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THE CONCEPT OF S-ALGEBRA AND ITS PROPERTIES

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Abstract. In this paper, a new type of algebra namely S-algebra is introduced. The partial ordering on S-algebra is introduced, some examples of S-algebras are given and some equivalent conditions for an S-algebra to become a distributive lattice are given by introducing a partial order S-algebra $x \leq y$, if $y \wedge x = x$. This partial ordering leads to some S-algebras. Congruences on S-algebra are introduced and some properties on congruences are proved. The concept of central element in an S-algebra is introduced. By using a central element a of S, S-algebra can be decomposed into two S-algebras and some important properties are emphasized.

Keywords: S-algebra; congruence; decomposition; central element.
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1. Introduction

Boolean logic has a wide applications in Computer science and Electronics. It is the main logic in Computer Languages . Lattice theory established to develop logic which is used in several sciences and technology. Distributive lattices are generalization of Boolean algebras. In this paper , a new concept namely S-algebra is introduced. It is neither a Distributive lattice

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nor a lattice but its satisfies some properties of these lattices.Infact its generalization of distributive lattices and also C-algebras.some examples of S-algebras are given and some equivalent conditions for an S-algebra to become a distributive lattice are given by introducing a partial order S-algebra . By Using this partial ordering , some S-algebras induced by the above partial ordering. Congruences on S-algebra are introduced and some properties on Congruences are proved.The concept of central element in an S-algebra is introduced. By using a central element a of S, S-algebra can be decomposed into two S-algebras and some important properties are emphasized.

2. PRELIMINARIES

Definition 2.1. Let A be an algebra and $\alpha,\beta \in Con(A)$. Then we have $\alpha o\beta = \{(x,y) \in A\times A \mid (x,z) \in \beta$ and $(z,y) \in \alpha$ for some $z \in A\}$.

Definition 2.2. Let A be an algebra and $\alpha,\beta \in Con(A)$. Then α and β are said to be permutable if $\alpha o\beta = \beta o\alpha$.

The following is a well known result.

Definition 2.3. Let A be an algebra. Then a subset L of $Con(A)$ is called permutable if any two congruences in L are permutable.

If A is any algebra and $\theta \in Con(A)$, then $A/\theta := \{a/\theta \mid a \in A\}$ is the quotient algebra with respect to the operations defined in [6], by $a/\theta \wedge b/\theta = (a\wedge b)/\theta$ and $a/\theta \vee b/\theta = (a\vee b)/\theta$. We write $\theta(a)$ for the element a/θ of A/θ.

Definition 2.4. Let A be an algebra and $\theta \in Con(A)$. Then the map $a \mapsto \theta(a)$ is called the natural map of A onto A/θ.

"If A is any algebra, then the congruences $A\times A$ and $\{(x,x) \mid x \in A\}$ are denoted by $\bigtriangledown_A$ and $\bigtriangleup_A$ respectively. Sometimes we refer to $\bigtriangleup_A$ as zero congruence on A."

Definition 2.5. Let A be an algebra and $\alpha \in Con(A)$. Then α is called a factor congruence or direct factor congruence if there exists $\beta \in Con(A)$ such that $\alpha \cap \beta = \bigtriangleup_A$ and $\alpha o\beta = \bigtriangledown_A$.

Definition 2.6. An algebra A is called (directly) indecomposable if A is not isomorphic to a direct product of two nontrivial algebras.

The following is a well known result, which characterize indecomposable algebras in terms of their congruences.

3. The Concept of S-Algebra

The variety of S-algebras is a generalisation of C-algebras,that is every C-algebra is an S-algebra but the converse need not be true since S-algebra is an algebra of type$(2,2)$ where as C-algebra is an algebra of type $(2,2,1)$.The unary operation in C-algebra is not there in S-algebra. According to our Knowledge the identities in S-algebra are independent .

Definition 3.1. An algebra $(S,\vee,\wedge)$ of type $(2,2)$ is called an S-algebra if it satisfies the following conditions;

(i): $x\wedge x=x, x\vee x=x$

(ii): $x\wedge(y\wedge z)=(x\wedge y)\wedge z, x\vee(y\vee z)=(x\vee y)\vee z$

(iii): $(x\wedge y)\vee(y\wedge x)=(y\wedge x)\vee(x\wedge y), (x\vee y)\wedge(y\vee x)=(y\vee x)\wedge(x\vee y)$

(iv): $x\wedge(x\vee y)=x, x\vee(x\wedge y)=x$

(v): $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z), x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$

(vi): $x\wedge y\wedge x=x\wedge y, x\vee y\vee x=x\vee y$

for all $x,y,z\in S$.

