SOLVING UNCONSTRAINED FUZZY PROBLEMS USING IH DIFFERENTIABILITY

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Abstract. In this paper, we develop new differentiability of fuzzy valued function termed improved Hukuhara(iH) differentiability. iH differentiability is used to solve unconstrained fuzzy optimization. The advantage of iH differentiability over-generalized Hukuhara differentiability is that it provides us with a unique solution to fuzzy optimization problems that remain fuzzy as time passes. We used the Newton approach to find the non-dominated solution in this case. We also provide an example to demonstrate the proposed method.

Keywords: fuzzy optimization; Newton method; Hukuhara differentiability.

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1. INTRODUCTION

Zadeh [16] first proposed the fuzzy set theory in 1965. Various scholars have explored fuzzy sets since then, and many applications have been discovered. Fuzzy optimization is one of them, and it deals with the uncertainty in the optimization problem. Bellman and Zadeh [1] introduced the notion of fuzzy optimization issues in 1970, in which they examined decision-making in a fuzzy context. Following that, a great deal of work has been done in this sector. We'll look

at a few of them here. Delgado et al [3], Kacprzyk[7], and Rommelfanger[11] provide a comprehensive overview of fuzzy optimization, including current breakthroughs and applications. Bachman [8] investigated probabilistic optimization problems. In fuzzy queuing theory, Abdalla and Buckley[2] presented Monte Carlo approaches. Wu [14], Pathak, and Pirzada [9] developed various optimality assumptions for fuzzy optimization problems.

In fuzzy optimization problem, differentiability is very significant. As a result, several scholars have looked into fuzzy differentiability in various methods. The concept of fuzzy hukuhara differentiability was proposed by Puri et.al[10]. To solve fuzzy differential equations, Ding[4], Kaleva[6], and Seikkala[12] used H differentiability. H differentiability was employed Wu.H.C[15] to identify the optimality criteria of fuzzy optimization problems. Bede and Stefanini[13] established a more general approach for fuzzy valued functions termed generalized Hukuhara(gH) differentiability. However, gH differentiability has the drawback of not providing a unique solution to the problem.

H differentiability of fuzzy functions is a highly limited concept. Consider the function $\tilde{\mathscr{F}} : X \to F_R$, F_R denotes the set of all fuzzy numbers on $\mathbb{R}$. Let $\tilde{\mathscr{F}}(x) = c.g(x)$, $c \in F_R$ and $g(x) \in \mathbb{R}$. The function is not H differentiable if $g'(x) < 0$. However, we can observe that the function above is gH differentiable. The only drawback of gH differentiability is that it does not give a singular value. To address this issue, we introduced a new notion known as the improved Hukuhara derivative. The benefit of this derivative is that it provides us with a unique answer. To obtain the solution of the function $\tilde{\mathscr{F}}(x) = c.g(x)$ when $g'(x) < 0$, we used the improved Hukuhara derivative instead of gH derivative.

The authors of the paper [9] used Hukuhara differentiability. However, the examples offered in the preceding paper did not meet the conditions of theorem 4.1 in[9]. In addition, in these situations, the objective functions are not H differentiable. As a result, we can deduce that the solutions obtained in [9] are not non-dominated..

We propose a new notion termed improved hukuhara differentiability in this study. Using the Newton technique and iH differentiability, we find the non-dominated solution to fuzzy optimization problems. In this work, we'll look at the examples in [9]. Clearly, it is iH differentiable, and the problem has a nondominated solution.

2. PRELIMINARIES

A fuzzy set on $\mathbb{R}$ is a mapping $\tilde{u} : \mathbb{R} \to [0,1]$. For each fuzzy set $\tilde{u}$, we denote its α level set as $[\tilde{u}]^{\alpha} = \{x \in \mathbb{R}/\tilde{u}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$. The support of $\tilde{u}$ is denoted by $supp(\tilde{u})$, where $supp(\tilde{u}) = \{x \in \mathbb{R}/\tilde{u}(x) > 0\}$. The closure of $supp(\tilde{u})$ defines the 0-level of $\tilde{u}$. i.e., $[\tilde{u}]^{0} = cl(supp(\tilde{u}))$, where cl denotes the closure of a set.

Definition 2.1. *A fuzzy set $\tilde{u}$ on $\mathbb{R}$ is said to be a fuzzy number if*

1. $\tilde{u}$ is normal and upper semi continuous.

2. The value of $\tilde{u}(\lambda x + (1 - \lambda)y)$ should be greater than or equal to $min\{\tilde{u}(x), \tilde{u}(y)\}, x, y \in \mathbb{R}, \lambda \in [0,1]$

3. $[\tilde{u}]^{0}$ should be compact.

Let F_R denote the set of all fuzzy numbers on $\mathbb{R}$. For any $\tilde{u} \in F_R$ we have $[\tilde{u}]^{\alpha} \in \mathscr{K}_c, \forall \alpha \in [0,1]$ where $\mathscr{K}_c$ denotes the space of all compact intervals in $\mathbb{R}$. The α level of a fuzzy number is given by $[\tilde{u}]^{\alpha} = [\underline{\tilde{u}}_{\alpha}, \overline{\tilde{u}}_{\alpha}], \underline{\tilde{u}}_{\alpha}, \overline{\tilde{u}}_{\alpha} \in \mathbb{R}, \forall \alpha \in [0,1]$. A triangular fuzzy number denoted by $\tilde{u} = (c_1, c_2, c_3)$ where $c_1 \leq c_2 \leq c_3$ has α cuts $[\tilde{u}]^{\alpha} = [c_1 + (c_2 - c_1)\alpha, c_3 - (c_3 - c_2)\alpha] \ \forall \alpha \in [0,1]$. For the fuzzy number $\tilde{u}, \tilde{v} \in F_R$, $\tilde{u}_{\alpha}$ and $\tilde{v}_{\alpha}$ are denoted by $[\underline{\tilde{u}}_{\alpha}, \overline{\tilde{u}}_{\alpha}]$ and $[\underline{\tilde{v}}_{\alpha}, \overline{\tilde{v}}_{\alpha}]$ respectively, for any real number λ and $\alpha \in [0,1]$, we define arithmetic operations using α level sets as follows.

