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η-RICCI YAMABE SOLITON ON LP-SASAKIAN MANIFOLDS

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Abstract. The objective of the present paper is to carry out η-Ricci Yamabe soliton on LP-Sasakian manifolds satisfying certain curvature conditions $R(\zeta, U_1) \cdot S = 0$ and $S(\zeta, U_1) \cdot R = 0$. The above mentioned team has study gradient η-Ricci Yamabe soliton and cyclic Ricci tensor on LP-Sasakian manifolds admit with η-Ricci Yamabe solitons. Further we have study the conditions for the η-Ricci Yamabe solitons to be shrinking, expanding or steady.

Keywords: Ricci Yamabe soliton; η-Ricci Yamabe soliton; LP-Sasakian manifold.

2010 AMS Subject Classification: 53D10, 53D15.

1. Introduction

In 1989, K.Matsumato [13] introduced the notion of LP-Sasakian manifold and it has been studied by U.C.De and et.al.,[7], A.A.Shaikh [16], Y.B.Maralabhaviand et.al., [20].

In 1988, Hamilton [10] initiated the notion of Ricci flow and Yamabe flow concurrently. The solution to the Ricci flow and Yamabe flow are known as Ricci soliton and Yamabe soliton respectively. Currently, G*ü*ler and Crasmareanu [8] initiated the study of a new geometric flow

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which is a scalar combination of Ricci flow and Yamabe flow under the name Ricci Yamabe map. This is also known as Ricci Yamabe flow of the type (p,q). The Ricci Yamabe flow is an evolution for the metrics on the Riemannian or semi-Riemannian manifolds defined by [8]

$$\frac{\partial}{\partial t}g(t) = -2pRic(t) + qR(t)g(t), \qquad g_0 = g(0). \tag{1.1}$$

A soliton to the Ricci Yamabe flow is known as Ricci Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. In 2020, Shivaprasanna G.S. and et.al., studied Ricci-Yamabe solitons on submanifolds[15]. To be precise a Ricci Yamabe soliton on Riemannian manifold (M,g) is a data (g,V,λ,p,q) satisfying

$$(L_V g)(U_1,U_2) + 2pS(U_1,U_2) + (2\lambda - qr)g(U_1,U_2) = 0, \tag{1.2}$$

where r, S and L_V is the scalar curvature, the Ricci tensor and the Lie-derivative along the vector field on M respectively and λ is a constant. Moreover the Ricci Yamabe soliton is said to be expanding, shrinking or steady accordingly as λ is negative, positive or zero respectively. Equation (1.2) is known as Ricci Yamabe soliton of (p,q)-type, which is a generalisation of Ricci and Yamabe soliton. The Ricci Yamabe soliton is p-Ricci soliton if $q = 0$ and q-Yamabe soliton if $p = 0$.

An extension of Ricci soliton is the notion of η-Ricci soliton explained by J. T. Cho and M. Kimura [5],[18] in 2009. The concept of η-Ricci Yamabe soliton of type (p,q) defined by Mohd. Danish Siddiqi and Mehmet Akif Akyol [14]

$$(L_V g)(U_1,U_2) + 2pS(U_1,U_2) + (2\lambda - qr)g(U_1,U_2) + 2\mu\eta(U_1)\eta(U_2) = 0. \tag{1.3}$$

The η-Ricci Yamabe soliton of type $(p,0)$ or $(1,0)$-type are $p-\eta$-Ricci soliton or η-Ricci soliton and $(0,q)$ or $(0,1)$-type are $q-\eta$-Yamabe soliton or η-Yamabe soliton for these particular cases refer [1, 2, 3, 4, 6, 9, 17].

In this paper we study η-Ricci Yamabe soliton in LP-Sasakian manifold.