Some examples of S-algebras are given in the following.

Example 3.2. Every Boolean algebra is an S-algebra.

Example 3.3. The three element set $S=\{r,s,t\}$ with operations $\wedge,\vee$
given by;

∧	r	s	t
r	r	s	t
s	s	s	s
t	t	t	t

∨	r	s	t
r	r	r	r
s	r	s	t
t	t	t	t

is an S-algebra.

In the following we introduced a partial ordering on S-algebra, this partial ordering leads to some S-algebras induced by this partial ordering. Given any two elements x, y in an S-algebra $(S, \vee, \wedge)$, we define $\leq$ on S by "$x \leq y$, if $y \wedge x = x$. "Through out this chapter, by S, we mean that it is an S-algebra $(S, \vee, \wedge)$ unless otherwise mentioned.

Lemma 3.4. *Let S be an S-algebra. Then $\leq$ is a partial ordering on S.*

Proof. It is easy to observe that $\leq$ satisfies the reflexivity. Let $x, y \in S$ such that $x \leq y$ and $y \leq x$. Then, we have $y \wedge x = x$ and $x \wedge y = y$. Now,

$$\begin{aligned} x &= y \wedge x \\ &= x \wedge y \wedge x \quad (\text{since } x \wedge y = y) \\ &= x \wedge y \qquad\quad (\text{by Def. } S - algebra) \\ &= y. \qquad\qquad\; (\text{since } x \wedge y = y) \end{aligned}$$

Therefore $\leq$ satisfies anti-symmetric. Let $x, y, z \in Z$ such that $x \leq y$ and $y \leq z$. Then $y \wedge x = x$ and $z \wedge y = y$. Now,

$$\begin{aligned} z \wedge x &= z \wedge y \wedge x \quad (\text{since } y \wedge x = x) \\ &= y \wedge x \qquad\quad (\text{since } z \wedge y = y) \\ &= x \qquad\qquad\;\; (\text{since } y \wedge x = x) \end{aligned}$$

Therefore $x \leq z$ and hence $\leq$ is a partial ordering on S. □

Lemma 3.5. *In a partial ordered set $(S, \leq)$, for any $x, y \in S$, we have the following;*

(i) If $x \leq y$, then $y \vee x = y$ and $x \wedge y \leq x$

(ii) If $x \leq y$, for any $z \in S$, (a) $z \wedge x \leq z \wedge y$ (b) $z \vee x \leq z \vee y$.

Proof. Let $x, y \in S$.

(i) If $x \leq y$, then $y \wedge x = x$. Now,

$$y \vee x = y \vee (y \wedge x) = y \text{ and}$$

$$x \wedge (x \wedge y) = (x \wedge x) \wedge y \; = x \wedge y$$

(ii) Suppose that $x \leq y$ and for any $z \in S$.

(a)

$$\begin{aligned}(z \wedge y) \wedge (z \wedge x) &= (z \wedge y \wedge z) \wedge x \\ &= z \wedge y \wedge x \\ &= z \wedge x \qquad \text{(since } y \wedge x = x)\end{aligned}$$

Therefore $z \wedge x \leq z \wedge y$.

(b)

$$\begin{aligned}(z \vee y) \wedge (z \vee x) &= z \vee (y \wedge x) \\ &= z \vee x \qquad \text{(since } y \wedge x = x)\end{aligned}$$

Therefore $z \vee x \leq z \vee y$. □

Lemma 3.6. *In a partial ordered set $(S, \leq)$, for any $x, y, z \in S$; we have the following; $x \leq y \implies x \vee (y \wedge z) = y \wedge (x \vee z)$.*

Theorem 3.7. *In an S-algebra S, for any $x, y, z \in S$, the following identity holds;*

$$x \wedge (y \vee z) = x \wedge [y \wedge (x \vee z)] \vee z$$

Theorem 3.8. *In an S-algebra S; for any $x, y, z \in S$, the following identity holds;*

$$x \vee (y \wedge z) = x \vee [y \vee (x \wedge z)] \wedge z$$

.