$$(2.1) \qquad [\tilde{u} + \tilde{v}]_{\alpha} = [(\underline{\tilde{u} + \tilde{v}})_{\alpha}, (\overline{\tilde{u} + \tilde{v}})_{\alpha}] = [\underline{\tilde{u}}_{\alpha} + \underline{\tilde{v}}_{\alpha}, \overline{\tilde{u}}_{\alpha} + \overline{\tilde{v}}_{\alpha}]$$

and

$$(2.2) \qquad [(\lambda \tilde{u})]_{\alpha} = [(\underline{\lambda \tilde{u}})_{\alpha}, (\overline{\lambda \tilde{u}})_{\alpha}] = \begin{cases} [\lambda . \underline{\tilde{u}}_{\alpha}, \lambda . \overline{\tilde{u}}_{\alpha}], if \lambda \geq 0 \\ [\lambda . \overline{\tilde{u}}_{\alpha}, \lambda . \underline{\tilde{u}}_{\alpha}], if \lambda < 0 \end{cases}$$

Definition 2.2. *Let $\tilde{u}, \tilde{v} \subset \mathbb{R}^n$. The Hausdorff metric d_H is defined as*

$d_H(\tilde{u}, \tilde{v}) = \max\{\sup\limits_{x \in \tilde{u}} \inf\limits_{y \in \tilde{v}} ||x - y||, \sup\limits_{y \in \tilde{v}} \inf\limits_{x \in \tilde{u}} ||x - y||\}.$

The metric d_F on F_R is defined by $d_F(\tilde{u}, \tilde{v}) = \sup\limits_{0 \leq \alpha \leq 1} \{d_H(\tilde{u}_{\alpha}, \tilde{v}_{\alpha})\}, \forall \tilde{u}, \tilde{v} \in F_R$. Since $\tilde{u}_{\alpha}, \tilde{v}_{\alpha}$ are compact intervals in $\mathbb{R}$,

$d_F(\tilde{u}, \tilde{v}) = \sup\limits_{0 \leq \alpha \leq 1} \max\{|\underline{\tilde{u}}_{\alpha} - \underline{\tilde{v}}_{\alpha}|, |\overline{\tilde{u}}_{\alpha} - \overline{\tilde{v}}_{\alpha}|\}$

Definition 2.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and $a_0 \in \tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We say a_0 is a cluster point of $\{x_n\}$ if every neighbourhood of a_0 contains infinite many items of $\{x_n\}_{n=1}^{\infty}$*

It is also known that $a_0 \in \tilde{\mathbb{R}}$ is a cluster poinmt of $\{x_n\}_{n=1}^{\infty} \iff \exists$ a subsequence $\{x_{nk}\}_{k=1}^{\infty}$ such that $\lim\limits_{k \to \infty} x_{nk} = a_0$

Definition 2.4. *Assume $\tilde{\mathscr{F}} : X \to F_R$ be a fuzzy valued function and $x_0 \in \mathbb{R}$. Suppose that $\tilde{\mathscr{F}}$ is well defined in the interval $(x_0, x_0 + \delta)$ for some positive number δ. For a number $a_0 \in \tilde{\mathbb{R}}$ if $\exists$ a sequence of positive real numbers $\{c_n\}_{n=1}^{\infty}$ satisfying the following conditions.*

(1) $\lim\limits_{n \to \infty} c_n = 0$

(2) $\{x_0 + c_n\}_{n=1}^{\infty} \subset (x_0, x_0 + \delta)$

(3) a_0 is a cluster point of the sequence $\left\{ \tilde{\mathscr{F}}(x_0 + c_n) \right\}_{n=1}^{\infty}$,

then we say that a_0 is a cluster point of $\tilde{\mathscr{F}}$ on the right of x_0. The set of cluster points of $\tilde{\mathscr{F}}$ on the right of x_0 is denoted by $C_{R(x_0)}(\tilde{\mathscr{F}})$. Similarly we can define the cluster point of $\tilde{\mathscr{F}}$ on the left of x_0 and set of cluster points of $\tilde{\mathscr{F}}$ on the left of x_0 is denoted by $C_{L(x_0)}(\tilde{\mathscr{F}})$.

Definition 2.5. *Let $\tilde{\mathscr{F}} : X \to F_R$ be a fuzzy valued function and $x_0 \in X$. Define the function $\psi_{\tilde{\mathscr{F}}} : \mathbb{R}|\{0\} \to \mathbb{R}$ by*

$$\psi_{\tilde{\mathscr{F}}}(h) = \frac{\tilde{\mathscr{F}}(x_0 + h) - \tilde{\mathscr{F}}(x_0)}{h}$$

where h satisfies $x_0 + h \in X$. The function $\psi_{\tilde{\mathscr{F}}}(h)$ is called the slope function of secants at x_0.

Lemma 2.1. *Suppose the fuzzy valued function $\tilde{\mathscr{F}}$ is well defined in the interval $[x_0, x_0 + \delta)$. If $C_{R(0)}(\psi_{\tilde{\mathscr{F}}}) \subset \mathbb{R}$, then $\tilde{\mathscr{F}}$ is right continuous at x_0.*

Definition 2.6. *Let $\tilde{\mathscr{F}}_1$ and $\tilde{\mathscr{F}}_2$ be two fuzzy functions and is well defined in $(x_0, x_0 + \delta)$. If the two functions satisfy the following conditions*

(1) $C_{R(x_0)}(\tilde{\mathscr{F}}_1) = C_{R(x_0)}(\tilde{\mathscr{F}}_2) = \{\underline{t}, \bar{t}\}$ where $\underline{t}, \bar{t} \in \mathbb{R}$ and $\underline{t} < \bar{t}$

(2) $\lim\limits_{h \to 0^+} min\{\tilde{\mathscr{F}}_1(x_0 + h), \tilde{\mathscr{F}}_2(x_0 + h)\} = \underline{t}$
$\lim\limits_{h \to 0^+} max\{\tilde{\mathscr{F}}_1(x_0 + h), \tilde{\mathscr{F}}_2(x_0 + h)\} = \bar{t}$

then $\tilde{\mathscr{F}}_1$ and $\tilde{\mathscr{F}}_2$ are right complementary at x_0. Similarly we can define $\tilde{\mathscr{F}}_1$ and $\tilde{\mathscr{F}}_2$ are left complementary at x_0.