2. Preliminaries

An odd-dimensional, differentiable manifold (M) is known as LP-Sasakian manifold [7, 13] , if it admits a (1,1)-tensor field ϕ , a contravariant vector field ζ , a 1-form η and Lorentzian metric g which satisfy

$$\eta(\zeta) = -1, \qquad g(U_1,\zeta) = \eta(U_1), \qquad \nabla_{U_1}\zeta = \phi U_1, \tag{2.1}$$

$$\phi^2 U_1 = U_1 + \eta(U_1)\zeta, \tag{2.2}$$

$$g(\phi U_1, \phi U_2) = g(U_1,U_2) + \eta(U_1)\eta(U_2), \tag{2.3}$$

$$(\nabla_{U_1}\phi)(U_2) = g(U_1,U_2)\zeta + \eta(U_2)U_1 + 2\eta(U_1)\eta(U_2)\zeta, \tag{2.4}$$

where ∇ denotes the covariant differentiation with respect to Lorentzian metric g.

In an LP-Sasakian manifold the following relations holds good:

$$\phi\zeta = 0, \qquad \eta\phi = 0, \qquad \eta(\phi U_1) = 0, \tag{2.5}$$

$$g(\phi U_1, U_2) = g(U_1, \phi U_2), \tag{2.6}$$

$$\Phi(U_1,U_2) = g(U_1, \phi U_2). \tag{2.7}$$

For all vector fields U_1 and U_2, the tensor field $\Phi(U_1,U_2)$ is symmetric (0, 2)-tensor field. Since the 1-form η is closed in an LP-Sasakian manifold, we have [12, 13]

$$(\nabla_{U_1}\eta)(U_2) = \Phi(U_1,U_2), \qquad \Phi(U_1,\zeta) = 0, \tag{2.8}$$

for all vector fields U_1 and U_2. An LP-Sasakian manifold M is known as η-Einstein if its Ricci tensor S is of the form

$$S(U_1,U_2) = l_1 g(U_1,U_2) + l_2\eta(U_1)\eta(U_2), \tag{2.9}$$

where l_1 and l_2 are functions on M.

Also the following relations hold in an LP-Sasakian manifold

$$R(U_1,U_2)U_3 = g(U_2,U_3)U_1 - g(U_1,U_3)U_2, \tag{2.10}$$

$$R(\zeta,U_1)U_2 = g(U_1,U_2)\zeta - \eta(U_2)U_1, \tag{2.11}$$

$$S(U_1,\zeta) = (n-1)\eta(U_1), \tag{2.12}$$

$$S(\phi U_1,\phi U_2) = S(U_1,U_2) + (n-1)\eta(U_1)\eta(U_2), \tag{2.13}$$

for any vector field U_1, U_2 and U_3.

In view of (1.3),(2.1), (2.6) and (2.7), we yield

$$S(U_1,U_2) = -\left(\tfrac{1}{p}\right)g(\phi U_1,U_2)-\left(\tfrac{2\lambda-qr}{2p}\right)g(U_1,U_2) \tag{2.14}$$

$$-\left(\tfrac{\mu}{p}\right)\eta(U_1)\eta(U_2), \tag{2.15}$$

$$QU_1 = -\left(\tfrac{1}{p}\right)\phi U_1-\left(\tfrac{2\lambda-qr}{2p}\right)U_1-\left(\tfrac{\mu}{p}\right)\eta(U_1)\zeta, \tag{2.16}$$

$$S(U_1,\zeta) = \left(\tfrac{qr-2\lambda-2\mu}{2p}\right)\eta(U_1). \tag{2.17}$$