Lemma 3.9. *In an S-algebra S, for any $x, y \in S$, $x \wedge y = y \wedge x \implies y \leq y \vee x$.*

Theorem 3.10. *An S-algebra S is a distributive lattice "iff" $x \vee y$ is an upper bound of x, y, for all $x, y \in S$*

Proof. Let S be an S-algebra. It is observe that if S is a distributive lattice; then $x \vee y$ is an upper bound of x, y.

Conversely, suppose that $x \vee y$ is an upper bound of x, y, for all $x, y \in S$. Then $x \leq x \vee y$ and $y \leq x \vee y$. That is $(x \vee y) \wedge x = x$ and $(x \vee y) \wedge y = y$. Now,

$$\begin{aligned}(x \vee y) \wedge (y \vee x) &= (x \vee y \vee x) \wedge (y \vee x) \\ &= [x \vee (y \vee x)] \wedge (y \vee x) \\ &= y \vee x\end{aligned}$$

$$\begin{aligned}(y\vee x)\wedge(x\vee y) &= (y\vee x\vee y)\wedge(x\vee y)\\ &= [y\vee(x\vee y)]\wedge(x\vee y)\\ &= x\vee y.\end{aligned}$$

Therefore $\vee$ is "commutative".

Similarly

$$\begin{aligned}(x\wedge y)\vee(y\wedge x) &= (x\wedge y)\vee(y\wedge x\wedge y)\\ &= (x\wedge y)\vee[y\wedge(x\wedge y)]\\ &= x\wedge y\end{aligned}$$

$$\begin{aligned}(y\wedge x)\vee(x\wedge y) &= (y\wedge x)\vee(x\wedge y\wedge x)\\ &= (y\wedge x)\vee[x\wedge(y\wedge x)]\\ &= y\wedge x. \qquad)\end{aligned}$$

Therefore $\wedge$ is commutative. Thus S is a distributive lattices. □

Theorem 3.11. *In an S-algebra S, if $x\vee y$ is an upper bound of x,y, for all $x,y\in S$, then $x\vee y$ is the supremum of x and y.*

Proof. Let $x,y\in S$ such that $x\vee y$ is an upper bound of x and y. That is $x\leq x\vee y$ and $y\leq x\vee y$. Let t be an upper bound of x and y. Then $x\leq t$ and $y\leq t$. So that $t\wedge x=x$ and $t\wedge y=y$. Now,

$$\begin{aligned}t\wedge(x\vee y) &= (t\wedge x)\vee(t\wedge y)\\ &= x\vee y. \qquad (\text{since } t\wedge x=x \text{ and } t\wedge y=y)\end{aligned}$$

Therefore $t\wedge(x\vee y)=x\vee y$ and hence $x\vee y\leq t$.

Thus $x\vee y$ is the supremum of x and y. □

Theorem 3.12. *An S-algebra S is distributive lattice if and only if the following holds.*

(i) $x\wedge(y\vee x)=x$ *for all* $x,y\in S$

(ii) $x\wedge y\leq y$, *for all* $x,y\in S$.

Proof. If S is a distributive lattice, then it is easy to observe that the conditions (i) and (ii) are trivial. On the other hand, assume that the conditions (i) and (ii) holds in a S-algebra S. Let

$x, y \in S$. Then

$$\begin{aligned}(x \vee y) \wedge (y \vee x) &= (x \vee y) \wedge (y \vee x \vee y) \\ &= (x \vee y) \wedge [y \vee (x \vee y)] \\ &= x \vee y\end{aligned}$$

and

$$\begin{aligned}(y \vee x) \wedge (x \vee y) &= (y \vee x) \wedge (x \vee y \vee x) \\ &= (y \vee x) \wedge [x \vee (y \vee x)] \\ &= y \vee x. \qquad \text{(by our assumption}(i))\end{aligned}$$

Therefore $x \vee y \leq y \vee x$ and $y \vee x \leq x \vee y$. Hence $x \vee y = y \vee x$.