3. DIFFERENTIABILITY OF FUZZY VALUED FUNCTIONS

Suppose X be a subset of $\mathbb{R}^n$ and a function $\tilde{\mathscr{F}} : X \to F_R$ is said to be a fuzzy function, for each $\alpha \in [0,1]$ we define the family of interval valued functions $\tilde{\mathscr{F}}_\alpha : X \to \mathscr{K}_c$ by $\tilde{\mathscr{F}}_\alpha(x) = [\tilde{\mathscr{F}}(x)]^\alpha$. For any $\alpha \in [0,1]$ we defined $\tilde{\mathscr{F}}_\alpha(x) = [\underline{\tilde{f}}_\alpha(x), \overline{\tilde{f}}_\alpha(x)]$ where the endpoint functions $\underline{\tilde{f}}_\alpha, \overline{\tilde{f}}_\alpha$ are called upper and lower functions of $\tilde{\mathscr{F}}$ respectively.

3.1. Hukuhara differentiability.
Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers. If $\exists$ a fuzzy number n such that $n \oplus \tilde{v} = \tilde{u}$, then n is called hukuhara difference of $\tilde{u}$ and $\tilde{v}$ and is denoted by $\tilde{u} \ominus_H \tilde{v}$.

Definition 3.1. *A fuzzy valued function* $\tilde{\mathscr{F}} : X \to F_R$ *is said to be H differentiable at* $x_0 \in X$ *there exists an element* $\tilde{\mathscr{F}}'(x_0) \in \mathscr{F}_\mathbb{R}$ *such that forall* $h > 0$ *there exists* $\tilde{\mathscr{F}}(x_0 + h) \ominus_H \tilde{\mathscr{F}}(x_0), \tilde{\mathscr{F}}(x_0) \ominus_H \tilde{\mathscr{F}}(x_0 - h)$ *exists and the limits* $\lim_{h \to 0^+} \dfrac{\tilde{\mathscr{F}}(x_0 + h) \ominus_H \tilde{\mathscr{F}}(x_0)}{h} = \lim_{h \to 0^-} \dfrac{\tilde{\mathscr{F}}(x_0) \ominus_H \tilde{\mathscr{F}}(x_0 - h)}{h} = \tilde{\mathscr{F}}'(x_0)$

3.2. Improved Hukuhara Derivative.
Let $\tilde{\mathscr{F}} : X \to F_R$ and consider the function $\tilde{\mathscr{F}}(x) = c.g(x)$ where $c \in F_R$ and $g(x) \in \mathbb{R}$. If $g'(x) < 0$, then the function above is not H differentiable. However, the provided function is gH differentiable, and we may use gH differentiability to determine the answer. The only drawback of gH differentiability is that it does not provide us with a unique solution. As a result, we developed a improved Hukuhara derivative (iH derivative). We can see that the provided function is iH differentiable, and we can use iH differentiability to get the answer. The key benefit of iH differentiability is that it provides us with a unique solution.

Definition 3.2. *A function* $\tilde{\mathscr{F}} : X \to F_R$ *is said to be iH differentiable at* $x_0 \in X$ *there exists an element* $\tilde{\mathscr{F}}'(x_0) \in F_R$ *such that for all* $h > 0$ *sufficiently small and the limits*

$$\lim_{h \to 0^+} \frac{\tilde{\mathscr{F}}(x_0 + h) \ominus_H \tilde{\mathscr{F}}(x_0)}{h} = \lim_{h \to 0^-} \frac{\tilde{\mathscr{F}}(x_0) \ominus_H \tilde{\mathscr{F}}(x_0 - h)}{h} = \tilde{\mathscr{F}}'(x_0)$$

where

$$\lim_{h \to 0^+} \frac{\tilde{\mathscr{F}}(x_0 + h) \ominus_H \tilde{\mathscr{F}}(x_0)}{h} = [\min\{\lim_{h \to 0} \frac{\underline{\tilde{f}}(x_0 + h) - \overline{\tilde{f}}(x_0)}{h}, \lim_{h \to 0} \frac{\underline{\tilde{f}}(x_0 + h) - \underline{\tilde{f}}(x_0)}{h},$$
$$\lim_{h \to 0} \frac{\overline{\tilde{f}}(x_0 + h) - \overline{\tilde{f}}(x_0)}{h}, \lim_{h \to 0} \frac{\overline{\tilde{f}}(x_0 + h) - \underline{\tilde{f}}(x_0)}{h}\}, \max\{\lim_{h \to 0} \frac{\underline{\tilde{f}}(x_0 + h) - \overline{\tilde{f}}(x_0)}{h},$$
$$\lim_{h \to 0} \frac{\underline{\tilde{f}}(x_0 + h) - \underline{\tilde{f}}(x_0)}{h}, \lim_{h \to 0} \frac{\overline{\tilde{f}}(x_0 + h) - \overline{\tilde{f}}(x_0)}{h}, \lim_{h \to 0} \frac{\overline{\tilde{f}}(x_0 + h) - \underline{\tilde{f}}(x_0)}{h}\}].$$

Also, $\lim_{h\to 0^-} \dfrac{\tilde{\bar{\mathscr{F}}}(x_0) \ominus_H \tilde{\bar{\mathscr{F}}}(x_0 - h)}{h} =$

$[\min\{\lim_{h\to 0} \dfrac{\tilde{\bar{f}}(x_0) - \bar{\bar{f}}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\tilde{f}(x_0) - \tilde{f}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\bar{\bar{f}}(x_0) - \bar{\bar{f}}(x_0 - h)}{h}$

$, \lim_{h\to 0} \dfrac{\bar{\bar{f}}(x_0) - \tilde{f}(x_0 - h)}{h}\}, \max\{\lim_{h\to 0} \dfrac{\tilde{f}(x_0) - \bar{\bar{f}}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\tilde{f}(x_0) - \underline{f}(x_0 - h)}{h}$

$, \lim_{h\to 0} \dfrac{\bar{\bar{f}}(x_0) - \bar{\bar{f}}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\bar{\bar{f}}(x_0) - \underline{f}(x_0 - h)}{h}\}]$

3.3. Existence of improved Hukuhara differentiability.

Consider the function $\tilde{\bar{\mathscr{F}}}(x) = c.g(x)$ where $c \in F_R$ and $g(x) \in \mathbb{R}$, if $g'(x) > 0$ then it is H differentiable.But if $g'(x) < 0$ then the hukuhara derivative does not exists . Thus we proposed the new derivative and if $g'(x) < 0$ the function is iH differentiable and we solve the function as follows:

