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APPLICATIONS OF LIE GROUP ANALYSIS TO A CORE GROUP MODEL FOR ISENTROPIC SUPER-DENSE STARS

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Abstract. In this paper, Lie group analysis have been applied to a core group model for isentropic super-dense stars assuming that, the interior of the star is filled up with a perfect fluid, the corresponding Einstein's equations constitute an ordinary differential equation of 2nd order involving a parameter *K.* Different values of *K* are responsible for different models of the fluid sphere. Optimal solutions have been obtained in terms of special functions, namely the confluent hypergeometric functions.

Keywords: GRG, Lie group applications, cosmology

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1. INTRODUCTION

Relativistic static fluid spheres are used very often to represent the interior of neutron stars. Such fluid spheres, in addition to satisfying the reality conditions, are required to possess negative gradients of density and pressure. In addition, the velocity of sound through the model should be less than that of light, and the adiabatic index (γ) should be larger than unity for the temperature to be decreasing away from the center.

Vaidya P.C and Tikekar R (1982) utilized the space - time with hypersurfaces t equal constant as spheroids to describe the gravitational field inside the superdense stars [2]. Physically, to prescribe the metric potential g_{11}(and hence the energy density) in the Schwarzschild coordinates and mathematically the Einstein's equation reduces to an ordinary differential equation of 2nd order which involves a parameter K and R with g_{44} as dependent

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variable. Vaidya et al (1982) obtained a closed form solution of the said differential equation for *K= -2* and analyzed it physically. Knutson (1984) had analyzed the Vaidya's model with respect to validity and stability. Tikekar (1990) discussed the model corresponding to K= -7 and found the maximum admissible mass [3].

Maharaja Leach P.G.L (1996) presented a new class of algebraic solution for all the negative integral value of *K* [4]. Patrick Wills (1990) analyzed the model for K=2 and K=-7 with various physical aspects [9].

Jasim et al (2000, 2003) obtained the most exact general solution with some restricted conditions [6], [7].

In the present article the author has solved the second order differential equation of Vaidya – Tikekar (1982) problem in general form using similarity transformation method (STM) to obtained an optimal solution by using the very powerful technique; Similarity transformation method with the help of confluent hypergeometric functions and MATLAB V.6.

2. BASIC EQUATIONS

The Vaidya-Tikeker space – time metric [2, 3]

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + e^{\nu(r)}dt^2 \tag{1}$$

Assuming that the physical hypersurfaces t = constant in a relativistic super-dense star has the geometry of a 3 – spheroid.

We should have $e^{\lambda(r)} = \dfrac{1 - K\left(r^2/R^2\right)}{1 - \left(r^2/R^2\right)}$ and $e^{\nu(r)} = c^2 y^2$ to satisfy the hypergeometric equation.

The metric (1) is regular and positive definite at all the points $r < R$ for *K<1*. In case *K=1*, this leads to a flat space while for *K=0,* leads to a well-known Schwarzschild's interior solution.

For *K<1*, the perfect fluid distribution assumes the form

$$\left(1 - K + Kx^2\right) y'' - Kxy' + K(K-1)y = 0 \tag{2}$$

where $x^2 = 1 - \dfrac{r^2}{R^2}$

Now, there are two cases: For $K<0$ a transformation of a type $x=\sqrt{\frac{K-1}{K}}X$ sends the equation (2) into the following compact form:

$$\left(1-X^2\right)y''+Xy'+\left(m^2-1\right)y=0\ ,\ K=2-m^2,\ m>\sqrt{2} \tag{3}$$

While for case 0<K<1, a transformation $x=\sqrt{\frac{1-K}{K}}X$ transforms the equation (2) to the following form:

$$\left(1+X^2\right)y''+Xy'-\left(m^2-1\right)y=0\ ,\ K=2-m^2,\ 1<m<\sqrt{2} \tag{4}$$