From (ii), we have $x \wedge y \leq y$. So that $y \wedge x \wedge y = x \wedge y$. Hence $y \wedge x = x \wedge y$. . Thus S is a distributive lattice. □

Lemma 3.13. *In an S-algebra S, if $x \wedge y$ is a lower bound of x and y, then $x \wedge y$ is the infimum of x and y, for all $x, y \in S$.*

Proof. Let $x, y \in S$ such that $x \wedge y$ is a lower bound of x and y. Then $x \wedge y \leq x$ and $x \wedge y \leq y$. Let t be a lower bound of x and y. Then $t \leq x, y$. That is $x \wedge t = y \wedge t = t$. Now,

$$\begin{aligned}(x \wedge y) \wedge t &= x \wedge (y \wedge t) \\ &= (x \wedge t) \qquad (\text{since } y \wedge t = t) \\ &= t. \qquad (\text{since } x \wedge t = t)\end{aligned}$$

Therefore $t \leq x \wedge y$. Hence $x \wedge y$ is the infimum of x and y. □

4. Some Properties of S-Algebra and Its Congruences

In this section we introduce congruence on S-algebra and some properties of these congruences are proved.

Definition 4.1. Let S be an S-algebra and $a \in S$; χ_a is defined as $\chi_a = \{(x, y) \mid a \wedge x = a \wedge y\}$.

Lemma 4.2. *Let S be an S-algebra and $a \in S$. Then χ_a is a congruence relation on S.*

Proof. Clearly χ_a satisfies" reflexive and symmetric." Let $(x, y) \in \chi_a$ and $(y, z) \in \chi_a$. Then $a \wedge x = a \wedge y$ and $a \wedge y = a \wedge z$. So that $a \wedge x = a \wedge z$. Therefore $(x, z) \in \chi_a$ and hence χ_a is an equivalence relation on S.

Let $(x,s),(y,t) \in \chi_a$. Then $a\wedge x = a\wedge s$, $a\wedge y = a\wedge t$. Now, $a\wedge(x\wedge y)$= $(a\wedge x)\wedge y$ = $(a\wedge s)\wedge y$ = $(a\wedge s\wedge a)\wedge y$ = $(a\wedge s)\wedge(a\wedge y)$ = $(a\wedge s)\wedge(a\wedge t)$ = $(a\wedge s\wedge a)\wedge t$= $(a\wedge s)\wedge t = a\wedge(s\wedge t)$

Therefore $(x\wedge y, s\wedge t) \in \chi_a$. Now, $a\wedge(x\vee y)$ = $(a\wedge x)\vee(a\wedge y)$ = $(a\wedge s)\vee(a\wedge t)$ (since $a\wedge x = a\wedge s, a\wedge y = a\wedge t$) = $a\wedge(s\vee t)$

Therefore $(x\vee y, s\vee t) \in \chi_a$ hence χ_a is compatible with binary operation $\vee,\wedge$.
Thus χ_a is congruence on S. □

Theorem 4.3. *The following are hold for any elements a,b of an S-algebra.*

(i) $\chi_a \cap \chi_b \subseteq \chi_{a\wedge b}$

(ii) If $a \leq b$, then $a\wedge b = b\wedge a$

(iii) $\chi_{a\wedge b} = \chi_{b\wedge a}$

(iv) $\chi_a o \chi_b \subseteq \chi_{a\wedge b} = \chi_{b\wedge a}$

(v) If $a \leq b$; then $\chi_b \subseteq \chi_a$.

Proof. For any $a,b \in S$.

(i) Let $(x,y) \in \chi_a \cap \chi_b$, then $a\wedge x = a\wedge y$ and $b\wedge x = b\wedge y$.

Now,

$(a\wedge b)\wedge x$ = $a\wedge(b\wedge x)$ = $a\wedge(b\wedge y)$ = $(a\wedge b)\wedge y$.

Therefore $(x,y) \in \chi_{a\wedge b}$ and hence $\chi_a \cap \chi_b \subseteq \chi_{a\wedge b}$.

(ii) If $a \leq b$, then we have $b\wedge a = a$.

Now, $a\wedge b$ = $a\wedge b\wedge a$ = $a\wedge a$ = a = $b\wedge a$.

Therefore $a\wedge b = b\wedge a$

(iii) Let $(x,y) \in \chi_{a\wedge b}$, then $a\wedge b\wedge\ x = a\wedge b\wedge y$.

Now,

$(b\wedge a)\wedge x$ = $(b\wedge a\wedge b)\wedge x$ = $b\wedge(a\wedge b\wedge x)$ = $b\wedge(a\wedge b\wedge y)$ = $(b\wedge a\wedge b)\wedge y)$ = $(b\wedge a)\wedge y$.