Collating (2.12) and (2.17), we obtain

$$\frac{qr-2\lambda}{2p}-\frac{\mu}{p}=n-1. \tag{2.18}$$

3. Gradient η-Ricci Yamabe Soliton on LP-Sasakian Manifold

Definition 3.1. *A Riemannian metric g on M is known as gradient η-Ricci Yamabe soliton if* [11]

$$\nabla\nabla f+S+\lambda g=0. \tag{3.1}$$

Let M be a LP-Sasakian manifold with g as a gradient η-Ricci Yamabe soliton. Then from equation (3.1), we have

$$\nabla_{U_1}Df+QU_1+\lambda U_1=0, \tag{3.2}$$

for any $U_1\in TM$, where D stands for the gradient operator with respect to g. From (3.2) it follows that

$$R(U_1,U_2)Df=(\nabla_{U_2}Q)U_1-(\nabla_{U_1}Q)U_2. \tag{3.3}$$

From (2.10), we yield

$$g(R(\zeta,U_1)Df,\zeta)=\eta(U_2)(\zeta f)-U_1 f. \tag{3.4}$$

Using (2.16), we obtain

$$\begin{aligned}(\nabla_{U_1}Q)U_2 &= -\left(\frac{1}{p}\right)[g(U_1,U_2)\zeta+2\eta(U_1)\eta(U_2)\zeta+\phi\nabla_{U_1}U_2]\\ &\quad+\left(\frac{q}{2p}\right)(U_1 r)-\left(\frac{\mu}{p}\right)[(\phi U_1)\zeta+\eta(U_2)\phi U_1].\end{aligned} \tag{3.5}$$

Interchanging U_1 and U_2, we write

$$(\nabla_{U_2}Q)U_1 = -\left(\frac{1}{p}\right)[g(U_1,U_2)\zeta + 2\eta(U_1)\eta(U_2)\zeta + \phi\nabla_{U_2}U_1] + \left(\frac{q}{2p}\right)(U_2 r) - \left(\frac{\mu}{p}\right)[(\phi U_2)\zeta + \eta(U_1)\phi U_2]. \tag{3.6}$$

In view of (3.5) and (3.6), we find that

$$(\nabla_{U_2}Q)U_1 - (\nabla_{U_1}Q)U_2 = -\left(\tfrac{1}{p}\right)[\phi\nabla_{U_2}U_1 - \phi\nabla_{U_1}U_2] + \left(\tfrac{q}{2p}\right)[(U_2 r) - (U_1 r)] - \left(\tfrac{\mu}{p}\right)[(\phi U_2)\zeta + \eta(U_1)\phi U_2 - (\phi U_1)\zeta - \eta(U_2)\phi U_1]. \tag{3.7}$$

Substituting $U_1 = \zeta$ in the above equation, we get

$$(\nabla_{U_2}Q)\zeta - (\nabla_\zeta Q)U_2 = -\left(\frac{1}{p}\right)[\phi\nabla_{U_2}\zeta - \phi\nabla_\zeta U_2] + \left(\frac{q}{2p}\right)[(U_2 r) - (\zeta r)] - \left(\frac{\mu}{p}\right)[(\phi U_2)\zeta - \phi U_2]. \tag{3.8}$$

Taking inner product with ζ, one has

$$g((\nabla_{U_2}Q)\zeta - (\nabla_\zeta Q)U_2, \zeta) = 0, \tag{3.9}$$

for any $U_2 \in TM$.

By virtue of (3.4) and (3.9), we find that

$$U_1 f = \eta(U_1)(\zeta f), \tag{3.10}$$

for any $U_1 \in TM$. Therefore $Df = (\zeta f)\zeta$.

Applying the covariant derivative with respect to U_1 and using (3.2) it follows that

$$S(U_1,U_2) = -(\zeta f)g(\phi U_1,U_2) - \lambda g(U_1,U_2) - U_1(\zeta f)\eta(U_2), \tag{3.11}$$

for any $U_1, U_2 \in TM$.