Assume $g'(x) < 0$ and $C = [\underline{c}, \bar{c}]$ then

$\lim_{h\to 0^+} \dfrac{\tilde{\bar{\mathscr{F}}}(x_0 + h) \ominus_H \tilde{\bar{\mathscr{F}}}(x_0)}{h} = [\min\{\lim_{h\to 0} \dfrac{\underline{cg}(x_0 + h) - c\bar{g}(x_0)}{h}, \lim_{h\to 0} \dfrac{\underline{cg}(x_0 + h) - \underline{cg}(x_0)}{h},$

$\lim_{h\to 0} \dfrac{\overline{cg}(x_0 + h) - \overline{cg}(x_0)}{h}, \lim_{h\to 0} \dfrac{\overline{cg}(x_0 + h) - \underline{cg}(x_0)}{h}\}, \max\{\lim_{h\to 0} \dfrac{\underline{cg}(x_0 + h) - \overline{cg}(x_0)}{h},$

$\lim_{h\to 0} \dfrac{\underline{cg}(x_0 + h) - \underline{cg}(x_0)}{h}, \lim_{h\to 0} \dfrac{\overline{cg}(x_0 + h) - \overline{cg}(x_0)}{h}, \lim_{h\to 0} \dfrac{\overline{cg}(x_0 + h) - \underline{cg}(x_0)}{h}\}].$

Since $g'(x) < 0$,the minimum and the maximum value are $-\bar{c}g'(x)$ and $-\underline{c}g'(x)$.

Also $\lim_{h\to 0^-} \dfrac{\tilde{\bar{\mathscr{F}}}(x_0) \ominus_H \tilde{\bar{\mathscr{F}}}(x_0 - h)}{h} =$

$[\min\{\lim_{h\to 0} \dfrac{\underline{cg}(x_0) - \overline{cg}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\underline{cg}(x_0) - \underline{cg}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\overline{cg}(x_0) - \overline{cg}(x_0 - h)}{h}$

$, \lim_{h\to 0} \dfrac{\overline{cg}(x_0) - \underline{cg}(x_0 - h)}{h}\}, \max\{\lim_{h\to 0} \dfrac{\underline{cg}(x_0) - \overline{cg}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\underline{cg}(x_0) - \underline{cg}(x_0 - h)}{h}$

$, \lim_{h\to 0} \dfrac{\overline{cg}(x_0) - \overline{cg}(x_0 - h)}{h}, \lim_{h\to 0} \dfrac{\overline{cg}(x_0) - \underline{cg}(x_0 - h)}{h}\}]$

Here $g(x_0) - g(x_0 - h)$ is negative thus the minimum and maximum value are $-\bar{c}g'(x)$ and $-\underline{c}g'(x)$.

Thus the left hand limit and right hand limit are equal. Hence $\tilde{\bar{\mathscr{F}}}(x) = c.g(x)$ is iH differentiable function.

Theorem 3.1. *Suppose the function* $\tilde{\bar{\mathscr{F}}} : X \to F_R$ *is iH differentiable then the interval valued function* $\tilde{\bar{\mathscr{F}}}_\alpha : X \to F_R$ *is iH differentiable for each* $\alpha \in [0, 1]$. *Moreover*

$$(3.1) \qquad\qquad [\tilde{\bar{\mathscr{F}}}'(x)]^\alpha = \tilde{\bar{\mathscr{F}}}'_\alpha(x)$$

.

Proof. The proof is obvious from the definition of iH differentiability.   □

Theorem 3.2. *Let $\tilde{\mathscr{F}} : X \to F_R$ is iH differentiable at x_0 if and only if one of the following cases holds:*

    (i) $(\underline{f}_\alpha)'_-(x_0), (\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_+(x_0)$ *exists and*

$$[\tilde{\mathscr{F}}'(x_0)]^\alpha = [min\{(\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_+(x_0)\}, max\{(\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_+(x_0)\}]$$
$$= [min\{(\underline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_-(x_0)\}, max\{(\underline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_-(x_0)\}]$$

   (ii) $\underline{f}$ *and* $\overline{f}$ *are right differentiable at* x_0. $\psi_{\underline{f}}(h)$ *and* $\psi_{\overline{f}}(h)$ *are left complementary at 0.ie*

$$C_{L(0)}(\psi_{\underline{f}}) = C_{L(0)}(\psi_{\overline{f}}) = \{\underline{t}, \overline{t}\} \text{ where } \underline{t}, \overline{t} \in \mathbb{R} \text{ and } \underline{t} < \overline{t}. \text{ Moreover,}$$
$$\tilde{\mathscr{F}}'(x_0) = [min\{(\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_+(x_0)\}, max\{(\underline{f}_\alpha)'_+(x_0), (\overline{f}_\alpha)'_+(x_0)\}] = [\underline{t}, \overline{t}]$$

  (iii) $\underline{f}$ *and* $\overline{f}$ *are left differentiable at* x_0. $\psi_{\underline{f}}(h)$ *and* $\psi_{\overline{f}}(h)$ *are right complementary at 0. ie*

$$C_{R(0)}(\psi_{\underline{f}}) = C_{R(0)}(\psi_{\overline{f}}) = \{\underline{t}, \overline{t}\} \text{ where } \underline{t}, \overline{t} \in \mathbb{R} \text{ and } \underline{t} < \overline{t}. \text{ Moreover,}$$
$$\tilde{\mathscr{F}}'(x_0) = [min\{(\underline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_-(x_0)\}, max\{(\underline{f}_\alpha)'_-(x_0), (\overline{f}_\alpha)'_-(x_0)\}] = [\underline{t}, \overline{t}]$$

  (iv) $\psi_{\underline{f}}(h)$ *and* $\psi_{\overline{f}}(h)$ *are both left complementary and right complementary at 0.ie*

$$C_{R(0)}(\psi_{\underline{f}}) = C_{R(0)}(\psi_{\overline{f}}) = C_{L(0)}(\psi_{\underline{f}}) = C_{L(0)}(\psi_{\overline{f}}) = \{\underline{t}, \overline{t}\} \text{ where } \underline{t}, \overline{t} \in \mathbb{R} \text{ and } \underline{t} < \overline{t}.$$
Moreover, $\tilde{\mathscr{F}}'(x_0) = [\underline{t}, \overline{t}]$

Next we are going to explain the partial derivative of $\tilde{\mathscr{F}}$ on $X \subset \mathbb{R}^n$. ie. $\tilde{\mathscr{F}}(x) = \tilde{\mathscr{F}}(x_1, ..., x_n) \in \mathscr{F}_\mathbb{R}$ for each $x = (x_1, ..., x_n) \in X$. Let $\tilde{\mathscr{F}} : X \to F_R$ then $\tilde{\mathscr{F}}(x)$ is denoted by $\tilde{\mathscr{F}}(x) = [\underline{f}(x), \overline{f}(x)]$. Also $\forall \ \alpha \in [0, 1]$, $\tilde{\mathscr{F}}_\alpha(x) = [\underline{f}_\alpha(x), \overline{f}_\alpha(x)]$.