Therefore $(x,y) \in \chi_{b\wedge a}$ and hence $\chi_{a\wedge b} \subseteq \chi_{b\wedge a}$

On the other side, let $(x,y) \in \chi_{b\wedge a}$;

Now,

$(a\wedge b)\wedge x = (a\wedge b\wedge a)\wedge x = a\wedge(b\wedge a\wedge x) = a\wedge(b\wedge a\wedge y) = (a\wedge b\wedge a)\wedge y) = (a\wedge b)\wedge y$

Therefore $(x,y)\in\chi_{a\wedge b}$. So that $\chi_{b\wedge a}\subseteq\chi_{a\wedge b}$ and hence $\chi_{a\wedge b}=\chi_{b\wedge a}$ (by (iii))

(iv) Let $(x,y)\in\chi_a o\chi_b$. Then there exists $t\in S$ such that $(x,t)\in\chi_a$ and $(t,y)\in\chi_b$. That is $a\wedge x=a\wedge t$ and $b\wedge t=b\wedge y$.

Now,

$(a\wedge b)\wedge x = (a\wedge b\wedge a)\wedge x = (a\wedge b)\wedge(a\wedge x) = (a\wedge b)\wedge(a\wedge t)$ (since $a\wedge x=a\wedge t$) $= (a\wedge b\wedge a)\wedge t = (a\wedge b)\wedge t = a\wedge(b\wedge t) = a\wedge(b\wedge y)$ (since $b\wedge t=b\wedge y$) $= (a\wedge b)\wedge y$.

There fore $(x,y)\in\chi_{a\wedge b}$. So that $\chi_a o\chi_b\subseteq\chi_{a\wedge b}$ and hence $\chi_a o\chi_b\subseteq\chi_{a\wedge b}=\chi_{b\wedge a}$. (by (iii))

(v) If $a\leq b$ then $b\wedge a=a$.

Let $(x,y)\in\chi_b$. Then we have $b\wedge x=b\wedge y$.

Now,

$a\wedge x = (b\wedge a)\wedge x$(since $b\wedge a=a$) $= (a\wedge b)\wedge x = a\wedge(b\wedge x) = a\wedge(b\wedge y)$ (since $b\wedge x=b\wedge y$) $= (a\wedge b)\wedge y = (b\wedge a)\wedge y = a\wedge y$. (since $b\wedge a=a$)

Therefore $a\wedge x=a\wedge y$ and hence $(x,y)\in\chi_a$.

Thus $\chi_b\subseteq\chi_a$. □

5. Decomposition of S Algebra by Using Partial Orderings

In this section ,for each $a\in S$,where S is an S-algebra , $S_a=\{a\wedge x/x\in S\}$ is a sub-algebra of S.The concept of Central element in S-algebra is introduced.By using this,if a is a central element of S then S is isomorphic to product of two sub-algebras.

For each element in an S-algebra S, we introduce a subalgebra of S.

Lemma 5.1. *Let S be an S-algebra and $a\in S$. Then $S_a=\{a\wedge x/x\in S\}$ is the subalgebra of S.*

Proof. Let S be an S-algebra and $x,y\in S$ such that

$(a\wedge x)\wedge(a\wedge y) = (a\wedge x\wedge a)\wedge y = (a\wedge x)\wedge y = a\wedge(x\wedge y)$

Therefore $(a\wedge x)\wedge(a\wedge y)\in S_a$. (since $x\wedge y\in S$)

Similarly, by Def of S, $(a\wedge x)\vee(a\wedge y)=a\wedge(x\vee y)\in S_a$. (since $x\vee y\in S$)

Hence S_a is a subalgebra of S. □

Theorem 5.2. *For any $a \in S$, a mapping γ_a from S to S_a defined by $\gamma_a(x) = a \wedge x$, for all $x \in S$ is a homomorphism. Moreover* $\dfrac{S}{Ker(\gamma_a)} \cong S_a$.