Theorem 3.1. *A gradient η-Ricci Yamabe soliton with potential vector field of gradient type, $V = Df$ satisfying $L_\zeta f = 0$ on a LP-Sasakian manifold is $S(U_1,U_2) = -(\zeta f)g(\phi U_1,U_2) - \lambda g(U_1,U_2) - U_1(\zeta f)\eta(U_2)$.*

Corollary 3.1. *If (3.2) defines a gradient η-Ricci Yamabe soliton on the LP-Sasakian manifold with $L_\zeta f = 0$, then M has constant scalar curvature and g is an η-Ricci soliton defined for any killing vector field V, which can be either expanding or shrinking accordingly as ζ is timelike or spacelike.*

In view of (1.3) and (2.12), we get

$$(L_V g)(U_1,U_2) = [-2p(n-1) - 2\lambda + qr]g(U_1,U_2) - 2\mu\eta(U_1)\eta(U_2) = 0. \tag{3.12}$$

Applying the covariant derivative with respect to U_3, we have

$$(\nabla_{U_3} L_V g)(U_1,U_2) = q(\nabla_{U_3} r)g(U_1,U_2) - 2\mu[(\nabla_{U_3}\eta)U_1\eta(U_2) + (\eta(U_1)\nabla_{U_3}\eta)U_2]. \tag{3.13}$$

We have from [19]

$$\begin{aligned}&(L_V\nabla_{U_1} g - \nabla_{U_1} L_V g - \nabla_{[V,U_1]} g)(U_2,U_3)\\ &= -g((L_V\nabla)(U_1,U_2),U_3) - g((L_V\nabla)(U_1,U_3),U_2),\end{aligned} \tag{3.14}$$

for any $U_1,U_2,U_3 \in TM$. Since g is parallel with respect to the Levi-Civita connection ∇, then (3.14) implies

$$(\nabla_{U_1} L_V g)(U_2,U_3) = g((L_V\nabla)(U_1,U_2),U_3) + g((L_V\nabla)(U_1,U_3),U_2). \tag{3.15}$$

As $L_V\nabla$ is a symmetric tensor of type $(1,2)$, i.e, $(L_V\nabla)(U_1,U_2) = (L_V\nabla)(U_2,U_1)$, then it follows from (3.15) that

$$\begin{aligned} g((L_V\nabla)(U_1,U_2),U_3) &= \tfrac{1}{2}[(\nabla_{U_1} L_V g)(U_2,U_3) + (\nabla_{U_2} L_V g)(U_1,U_3)\\ &\quad +(\nabla_{U_3} L_V g)(U_1,U_2)].\end{aligned} \tag{3.16}$$

Using (3.13) in (3.15), one has

$$\begin{aligned} 2g((L_V\nabla)(U_1,U_2),U_3) &= q(\nabla_{U_1} r)g(U_2,U_3) - 2\mu[g(U_1,\phi U_2)\eta(U_3)\\ &\quad +g(U_1,\phi U_3)\eta(U_2)] + q(\nabla_{U_2} r)g(U_1,U_3)\\ &\quad -2\mu[g(U_2,\phi U_1)\eta(U_3) + g(U_2,\phi U_3)\eta(U_1)]\\ &\quad +q(\nabla_{U_3} r)g(U_1,U_2) - 2\mu[g(U_3,\phi U_1)\eta(U_2)\\ &\quad +g(U_3,\phi U_2)\eta(U_1)].\end{aligned} \tag{3.17}$$

Remove U_3 from the above equation we yield

$$\begin{aligned} 2g((L_V\nabla)(U_1,U_2)) &= q[(\nabla_{U_1}r)U_2+(\nabla_{U_2}r)U_1+(Dr)g(U_1,U_2)] \\ &\quad -4\mu[g(U_1,\phi U_2)\zeta+\phi U_1\eta(U_2)+\phi U_2\eta(U_1)], \end{aligned} \tag{3.18}$$

where $(U_1\alpha)=g(D\alpha,U_1)$, for D the gradient operator with respect to g. Adopting $U_2=\zeta$ in the foregoing equation and using $r=constant$ (hence, $(Dr)=0$ and $(\zeta r)=0$), we yield