Equation (1), together with the Einstein's field equations for perfect fluid distributions yields:

$$R_{ij}-\frac{1}{2}Rg_{ij}=-\frac{8\pi G}{c^4}\left[(c^2\rho+P)\upsilon_i\upsilon_j-Pg_{ij}\right] \tag{5}$$

The corresponding expressions for energy density ρ, pressure P, flow vector v^i can be furnished as follows:

$$\frac{8\pi G\rho}{c^2}=\frac{3(1-K)}{R^2}\left(1-\frac{Kr^2}{3R^2}\right)\left(1-\frac{Kr^2}{R^2}\right)^{-2} \tag{6}$$

$$\frac{8\pi GP}{c^4}=\left(1-\frac{r^2}{R^2}\right)\left(1-\frac{Kr^2}{R^2}\right)^{-1}\left(\frac{2y'}{ry}+\frac{1}{r^2}\right)-\frac{1}{r^2} \tag{7}$$

$$v^i=\left(0,00,y^{-1}\right) \tag{8}$$

where y satisfies the isotropy condition $\left(T_1^1=T_2^2=T_3^3\right)$

The expressions (6)-(8) with (3) are said to represent physically plausible model for a star provided that [8]:

- The matter density ρ and the fluid pressure P should be positive throughout the distribution.
- The gradients $\frac{d\rho}{dr}$ & $\frac{dP}{dr}$ should be negative with increasing radius.
- The weak energy (WEC) and strong energy (SEC) should be positive.
- The adiabatic sound speed should not exceed the speed of light as implication of causality fulfilment.

The energy density gradient can be obtained by differentiating (6) with respect to *r* as:

$$\frac{8\pi G}{c^2}\frac{d\rho}{dr} = \frac{2rK(1-K)}{R^2}\left(1-\frac{Kr^2}{3R^2}\right)^{-3}\left(5-\frac{Kr^2}{R^2}\right) \tag{9}$$

It is clear that the energy density gradient ceases to be negative as one goes inside the region $\left(r^2 < R^2 / K\right)$. Therefore, it is clear that $\left(\rho > 0,\ {d\rho}/{dr} < 0\right)$ to get and ensure that K is negative.

3. MATHEMATICAL FORMULATION [1]

We consider a one - parameter Lie group of transformation

$$x^* = x + \varepsilon X(x,y) + O\left(\varepsilon^2\right)$$

$$y^* = y + \varepsilon Y(x,y) + O\left(\varepsilon^2\right) \tag{10}$$

Then the extended transformation, will be

$$P^* = P + \zeta(x,y;p) + O\left(\varepsilon^2\right)$$

$$q^* = q + \xi(x,y;p,q) + O\left(\varepsilon^2\right) \tag{11}$$

Where

$$P = \frac{dy}{dx}, q = \frac{d^2y}{dx^2}, P^* = \frac{dy*}{dx*}, q^* = \frac{d^2y^*}{dx^{*2}} \tag{11a}$$

We derive ζ and ξ as follows:

$$\zeta = Yx + PY_y - PX_x - P^2X_y$$

$$\xi = Y_{xx} + 2PY_{yx} + P^2Y_{yy} - PX_{xx} - 2P^2X_{yx} - P^3X_{yy} - qX_x - qPX_y \tag{12}$$

The infinitesimal generator $D^{(1)}$ and $D^{(2)}$ are:

$$D^{(1)} = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p} \tag{13}$$

$$D^{(2)} = D^{(1)} + \xi\frac{\partial}{\partial q} \tag{14}$$

4. GROUP – INVARIANT SOLUTION

Equation (3) can be written in the primed form with the use of equations (9) and (10) for the primed variables and equating coefficients of $p^n q^m$ give the infinitesimal elements *(X,Y)* leaving equation (3) to be invariant.