Proof. For any $a \in S$, define a map $\gamma_a : S \longrightarrow S_a$ by $\gamma_a(x) = a \wedge x$, for all $x \in S$. Now, for any $x, y \in S$,

$$\begin{aligned} x = y \Rightarrow \quad & a \wedge x = a \wedge y \\ \Rightarrow \quad & \gamma_a(x) = \gamma_a(y). \end{aligned}$$

Therefore γ_a is well defined. Let $x, y \in S$. Then

$$\begin{aligned} \gamma_a(x \wedge y) &= a \wedge (x \wedge y) \\ &= (a \wedge x) \wedge y \\ &= (a \wedge x \wedge a) \wedge y \\ &= (a \wedge x) \wedge (a \wedge y) \\ &= \gamma_a(x) \wedge \gamma_a(y). \end{aligned}$$

Similarly,

$$\begin{aligned} \gamma_a(x \vee y) &= a \wedge (x \vee y) \\ &= (a \wedge x) \vee (a \wedge y) \\ &= \gamma_a(x) \vee \gamma_a(y). \end{aligned}$$

Therefore γ_a is homomorphism. Let $z \in S_a$. Then $z = a \wedge x$ for some $x \in S$. So that $\gamma_a(x) = a \wedge x = z$. Therefore γ_a is an onto homomorphism. Now,

$$\begin{aligned} Ker\gamma_a = \; & \{(x, y) \mid \gamma_a(x) = \gamma_a(y)\} \\ = \; & \{(x, y) \mid a \wedge x = a \wedge y\} \\ = \; & \chi_a. \end{aligned}$$

Therefore $Ker(\gamma_a) = \chi_a$. Hence by the homomorphism theorem, we get $\dfrac{S}{Ker(\gamma_a)} \cong S_a$. □

Definition 5.3. An S-algebra S is said to be S-algebra with T, if there exists $T \in S$ such that $T \wedge x = x \wedge T = x$, for all $x \in S$.

In this case, T is called meet identity.

Definition 5.4. An S-algebra S is said to be S-algebra with F, if there exists $F \in S$ such that $F \vee x = x \vee F = x$, for all $x \in S$.

In this case, F is called join identity.

Lemma 5.5. *If F is join identity in S-algebra, then $F \wedge x = F$.*

Proof. Let $x \in s$, and F is join identity. Then we have $F \vee x = x \vee F = x$.

Now,

$$\begin{aligned} F \wedge x &= F \wedge (F \vee x) \quad (\text{since } F \vee x = x) \\ &= F. \end{aligned}$$

Therefore $F \wedge x = F$. □

Theorem 5.6. *Let S be an S-algebra with T, F .Then $\chi_T = \Delta$, $\chi_F = S \times S$.*

Proof. Let $x, y \in S$. Then

$$\begin{aligned} \chi_T &= \{(x,y) \mid T \wedge x = T \wedge y\} \\ &= \{(x,y) \mid x = y\} \\ &= \Delta. \end{aligned}$$

and

$$\begin{aligned} \chi_F &= \{(x,y) \mid F \wedge x = F \wedge y\} \\ &= \{(x,y) \mid F = F\} \\ &= S \times S. \end{aligned}$$

□

Definition 5.7. An element a of an S-algebra with T, F is said to be a central element of S, if it obeys the below conditions;

(i) There exists $b \in S$ such that $a \wedge b = b \wedge a = F$ and $a \vee b = T$.

(ii) If $a \wedge x = a \wedge y$ and $b \wedge x = b \wedge y$, then $x = y$.

Theorem 5.8. *For any central element a of S, there exists $b \in S$ such that $\chi_a \cap \chi_b = \Delta$ and $\chi_a o \chi_b = S \times S$.*

Proof. Let $(x,y) \in \chi_a \cap \chi_b$.

Then $a \wedge x = a \wedge y$ and $b \wedge x = b \wedge y$. So that $x = y$. (since a is central element)

Therefore $(x,y) \in \Delta$ hence we get $\chi_a \cap \chi_b \subseteq \Delta$. Clearly we have $\Delta \subseteq \chi_a \cap \chi_b$. Hence $\chi_a \cap \chi_b = \Delta$.