$$(L_V\nabla)(U_1,\zeta)=0. \tag{3.19}$$

Adopting the covariant derivative of (3.19) with respect to U_2, we write

$$(\nabla_{U_2}L_V\nabla)(U_1,\zeta)=-(L_V\nabla)(U_1,U_2). \tag{3.20}$$

Again from [19]

$$(L_VR)(U_1,U_2,\zeta)=0, \tag{3.21}$$

for any $U_1,U_2\in TM$. Substituting $U_2=\zeta$ in (3.12) which follows that

$$(L_Vg)(U_1,\zeta)=[-2p(n-1)-2\lambda+qr+2\mu]\eta(U_1). \tag{3.22}$$

Lie-differentiating the equation (2.1) along V and by virtue of (3.22), we have

$$(L_V\eta)(U_1)-g(L_V\zeta,U_1)-[-2p(n-1)-2\lambda+qr+2\mu]\eta(U_1)=0. \tag{3.23}$$

Assuming $U_1=\zeta$ in the above equation, we get

$$\eta(L_V\zeta)=2p(n-1)+2\lambda-qr-2\mu. \tag{3.24}$$

Theorem 3.2. *If LP-Sasakian manifold with scalar curvature admits η-Ricci Yamabe soliton with potential vector field V, then $L_V\zeta$ is g-orthogonal to ζ provided $\lambda=\frac{qr}{2}-[p(n-1)-\mu]$.*

Corollary 3.2. *If LP-Sasakian manifold admits η-Ricci Yamabe soliton with potential vector field V, $L_V\zeta$ is g-orthogonal to ζ. Then η-Ricci Yamabe soliton is shrinking if $\frac{qr}{2}>[p(n-1)-\mu]$, expanding if $\frac{qr}{2}<[p(n-1)-\mu]$ or steady if $\frac{qr}{2}=[p(n-1)-\mu]$.*

If $q = 0$ then we have $\lambda = \mu - p(n-1)$.

Thus we can state the result as

Corollary 3.3. *If LP-Sasakian manifold admits η-Ricci Yamabe soliton with potential vector field V, $L_V\zeta$ is g-orthogonal to ζ. then $p-\eta$-Ricci soliton is shrinking if $\mu > p(n-1)$, expanding if $\mu < p(n-1)$ or steady if $\mu = p(n-1)$.*

If $p = 0$ then we have $\lambda = \frac{qr}{2} + \mu$.

Thus we can state the result as

Corollary 3.4. *If LP-Sasakian manifold admits η-Ricci Yamabe soliton with potential vector field V, $L_V\zeta$ is g-orthogonal to ζ. then $q-\eta$-Yamabe soliton is shrinking if $\frac{qr}{2} + \mu > 0$, expanding if $\frac{qr}{2} + \mu < 0$ or steady if $\mu = -\frac{qr}{2}$.*

4. An η-Ricci Yamabe Soliton on LP-Sasakian manifold Satisfying $R(\zeta, U_1) \cdot S = 0$

Let M be a LP-Sasakian manifold satisfies the condition $R(\zeta, U_1) \cdot S = 0$ and we can write it as

$$S(R(\zeta, U_1)U_2, V_1) + S(U_2, R(\zeta, U_1)V_1) = 0. \tag{4.1}$$

By virtue of (2.17) and (4.1), we write

$$\begin{aligned} \left(\frac{qr - 2\lambda - 2\mu}{2p}\right)[\eta(U_2)g(U_1, V_1) \quad &+ \eta(V_1)g(U_1, U_2)] - [\eta(U_2)S(U_1, V_1) \\ &+ \eta(V_1)S(U_1, U_2)] = 0. \end{aligned} \tag{4.2}$$

Assume $V_1 = \zeta$ in (4.2) to have

$$S(U_1, U_2) = \left(\frac{qr - 2\lambda - 2\mu}{2p}\right) g(U_1, U_2). \tag{4.3}$$