Now, we find the determining equations for X and Y as follows:

$$q+\left[Y_{xx}++2PY_{yx}+p^2Y_{yy}-pX_{xx}-2p^2Y_{yx}-p^3X_{yy}-qX_x-qpX_y\right]\varepsilon+O\left(\varepsilon^2\right)$$
$$-qx^2-x^2Y_{xx}\varepsilon-2px^2Y_{yx}\varepsilon-p^2x^2Y_{yy}\varepsilon+x^2pX_{xx}\varepsilon+x^2p^3X_{yy}\varepsilon+2X^2p^2X_{yx}\varepsilon$$
$$+x^2qpX_y\varepsilon+x^2qX_x\varepsilon+O\left(\varepsilon^2\right)-2xqX\varepsilon+O\left(\varepsilon^2\right)+xp+xY_x\varepsilon-xpX_x\varepsilon-xp^2X_y\varepsilon$$
$$+O\left(\varepsilon^2\right)+pX\varepsilon+O\left(\varepsilon^2\right)+y+Y\varepsilon-ky-kY\varepsilon+O\left(\varepsilon^2\right)=0 \tag{15}$$

We find a primary set of determining equations as follows:

Monomials	Coefficient	
p	$2Y_{yx}-X_{xx}-2x^2Y_{yx}+x^2X_{xx}-xX_x-X=0$	(16)
P^2	$Y_{yy}-2X_{yx}-x^2Y_{yy}+2x^2X_{yx}-xX_y=0$	(17)
P^3	$-X_{yy}-x^2X_{yy}=0$	(18)
q	$-X_x+x^2X_x-2xX=0$	(19)
q p	$-X_y+x^2X_y=0$	(20)

and the terms for ε free of the derivatives in extended equation(15) have been found as:

$Y_{xx}-x^2Y_{xx}+xY_x+Y-kY=0$, which implies to

$$\left(1-x^2\right)Y_{xx}+xY_x+(1-k)Y=0 \tag{21}$$

Equation (20) $\Rightarrow\ \left(x^2-1\right)X_y=0\Rightarrow X_y=0$

$$X(x,y)=\alpha+H(x)$$
$$X_x=H'(x) \tag{22}$$

Substituting equation (22) in equation (19), yields

$$H(x)=\frac{x^2-1}{x^2-1}=1 \tag{23}$$

Since $X(x)=\frac{x^2-1}{2x}H'(x)$, substituting equation (23) in equation (19), we get

$$X(x,y)=zero$$

Now, rewriting equation (15) using the result obtained above, the second set of the determining equations has the following:

Monomials	Coefficient
p	$2Y_{yx}-2x^2Y_{yx}=0$ (24)
P^2	$Y_{yy}-x^2Y_{yy}$ (25)

From equation (25), we have

$$(1-x^2)Y_{yy}=0\Rightarrow Y_{yy}=0 \tag{26}$$

Equation (26) leads to $Y(x,y)=c_1y+R(x)$

at $c_1=0$, equation (25) gives

$Y(x,y)=R(x)$

i.e $Y(x,y)$ is a function of x only

The second order deferential equation (3) is invariant to the twice- extended group $x^*=x$

$y^*=y+Y(x)\varepsilon+O(\varepsilon^2)$

where

$$Y(x)=c_1(x^2-1)^{3/4}P1\left(\sqrt{2-k}-\frac{1}{2},\frac{3}{2};x\right)+c_2(x^2-1)^{3/4}P_2\left(\sqrt{2-k}-\frac{1}{2},\frac{3}{2};x\right)$$

This is of the type of confluent hypergeometric function which is well defined provided *c* is non-negative integer, and converges for $|x|<1$ and any solutions in other regions are obtainable by analytic continuation of these solutions [5, 10]

$$M(a,c,x)={}_1P_1(a,c,x)=\lim_{b\to\infty}{}_2P_1\left(a,b,c,\frac{x}{b}\right)=1+\frac{a}{1!c}x+\frac{a(a+1)}{c(c+1)2!}x^2+\ldots\ldots \tag{27}$$

Therefore we find

$P\left(\frac{3}{2},\frac{3}{2};x\right)$ when $k=-2$

$P\left(\frac{5}{2},\frac{3}{2};x\right)$ when $\mathrm{k}=-7$

$P\left(\frac{7}{2},\frac{3}{2};x\right)$ when $\mathrm{k}=-14$

Now, there exists a simple relation between the confluent hypergeometric with different parameters by ± 1.