Definition 3.3. *Suppose the fuzzy function $\tilde{\mathscr{F}}$ defined on $X \subset \mathbb{R}^n$ and let $x_0 = (x_1^{(0)}, ..., x_n^{(0)})$ be a fixed element of X. Consider the function $\tilde{d}_i(x_i) = \tilde{\mathscr{F}}(x_1^{(0)}, ..., x_i - 1^{(0)}, x_i, x_i + 1^{(0)}, ..., x_n^{(0)})$. If $\tilde{d}_i$ is iH differentiable at $x_i^{(0)}$, then clearly $\tilde{\mathscr{F}}$ has the ith partial iH derivative at x_0. ie.*

$$\left(\frac{\partial \tilde{\mathscr{F}}}{\partial x_i}\right)(x_0) = (\tilde{d}_i)'(x_i^{(0)})$$

Definition 3.4. *Let $\tilde{\mathscr{F}}$ be a fuzzy function defined on X and let $x_0 = (x_1^{(0)}, ..., x_n^{(0)}) \in X$ be fixed. We say that $\tilde{\mathscr{F}}$ is iH differentiable at x_0 if all the partial iH derivatives $\left(\frac{\partial \tilde{\mathscr{F}}}{\partial x_1}\right)(x_0), ..., \left(\frac{\partial \tilde{\mathscr{F}}}{\partial x_n}\right)(x_0)$ exist on some neighbourhood of x_0 and are continuous at x_0.*

If $\tilde{\mathscr{F}}$ is iH differentiable at x_0 then $\left(\frac{\partial \tilde{\mathscr{F}}}{\partial x_i}\right)(x_0)$ is a fuzzy number. Thus $\forall \alpha \in [0,1]$ we denote,

$$\left[\frac{\partial \tilde{\mathscr{F}}}{\partial x_i}(x_0)\right]^\alpha = \frac{\partial \tilde{\mathscr{F}}_\alpha}{\partial x_i}(x_0) = \left[\frac{\partial \underline{f}_\alpha}{\partial x_i}(x_0), \frac{\partial \overline{f}_\alpha}{\partial x_i}(x_0)\right], \forall \; \alpha \in [0,1].$$

Now we define a proposition which will be used in our main result.

Proposition 3.1. *If $\tilde{\mathscr{F}} : X \to F_R$ is iH differentiable at $x_0 \in X$ then, $\forall \; \alpha \in [0,1]$, $\underline{f}_\alpha + \overline{f}_\alpha : X \to \mathbb{R}$ is differentiable at x_0. Moreover*

$$(3.2) \qquad\qquad \frac{\partial \underline{\tilde{\mathscr{F}}_\alpha}}{\partial x_i}(x_0) + \frac{\partial \overline{\tilde{\mathscr{F}}_\alpha}}{\partial x_i}(x_0) = \frac{\partial (\underline{f}_\alpha + \overline{f}_\alpha)}{\partial x_i}(x_0)$$

Proof. The proof follows directly from Theorem 3.2 $\qquad\qquad\qquad\qquad$ $\square$

Now we can define the gradient of a fuzzy function.

Definition 3.5. *The gradient of $\tilde{\mathscr{F}} : X \to F_R$ at x_0, $\tilde{\nabla}\tilde{\mathscr{F}}(x_0)$, is defined as*

$$(3.3) \qquad\qquad \tilde{\nabla}\tilde{\mathscr{F}}(x_0) = \left(\left(\frac{\partial \tilde{\mathscr{F}}}{\partial x_1}\right)(x_0), ..., \left(\frac{\partial \tilde{\mathscr{F}}}{\partial x_n}\right)(x_0)\right)$$

where $(\frac{\partial \tilde{\mathscr{F}}}{\partial x_j})(x_0)$ denotes the jth partial iH derivative of $\tilde{\mathscr{F}}$ at x_0

Here $\tilde{\nabla}\tilde{\mathscr{F}}(x)$ denote the n-dimensional fuzzy vector. $\tilde{\nabla}$ represents the gradient of a fuzzy function whereas ∇ represents the gradient of a real valued function.

Definition 3.6. *Let the fuzzy function be $\tilde{\mathscr{F}} : X \subset \mathbb{R}^n \to F_R$. Assume that $x^0 \in X$ such that $\tilde{\nabla}\tilde{\mathscr{F}}$ is itself iH differentiable at x^0. ie for each i, $\frac{\partial \tilde{\mathscr{F}}}{\partial x_i} : X \to F_R$ is iH differentiable at x^0. The iH partial derivative of $\frac{\partial \tilde{\mathscr{F}}}{\partial x_i}$ is denoted by*

$D_{ij}^2 \tilde{\mathscr{F}}(x^0)$ *or* $\frac{\partial^2 \tilde{\mathscr{F}}}{\partial x_i x_j}(x^0)$ *if $i \neq j$*

and

$D_{ii}^2 \tilde{\mathscr{F}}(x^0)$ *or* $\frac{\partial^2 \tilde{\mathscr{F}}}{\partial x_i^2}(x^0)$ *if $i = j$*

If $\tilde{\mathscr{F}}$ is twice iH differentiable at each $x^0 \in X$ then we say that $\tilde{\mathscr{F}}$ is twice iH differentiable on X and if for each i,j=1,2,...,n the cross partial derivative $\frac{\partial^2 \tilde{\mathscr{F}}}{\partial x_i x_j}$ is continuous from $X \to F_R$ then we say that $\tilde{\mathscr{F}}$ is twice continuously iH differentiable on X.

Next we define p-times continuously iH differentiability of fuzzy valued functions similar to the above definiton.

ie. $\tilde{\mathscr{F}} : X \to F_R$ is p-times continuously iH differentiable on X if and only if all the partial iH derivatives of order $p \in \mathbb{N}$ exist and are continuous.