In view of (2.15), equation (4.3) implies

$$-\left(\frac{1}{p}\right) g(U_1, \phi U_2) + \left(\frac{\mu}{p}\right)[g(U_1, U_2) - \eta(U_1)\eta(U_2)] = 0. \tag{4.4}$$

Replace U_2 by ϕU_2 and using (2.2), we obtain

$$\left(\frac{\mu}{p}\right) g(U_1, \phi U_2) - \left(\frac{1}{p}\right)[g(U_1, U_2) - \eta(U_1)\eta(U_2)] = 0. \tag{4.5}$$

On adding (4.4) and (4.5), one has

$$\left(\frac{\mu}{p}-\frac{1}{p}\right)[g(U_1,\phi U_2)-g(U_1,U_2)]-\left(\frac{\mu}{p}+\frac{1}{p}\right)\eta(U_1)\eta(U_2)=0. \tag{4.6}$$

Taking $U_1=\phi U_1$, we obtain

$$\left(\frac{\mu}{p}-\frac{1}{p}\right)[g(U_1,\phi U_2)+g(U_1,U_2)+\eta(U_1)\eta(U_2)]=0, \tag{4.7}$$

which implies $\mu=1$ and from (2.18) $\lambda=\frac{qr}{2}-np$.

Theorem 4.3. *Let η-Ricci Yamabe soliton on LP-Sasakian manifold (μ,ϕ,ζ,g) satisfying $R(\zeta,U_1)\cdot S=0$, then $\mu=1$ and $\lambda=\frac{qr}{2}-np$.*

In view of (2.15) and (4.4), we yield

$$S(U_1,U_2)=\left(\frac{qr-2\lambda}{2p}-\frac{\mu}{p}\right)g(U_1,U_2).$$

Thus we can state the result as

Theorem 4.4. *Let η-Ricci Yamabe soliton on LP-Sasakian manifold (μ,ϕ,ζ,g) satisfying $R(\zeta,U_1)\cdot S=0$, then (M,g) is Einstein manifold.*

5. η-Ricci Yamabe Soliton on LP-Sasakian manifold Satisfying $S(\zeta,U_1)\cdot R=0$

Let M be a LP-Sasakian manifold satisfies the condition $S(\zeta,U_1)\cdot R=0$ and it follows that

$$\begin{aligned}&S(U_1,R(U_2,U_3)V_1)\zeta-S(\zeta,R(U_2,U_3)V_1)U_1+S(U_1,U_2)R(\zeta,U_3)V_1\\&-S(\zeta,U_2)R(U_1,U_3)V_1+S(U_1,U_3)R(U_2,\zeta)V_1-S(\zeta,U_3)R(U_2,U_1)V_1\\&+S(U_1,V_1)R(U_2,U_3)\zeta-S(\zeta,V_1)R(U_2,U_3)U_1=0.\end{aligned} \tag{5.1}$$

Taking inner product with ζ, we write

$$\begin{aligned}&-S(U_1,R(U_2,U_3)V_1)-S(\zeta,R(U_2,U_3)V_1)\eta(U_1)+S(U_1,U_2)\eta(R(\zeta,U_3)V_1)\\&-S(\zeta,U_2)\eta(R(U_1,U_3)V_1)+S(U_1,U_3)\eta(R(U_2,\zeta)V_1)-S(\zeta,U_3)\eta(R(U_2,U_1)V_1)\\&+S(U_1,V_1)\eta(R(U_2,U_3)\zeta)-S(\zeta,V_1)\eta(R(U_2,U_3)U_1)=0.\end{aligned} \tag{5.2}$$

Assuming $V_1 = \zeta = U_3$ in (5.2), which implies

$$-S(U_1,R(U_2,\zeta)\zeta)-S(\zeta,R(U_2,\zeta)\zeta)\eta(U_1)+S(U_1,U_2)\eta(R(\zeta,\zeta)\zeta)$$

$$-S(\zeta,U_2)\eta(R(U_1,\zeta)\zeta)+S(U_1,\zeta)\eta(R(U_2,\zeta)\zeta)-S(\zeta,\zeta)\eta(R(U_2,U_1)\zeta)$$

$$+S(U_1,\zeta)\eta(R(U_2,\zeta)\zeta)-S(\zeta,\zeta)\eta(R(U_2,\zeta)U_1)=0. \tag{5.3}$$