For, $x \neq y$, consider $z = (a \wedge x) \vee (b \wedge y)$

Now,

$$\begin{aligned} a \wedge z &= a \wedge [(a \wedge x) \vee (b \wedge y)] \\ &= (a \wedge a \wedge x) \vee (a \wedge b \wedge y) \\ &= (a \wedge x) \vee (F \wedge y) && \text{(since } a \text{ is central element)} \\ &= (a \wedge x) \vee F \\ &= a \wedge x. \end{aligned}$$

Therefore $(x,z) \in \chi_a$. Similarly,

$$\begin{aligned} b \wedge z &= b \wedge [(a \wedge x) \vee (b \wedge y)] && \text{(since } z = (a \wedge x) \vee (b \wedge y)) \\ &= (b \wedge a \wedge x) \vee (b \wedge b \wedge y) \\ &= (F \wedge x) \vee (b \wedge y) && \text{(since } a \text{ is central element)} \\ &= F \vee (b \wedge y) \\ &= b \wedge y. \end{aligned}$$

Therefore $(z,y) \in \chi_b$. So that $(x,y) \in \chi_a o\ \chi_b$ and hence $\chi_a o\ \chi_b \supseteq S \times S$.

Clearly, we have that $\chi_a o\ \chi_b \subseteq S \times S$. So that $\chi_a o \chi_b = S \times S$.

Thus χ_a, χ_b are factor congruences on S. □

Theorem 5.9. *If a is central element of S, then there exists $b \in S$ such that $S \cong S_a \times S_b$.*

Proof. Define a map $h : S \longrightarrow S_a \times S_b$ such that $h(x) = (\gamma_a(x), \gamma_b(x))$. Then,

$$\begin{aligned} h[x \vee y] &= (\gamma_a[x \vee y], \gamma_b[x \vee y]) \\ &= (a \wedge [x \vee y], b \wedge [x \vee y]) && \text{(since } \gamma_a(x) = a \wedge x) \\ &= ((a \wedge x) \vee (a \wedge y), (b \wedge x) \vee (b \wedge y)) \\ &= (\gamma_a(x) \vee \gamma_a(y), \gamma_b(x) \vee \gamma_b(y)) \\ &= (\gamma_a(x), \gamma_b(x)) \vee (\gamma_a(y), \gamma_b(y) \\ &= h(x) \vee h(y). \end{aligned}$$

and

$$\begin{aligned} h(x\wedge y) &= (\gamma_a(x\wedge y),\gamma_b(x\wedge y)) \\ &= (a\wedge(x\wedge y), b\wedge(x\wedge y)) \qquad \text{(since } \gamma_a(x)=a\wedge x) \\ &= ((a\wedge x)\wedge y,(b\wedge x)\wedge y)) \\ &= ((a\wedge x\wedge a)\wedge y,(b\wedge x\wedge b)\wedge y) \\ &= ((a\wedge x)\wedge(a\wedge y),(b\wedge x)\wedge(b\wedge y)) \\ &= (\gamma_a(x)\wedge\gamma_a(y),\gamma_b(x)\wedge\gamma_b(y)) \\ &= (\gamma_a(x),\gamma_b(x))\wedge(\gamma_a(y),\gamma_b(y)) \\ &= h(x)\wedge h(y). \end{aligned}$$

Therefore h is a homomorphism.

Let $x,y\in S$ such that

$$\begin{aligned} &h(x)=h(y) \\ &\Rightarrow ((\gamma_a(x),\gamma_b(x))=(\gamma_a(y),\gamma_b(y)) \\ &\Rightarrow (a\wedge x, b\wedge x)=(a\wedge y, b\wedge y) \\ &\Rightarrow a\wedge x=a\wedge y \text{ and } b\wedge x=b\wedge y \end{aligned}$$

Then we have $a\wedge x=a\wedge y$ and $b\wedge x=b\wedge y$.

Therefore $x=y$ (since a is central element). Hence h is one-one.

Let $(x,y)\in S_a\times S_b$. Then $x=a\wedge t$ and $y=b\wedge s$, for some $s,t\in S$.

Therefore $a\wedge x=a\wedge a\wedge t=a\wedge t=x$ and $b\wedge y=b\wedge b\wedge s=b\wedge s=y$.

So that $a\wedge x=x$ and $b\wedge y=y$. Now,

$$\begin{aligned} h(x\vee y) &= (\gamma_a(x\vee y),\gamma_(x\vee y)) \\ &= (a\wedge(x\vee y), b\wedge(x\vee y)) \\ &= ((a\wedge x)\vee(a\wedge y),(b\wedge x)\vee(b\wedge y)) \\ &= (x\vee F, F\vee y) \qquad \text{(since } a\wedge y=b\wedge y=F) \\ &= (x,y). \end{aligned}$$

Therefore $(x,y)\in S_a\times S_b$. Hence h is onto.

Thus $S\cong S_a\times S_b$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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