From equation (27), we found k for fixed $c=\frac{3}{2}$, and different value of a, such that

$$(c-a)_1P_1(a,c;x)=(c-a)_1P_1(a-1,c;x)-a(1-x)_1P_1(a+1,c;x)$$

This yield to

$$-_1P_1\left(\frac{5}{2},\frac{3}{2};x\right)=-_1P_1\left(\frac{3}{2},\frac{3}{2};x\right)-\frac{5}{2}(1-x)_1P_1\left(\frac{7}{2},\frac{3}{2};x\right)$$

When $a=5/2$, $c=3/2$, $k=-7$

$$-_1P_1\left(\frac{3}{2},\frac{3}{2};x\right)={}_1P_1\left(\frac{5}{2},\frac{3}{2};x\right)-\frac{5}{2}(1-x)_1P_1\left(\frac{7}{2},\frac{3}{2};x\right)$$

In the similar manner, when $a=7/2$ and $c=3/2$, $k=-14$

This yield to

${}_1P_1\left(\frac{3}{2},\frac{3}{2};x\right)=e^x$ by using MATLAB program V.6.

5. LIE'S REDUCTION THEOREM AND ITS APPLICATION

Let the general form of the second order Ordinary Differential Equation is $w(x,y,\dot{y},\ddot{y})=0$, which is always written as a pair coupled first order ordinary differential equation as follows [4]

$$\dot{y}=u$$

$$w(x,y,u,\dot{u})=0 \tag{28}$$

Equation (28) determines a two – parameter family of curves in 3 – dimensional space, which is invariant to the once-extended group (X,Y,ζ); the transformations of the group carry each of these curves into other curves of the family.

Each one – parameter family of curves defines a surface in (x,y,u) - space and denoted by equation $\phi(x,y,u,c)=0$, which is invariant.

i.e $0=\phi\left(x^{*},y^{*},u^{*},c\right)=\phi(X,Y,U;c)$ (29)

and satisfies $X\phi_{x},+Y\phi_{y}+\zeta\phi_{u}=0$ (30)

The characteristics equations of which are

$$\frac{dx}{X(x,y)}=\frac{dy}{Y(x,y)}=\frac{du}{\zeta(x,y,\dot{y})} \tag{31}$$

If $P(x,y)$ and $q(x,y)$ are two integrals of equation (29) the general solution for ϕ is an arbitrary function G of P and q. The function $P(x,y)$ being an integral of the first pair of equation (29) is a group invariant, the function $q(x,y,u)=q(x,y,\dot{y})$ which is an invariant of the once extended group called a first differential invariant.

(If we adopt the invariant P and first differential invariant q as new variables, the second – order differential equation $w(x,y,\dot{y},\ddot{y})=0$ will reduce to a first order differential equation in P and q).

Now, if we write differential equation (3) in the form $u(x,y,\dot{y},\ddot{y})=0$ where

$$u(x,y,\dot{y},\ddot{y})=\ddot{y}+\frac{x}{1-x^{2}}\dot{y}+\frac{1-k}{1-x^{2}}y=0$$

Then u satisfies the condition $Xu_{x}+Yu_{y}+\zeta u_{\dot{y}}+\xi u_{\ddot{y}}=0$

The infinitesimal coefficients of the group are**:**

$$X(x,y)=0 \quad , \quad Y(x,y)=Y(x) \quad , \quad \zeta=\dot{y}(x) \quad \text{and} \quad \xi=\ddot{y}(x)$$

Direct substitution now shows that

$$Xu_{x}+Yu_{y}+\zeta u_{\dot{y}}+\xi u_{\ddot{y}}=\ddot{Y}+P(x)\dot{Y}+Q(x)Y=0 \quad \text{as required}$$