If $\tilde{\mathscr{F}}$ is iH differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ need not be differentiable. But by Proposition 3.1 we have $\underline{f}_\alpha + \overline{f}_\alpha$ is always differentiable for all $\alpha \in [0,1]$. This property holds for the p-times iH differentiability of $\underline{f}_\alpha + \overline{f}_\alpha$.

Proposition 3.2. *If $\tilde{\mathscr{F}} : X \to F_R$ is p-times iH differentiable at $x_0 \in X$ then for each $\alpha \in [0,1]$ the real valued function $\underline{f}_\alpha + \overline{f}_\alpha : X \to \mathbb{R}$ is p-times differentiable at x_0.*

Proof. It is clear from Proposition 3.1   □

4. iH DIFFERENTIABILITY IN FUZZY OPTIMIZATION

Results below shows the order relation on F_R. Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers such that the intervals $[\tilde{u}]^\alpha = [\underline{\tilde{u}}_\alpha, \overline{\tilde{u}}_\alpha]$ and $[\tilde{v}]^\alpha = [\underline{\tilde{v}}_\alpha, \overline{\tilde{v}}_\alpha] \ \forall \ \alpha \in [0,1]$, then,

$$(4.1) \qquad \tilde{u} \preceq \tilde{v} \qquad iff \quad [\tilde{u}]^\alpha \preceq [\tilde{v}]^\alpha, \quad \forall \ \alpha \in [0,1];$$

which is equal to $\underline{\tilde{u}}_\alpha \preceq \underline{\tilde{v}}_\alpha$ and $\overline{\tilde{u}}_\alpha \preceq \overline{\tilde{v}}_\alpha, \forall \ \alpha \in [0,1]$.

$$(4.2) \qquad \tilde{u} \preceq \tilde{v} \qquad iff \quad \tilde{u} \preceq \tilde{v} \quad and \quad \tilde{u} \neq \tilde{v}$$

which is equal to $[\tilde{u}]^\alpha \preceq [\tilde{v}]^\alpha, \forall \ \alpha \in [0,1]$ and there exists $\alpha^* \in [0,1]$ such that $\underline{\tilde{u}}_{\alpha*} \prec \underline{\tilde{v}}_{\alpha*}$ or $\overline{\tilde{u}}_{\alpha*} \prec \overline{\tilde{v}}_{\alpha*}$.

$$(4.3) \qquad \tilde{u} \prec \tilde{v} \qquad iff \quad \tilde{u} \preceq \tilde{v} \quad and \quad [\tilde{u}]^\alpha \neq [\tilde{v}]^\alpha, \quad \forall \ \alpha \in [0,1]$$

which is equal to $\underline{\tilde{u}}_\alpha \prec \underline{\tilde{v}}_\alpha$ and $\overline{\tilde{u}}_\alpha \prec \overline{\tilde{v}}_\alpha, \forall \ \alpha \in [0,1]$.

Since $\prec, \preceq$ denote the partial ordering on F_R, we may follow the same solution procedure used in multiobjective problem.

Now we write the mathematical expression for the fuzzy valued optimization problem

$$(4.4) \qquad \min \tilde{\mathscr{F}}(x)$$

$$x \in X \subset \mathbb{R}^n$$

where $\tilde{\mathscr{F}} : X \rightarrow F_R$.

Definition 4.1. *Let $x^* \in X \subset \mathbb{R}^n$ be a non dominated solution of fuzzy optimization problem (4.4) if there exists no $x \in N_{\varepsilon}(x^*) \cap X$ such that $\tilde{\mathscr{F}}(x) \preceq \tilde{\mathscr{F}}(x^*)$, where $N_{\varepsilon}(x^*)$ is a ε- neighbourhood of x^*.*

Next we go through the drawbacks of a theorem proved by Pirzada and Pathak . This theorem is applied in all the problems of [9]

Theorem 4.1. *Let $\tilde{\mathscr{F}} : X \rightarrow F_R$ be a fuzzy function. If x^* is a locally non dominated solution of (4.4) and for any direction d, for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{\mathscr{F}}(x^* + \lambda.d)$ and $\tilde{\mathscr{F}}(x^*)$ are comparable, then x^* is a local minimizer of the real valued functions $\underline{f}_{\alpha}$ and $\overline{f}_{\alpha}$, $\forall$ $\alpha \in [0, 1]$.*

In Theorem 4.1, the statement x^* is a locally non dominated solution of (4.4) and for any direction d, for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{\mathscr{F}}(x^* + \lambda.d)$ and $\tilde{\mathscr{F}}(x^*)$ are comparable, is very restrictive.

For example, consider a function $\tilde{\mathscr{F}}(x) = (-1, 0, 1)x$. We can see that the above conditions are not satisfied . Moreover $\forall$ x, y, $\tilde{\mathscr{F}}(x)$ and $\tilde{\mathscr{F}}(y)$ are not comparable. But 0 is a nondominated solution of $\tilde{\mathscr{F}}$.

Again in Theorem 4.1, the statement, x^* is a local minimizer of the real valued functions $\underline{f}_{\alpha}$ and $\overline{f}_{\alpha}$, $\forall$ $\alpha \in [0, 1]$ is restricitive. By this condition they stated that it is an ideal point and it is very difficult to find an ideal point for a fuzzy valued function. Thus all the examples considered in [9] do not have any ideal point. Consider an example in [9].

$\tilde{\mathscr{F}}(x_1, x_2) = (-1, 1, 3).x_1^2 + (0, 1, 2).x_1 x_2 + (1, 2, 4).x_2^2$ where $x_1, x_2 \in \mathbb{R}$. Here $x^* = (0, 0)$is a non dominated solution of $\tilde{\mathscr{F}}$ but it is not a local minimizer of the the end point function $\underline{f}_0$ since $\underline{f}_0(\varepsilon, 0) < 0 = \underline{f}_0(0, 0)$ for all $\varepsilon > 0$ near 0.

Thus we rewrite the above statement using the sum of the end point functions.

