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0.7196 & 0 & 0.2804 \end{bmatrix}$	$v = 0.2242$

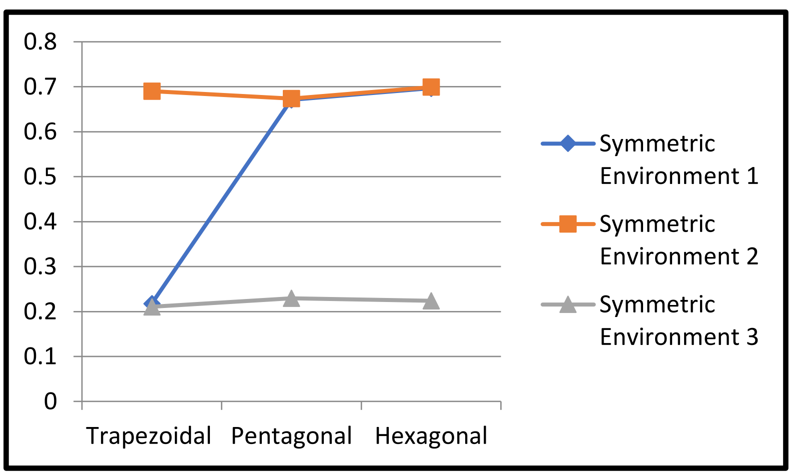


FIGURE 10. Values of the Game for IDBGTP

Therefore, the Value of the Game for given IDGTP = Max { Max {0.2178, 0.6903, 0.2106}, Max {0.6706, 0.6736, 0.2294}, Max {0.6969, 0.6991, 0.2242} } = Max {0.6903, 0.6736, 0.6991} = 0.6991.

It is observed from the above three examples for pure and mixed strategies that we can get maximum gain for player I and minimum loss for player II in the 2^{nd} symmetric fuzzy environment with regard to the three different shapes of fuzzy numbers taken in this study as per the utilization of its new centroids for defuzzification. Hence the study suggests that if we solve the matrix game with payoffs of interval number in the second symmetric fuzzy environment with regard to any shape of fuzzy number, it will give the effective optimum solution as per the proposed ranking function of new centroids of trapezoidal, pentagonal and hexagonal fuzzy numbers.

6. CONCLUSION

The matrix games with fuzzy payoffs have been widely studied in the literature. But, very few of the researchers have used interval numbers for representing the payoffs in game theory studied under uncertainty. Moreover, in finding the solution of interval data based game theoretical problems, they used interval arithmetic for its calculation based on traditional methods. In recent years, fuzzy set theory has got the recognition among many researchers for solving the optimization problems with imprecise information. The reason for utilizing this theory is that it has different shapes of membership functions for representing the vagueness. But some particular shape of membership functions only are being used by the researchers in frequent manner. Considering these things in our mind, we have proposed a new algorithm through connecting interval number with fuzzy number of its different shapes in order to find the effective solution of game theoretical problems which is the one of the special theory of optimization that deals the competitive situation with general features. Furthermore, the study has made an attempt to analyse the solutions of interval matrix games under various symmetric fuzzy environment with the help of various forms of fuzzy number.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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