Now, by using the characteristic equation of which are

$$\frac{dx}{X(x,y)}=\frac{dy}{y(x)}=\frac{d\dot{y}}{\dot{Y}(x)}\Rightarrow\frac{dx}{0}=\frac{dy}{Y(x)}=\frac{d\dot{y}}{\dot{Y}(x)}$$

From the first equality, we get:

$\frac{2}{5}xe^{x}\left(x^{2}-1\right)^{3/4}+\frac{3}{5}e^{x}(-1)^{3/4}=a$ where a is a constant, x is hypergeometric

$\left\{\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2}\right],x^{2}\right\}$ and $Y(x)=\left(x^{2}-1\right)^{3/4}e^{x}$

The greater generality of solution in terms of hypergeometric functions outweighs tractability of the solution presented [4].

Substituting this value of x in the second and third term tells us that we can treat Y and $\dot{Y}$ as constants when integrating the second equality

$$\frac{dy}{Y(x)}=\frac{d\dot{y}}{\dot{Y}(x)}\Rightarrow\frac{dy}{c-\frac{3}{2x}e^{x}(-1)^{3/4}}=\frac{d\dot{y}}{c+\frac{3}{2}xe^{x}\left(x^{2}-1\right)^{-1/4}}$$

$$\int\left[c-\frac{3}{2x}e^{x}(-1)^{3/4}\right]d\dot{y}=\int\left[c+\frac{3}{2}xe^{x}\left(x^{2}-1\right)^{-1/4}\right]dy$$

$$\Rightarrow\left[c-\frac{3}{2x}e^{x}(-1)^{3/4}\right]\dot{y}=\left[c+\frac{3}{2}xe^{x}\left(x^{2}-1\right)^{-1/4}\right]y$$

$\Rightarrow Y(x)\dot{y}-y\dot{Y}=b$; b is a constant

Thus $\frac{2}{5}xe^{x}\left(x^{2}-1\right)^{3/4}+\frac{3}{5}e^{x}(-1)^{3/4}=\rho$ is an invariant (32)

and $\left(c-\frac{3}{2x}e^{x}(-1)^{3/4}\right)\dot{y}-\left(c+\frac{3}{2}xe^{x}\left(x^{2}-1\right)^{-1/4}\right)y=q$ (33)

Differentiating equation (33), we get

$$\frac{dq}{dx}=\ddot{y}Y-y\ddot{Y}=R(x)Y(x)-P(x)q$$

$$\frac{dq}{dx}=R(x)Y(x)-P(x)q$$

$$\Rightarrow q=D\sqrt{x^{2}-1} \tag{34}$$

Substitute (34) in (33), we get

$\dot{y}-\frac{\dot{Y}}{Y}y=\frac{D\sqrt{x^{2}-1}}{Y}$, which implies to

$$y=D\left[\frac{1}{2}e^{x+2}\left(x^{2}-1\right)^{3/4}E_{i}(1,2x+2)-\frac{1}{2}e^{x-2}\left(x^{2}-1\right)^{3/4}E_{i}(1,2x-2)\right]$$

Such that $E_i(x)$ is called (Cauchy-Newton function) and defined by $E_i(x)=\int_{-\infty}^{x}\frac{e^{t}}{t}dt$

6. CONCLUSIONS

The space time metrics with hypersurfaces t= constant as spheroid involve two parameters *K* and *R* is describe the interior of a star. The corresponding Einstein's equation gives rise an ordinary differential equation of 2nd order involving a parameter *K*. Lie's group method has been applied to get a very interesting optimal solutions to feed as some very remarkable models at K=-2, -7, -14 and so on depending on continuation of solutions for confluent hypergeometric series:

$$M(a,c,x) = {}_1P_1(a,c,x) = \lim_{b\to\infty} {}_2P_1\left(a,b,c,\frac{x}{b}\right) = 1 + \frac{a}{1!c}x + \frac{a(a+1)}{c(c+1)2!}x^2 + ...$$

Assuming different values of *c* has generated different values of *K* to represent different star models. The physical analyses are not investigated in this article.

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