Theorem 4.2. *Suppose $\tilde{\mathscr{F}} : X \rightarrow F_R$ be a fuzzy function. If x^* is a local minimizer of the real valued function $\underline{f}_{\alpha} + \overline{f}_{\alpha}$, $\forall$ $\alpha \in [0, 1]$, then x^* is a locally non dominated solution of (4.4).*

Proof. We assume that x^* is not a locally non dominated solution of (4.4). Then there exists

$x \in N_{\varepsilon}(x^*)$ such that $\tilde{\mathscr{F}}(x) \prec \tilde{\mathscr{F}}(x^*)$. Thus there exists $\alpha^* \in [0,1]$ such that

$\underline{f}_{\alpha^*}(x) \leq \underline{f}_{\alpha^*}(x^*)$ or $\overline{f}_{\alpha^*}(x) \leq \overline{f}_{\alpha^*}(x^*)$. So $(\underline{f}_{\alpha^*} + \overline{f}_{\alpha^*})(x) < (\underline{f}_{\alpha^*} + \overline{f}_{\alpha^*})(x^*)$. Thus x^* is a local

minimizer of $\underline{f}_{\alpha} + \overline{f}_{\alpha}$. Hence the proof.   □

5. NEWTON METHOD TO FIND THE NON DOMINATED SOLUTION

Here we are going to define Newton method to find non-dominated solution of (4.4). Sup-

pose that at each measurement point $x^{(k_*)}$ we can find out the values of $\tilde{\mathscr{F}}(x^{(k_*)}), \tilde{\nabla}\tilde{\mathscr{F}}(x^{(k_*)})$

and $\tilde{\nabla}^2\tilde{\mathscr{F}}(x^{(k_*)})$. By considering the Proposition 1 and Proposition 2 we can find the values

of $\underline{f}_{\alpha}(x^{(k_*)}), \overline{f}_{\alpha}(x^{(k_*)}), \nabla(\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)})$ and $\nabla^2(\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)}) \ \forall \alpha \in [0,1]$. Thus by using

Taylor's formula the quadratic real valued function h_{α} can be obtained as

$h_{\alpha}(x) = (\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)}) + \nabla(\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)}).(x - x^{(k_*)}) + \{\frac{1}{2}(x - x^{(k_*)})^T.\nabla^2(\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)}).$

$(x - x^{(k_*)})\}$ for all $\alpha \in [0,1]$.

If x^* is a minimizer of $\underline{f}_{\alpha} + \overline{f}_{\alpha}$, for all $\alpha \in [0,1]$, given $x^{(k_*)}$ we try to approximate a minimizer

of $\underline{f}_{\alpha} + \overline{f}_{\alpha}$ by finding a minimizer of $h_{\alpha} \ \forall \alpha \in [0,1]$. From first order necessary condition for

h_{α} we have

$\nabla h_{\alpha}(x^*) = 0$, for all $\alpha \in [0,1]$.

$$(5.1) \qquad \implies \int_0^1 \nabla(\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)})d\alpha + \int_0^1 \nabla^2(\underline{f}_{\alpha} + \overline{f}_{\alpha})(x^{(k_*)})d\alpha(x - x^{(k_*)}) = 0$$

Define the real valued function $\phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi(x) = \int_0^1 (\underline{f}_{\alpha} + \overline{f}_{\alpha})(x)d\alpha.$$

If the function $\tilde{\mathscr{F}}$ is twice continuously iH differentiable then by Proposition 2 , $\underline{f}_{\alpha} + \overline{f}_{\alpha})$

is twice continuously iH differentiable $\forall \ \alpha \in [0,1]$. Hence ϕ is also twice continuously iH

differentiable. Thus from (5.1) we have

$$(5.2) \qquad\qquad\qquad \nabla\phi(x^{(k_*)}) + \nabla^2\phi(x^{(k_*)}).(x - x^{(k_*)}) = 0$$

substituting $x = x^{(k_*+1)}$ in (5.2) we get

$$(5.3) \qquad\qquad\qquad x^{(k_*+1)} = x^{(k_*)} - \nabla\phi(x^{(k_*)}).[\nabla^2\phi(x^{(k_*)})]^{-1}$$

where $[\nabla^2\phi(x^{(k_*)})]^{-1}$ is the inverse of matrix $\nabla^2\phi(x^{(k_*)})$. The method is stoped when $||x^{(k_*+1)} - x^{(k_*)}|| < \varepsilon$, where ε is a pre-defined termination scalar.

Using (5.3) we can find the stationary points of $\underline{f}_\alpha + \overline{f}_\alpha \ \forall \ \alpha \in [0,1]$. If we want to check if these points are minimizers of $\underline{f}_\alpha + \overline{f}_\alpha$, we need second order sufficient condition or convexity or generalized convexity of $\underline{f}_\alpha + \overline{f}_\alpha \forall \ \alpha \in [0,1]$.

5.1. Convergence. Now we discuss about the convergence of the above method.

Theorem 5.1. *Assume $\tilde{\mathscr{F}}$ is three times continuously iH differentiable on $\mathbb{R}^n$ and $x^* \in \mathbb{R}$ is a point such that*

(i) $\nabla\phi(x^*) = 0$

(ii) $\nabla^2\phi(x^*) is invertible;$

then $\forall x^{(0)}$ sufficiently close to x^* , the Newton method is well defined $\forall \ k_*$ and converges to x^* with order of convergence at least 2.

Proof. Since the fuzzy function $\tilde{\mathscr{F}}$ is three times continuously iH differentiable then ϕ is three times continuously differentiable function. Thus the proof is same as in the proof of Newton method.   □

5.2. Examples. In this section we consider the examples given in [9] and we correct it using the proposed method.

Example 5.1. *Consider the following nonlinear fuzzy optimization problem,*

$\min \tilde{\mathscr{F}}(x_1,x_2) = \tilde{1}.x_1^3 \oplus \tilde{2}.x_2^3 \oplus \tilde{1}.x_1.x_2,$

$x_1,x_2 \in \mathbb{R}$ *and* $\tilde{1} = (-1,1,3), \tilde{2} = (1,2,3)$ *are triangular fuzzy numbers.*

$[\tilde{\mathscr{F}}(x_1,x_2)]^\alpha = [\tilde{1}]^\alpha x_1^3 + [\tilde{2}]^\alpha x_2^3 + [\tilde{1}]^\alpha x_1 x_2$

$= [-1+2\alpha, 3-2\alpha]x_1^3 + [1+\alpha, 3-\alpha]x_2^3 + [-1+2\alpha, 3-\alpha]x_1 x_2$

$$= \begin{cases} \begin{aligned} &[(-1+2\alpha)x_1^3, (3-2\alpha)x_1^3] + [(1+\alpha)x_2^3, (3-\alpha)x_2^3] + \\ &[(-1+2\alpha)x_1x_2, (3-2\alpha)x_1x_2], && if\, x_1 \geq 0, x_2 \geq 0; \\[4pt] &[(-1+2\alpha)x_1^3, (3-2\alpha)x_1^3] + [(3-\alpha)x_2^3, (1+\alpha)x_2^3] + \\ &[(3-2\alpha)x_1x_2, (-1+2\alpha)x_1x_2], && if\, x_1 \geq 0, x_2 < 0; \\[4pt] &[(3-2\alpha)x_1^3, (-1+2\alpha)x_1^3] + [(1+\alpha)x_2^3, (3-\alpha)x_2^3] + \\ &[(-1+2\alpha)x_1x_2, (3-2\alpha)x_1x_2], && if\, x_1 < 0, x_2 < 0; \\[4pt] &[(3-2\alpha)x_1^3, (-1+2\alpha)x_1^3] + [(1+\alpha)x_2^3, (3-\alpha)x_2^3] + \\ &[(3-2\alpha)x_1x_2, (-1+2\alpha)x_1x_2], && if\, x_1 < 0, x_2 \geq 0. \end{aligned} \end{cases}$$

The endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are defined by

$$\underline{f}_\alpha(x_1,x_2) = \begin{cases} (-1+2\alpha)x_1^3 + (1+\alpha)x_2^3 + (-1+2\alpha)x_1x_2, & if\, x_1 \geq 0, x_2 \geq 0; \\ (-1+2\alpha)x_1^3 + (3-\alpha)x_2^3 + (3-2\alpha)x_1x_2, & if\, x_1 \geq 0, x_2 < 0; \\ (3-2\alpha)x_1^3 + (1+\alpha)x_2^3 + (-1+2\alpha)x_1x_2, & if\, x_1 < 0, x_2 < 0; \\ (3-2\alpha)x_1^3 + (1+\alpha)x_2^3 + (3-2\alpha)x_1x_2, & if\, x_1 < 0, x_2 \geq 0. \end{cases}$$

and

$$\overline{f}_\alpha(x_1,x_2) = \begin{cases} (3-2\alpha)x_1^3 + (3-\alpha)x_2^3 + (3-2\alpha)x_1x_2, & if\, x_1 \geq 0, x_2 \geq 0; \\ (3-2\alpha)x_1^3 + (1+\alpha)x_2^3 + (-1+2\alpha)x_1x_2, & if\, x_1 \geq 0, x_2 < 0; \\ (-1+2\alpha)x_1^3 + (3-\alpha)x_2^3] + (3-2\alpha)x_1x_2, & if\, x_1 < 0, x_2 < 0; \\ (-1+2\alpha)x_1^3 + (3-\alpha)x_2^3 + (-1+2\alpha)x_1x_2, & if\, x_1 < 0, x_2 \geq 0. \end{cases}$$

Example 4.1 in [9], Pirzada and Pathak have obtained different endpoint functions which is incorrect.

we can clearly see that the end point functions obtained in our method are not differentiable and hence $\tilde{\mathcal{F}}$ is not H differentiable. Thus we cannot apply the method given by Pirzada and Pathak.

But it clear that $\tilde{\mathcal{F}}$ is three times iH differentiable and also $\underline{f}_\alpha + \overline{f}_\alpha$ is three times differentiable. Hence

$(\underline{f}_\alpha + \overline{f}_\alpha)(x_1, x_2) = 2x_1^3 + 4x_2^3 + 2x_1 x_2, \ \forall \ \alpha \in [0,1].$

Thus $\phi(x) = \int_0^1 (\underline{f}_\alpha + \overline{f}_\alpha)(x_1, x_2)d\alpha = 2x_1^3 + 4x_2^3 + 2x_1 x_2,$

$$\nabla \phi(x_1, x_2) = \begin{pmatrix} 6x_1^2 + 2x_2 \\ 12x_2^2 + 2x_1 \end{pmatrix}$$

and

$$\nabla^2 \phi(x_1, x_2) = \begin{pmatrix} 12x_1 & 2 \\ 2 & 24x_2 \end{pmatrix}$$

We get a sequence $\{x^{(k)}\}, k = 1, 2, \ldots$ using the equation

(5.4) $\qquad\qquad x^{(k+1)} = x^{(k)} - \nabla\phi(x^{(k)}).[\nabla^2\phi(x^{(k)})]^{-}1$

and $x^* = (0,0)$ is a stationary point of $\underline{f}_\alpha + \overline{f}_\alpha \ \forall \ \alpha \in [0,1]$ with accuracy 10^{-3}. Also $\underline{f}_\alpha + \overline{f}_\alpha$ is not invex. So $x^* = (0,0)$ is not a non dominated solution since $\tilde{\mathscr{F}}(0, -\varepsilon) \prec \tilde{\mathscr{F}}(0,0)$. Thus we can conclude that the solution of example 4.1 in [9] is wrong.

Example 5.2. *Consider the problem*

$min\tilde{\mathscr{F}}(x_1, x_2) = (-1, 1, 3)x_1^2 \oplus (0, 1, 2)x_1 x_2 \oplus (1, 2, 4)x_2^2, x_1, x_2 \in \mathbb{R}$

Here we can see that $\tilde{\mathscr{F}}$ is not H differentiable and cannot apply Newton method proposed in [9]. But it is three times iH differentiable and $\underline{f}_\alpha + \overline{f}_\alpha(x_1, x_2) = 2x_1^2 + 2x_1 x_2 + (5 - \alpha)x_2^2$. Using Newton method we find the solution by considering the initial point as $x_0 = (2, -2)$. Using equation (5.3) we calculate the sequence $\{x^{(k)}\}, k = 1, 2, \ldots$ and we get $x^* = (0,0)$ as a stationary point of $\underline{f}_\alpha + \overline{f}_\alpha$ forall α with accuracy 10^{-3}. Since $\underline{f}_\alpha + \overline{f}_\alpha$ is convex, is a minimizer of $\underline{f}_\alpha + \overline{f}_\alpha$. Thus by theorem 4.2 x^* is a non dominated solution of $\tilde{\mathscr{F}}$.

6. CONCLUSION

We introduced a new notion called iH differentiability of fuzzy valued functions in this study. The advantage of iH differentiability over gH differentiability is that it provides us with a unique solution. We also looked at the examples in [9] and discovered that the functions are not H differentiable. As a result, we can't use Pirzada and Pathak's [9] Newton approach. However,

the same examples in [9] are iH differentiable, resulting in a non-dominated solution to the fuzzy valued issues.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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