By virtue of (2.10), (2.15) and (5.3), we find that

$$S(U_1,U_2)=\left(\frac{2\lambda+2\mu-qr}{2p}\right)[g(U_1,U_2)+2\eta(U_1)\eta(U_2)]. \tag{5.4}$$

Thus we can state the result as

Theorem 5.5. *If η-Ricci Yamabe soliton on LP-Sasakian manifold (μ,ϕ,ζ,g) satisfying $S(\zeta,U_1)\cdot R=0$, then the manifold is η-Einstein.*

Contracting (5.4) with (2.9), we obtain

$$\lambda = pl_1+\frac{qr}{2}-\mu. \tag{5.5}$$

Thus we can state the result as

Theorem 5.6. *If LP-Sasakian manifold (μ,ϕ,ζ,g) satisfying $S(\zeta,U_1)\cdot R=0$, then the η-Ricci Yamabe soliton is shrinking, expanding or steady accordingly as $pl_1+\frac{qr}{2}>\mu$, $pl_1+\frac{qr}{2}<\mu$ or $pl_1+\frac{qr}{2}=\mu$.*

If $q=0$ then from (5.5), one has

$$\lambda = pl_1-\mu.$$

Corollary 5.5. *If LP-Sasakian manifold (μ,ϕ,ζ,g) satisfying $S(\zeta,U_1)\cdot R=0$, then the $p-\eta$-Ricci soliton is shrinking, expanding or steady accordingly as $pl_1>\mu$, $pl_1<\mu$ or $pl_1=\mu$.*

If $p=0$ then from (5.5), we yield

$$\lambda=\frac{qr}{2}-\mu.$$

Corollary 5.6. *If LP-Sasakian manifold (μ,ϕ,ζ,g) satisfying $S(\zeta,U_1)\cdot R=0$, then the $q-\eta$-Yamabe soliton is shrinking, expanding or steady accordingly as $\frac{qr}{2}>\mu$, $\frac{qr}{2}<\mu$ or $\frac{qr}{2}=\mu$.*

6. η-Ricci Yamabe Solitons on LP-Sasakian manifold with Cyclic Ricci Tensor

Definition 6.2. *The Ricci tensor S is said to be the cyclic Ricci tensor if M satisfies the following* [1]

$$(\nabla_{U_1}S)(U_2,U_3)+(\nabla_{U_2}S)(U_3,U_1)+(\nabla_{U_3}S)(U_1,U_2)=0, \tag{6.1}$$

for any $U_1,U_2,U_3 \in TM$.

Definition 6.3. *The Ricci tensor S is said to be the cyclic* η*-recurrent Ricci tensor if M satisfies the following* [1]

$$\begin{aligned}&(\nabla_{U_1}S)(U_2,U_3)+(\nabla_{U_2}S)(U_3,U_1)+(\nabla_{U_3}S)(U_1,U_2)\\ &=\eta(U_1)S(U_2,U_3)+(\eta(U_2)S(U_3,U_1)+(\eta(U_3)S(U_1,U_2),\end{aligned} \tag{6.2}$$

for any $U_1,U_2,U_3 \in TM$.

Using (2.15) in $(\nabla_{U_1}S)(U_2,U_3)=\nabla_{U_1}(S(U_2,U_3)-S(\nabla_{U_1}U_2,U_3)-S(U_2,\nabla_{U_1}U_3)$, we get

$$(\nabla_{U_1}S)(U_2,U_3)=-\frac{1}{p}g((\nabla_{U_1}\phi)U_2,U_3)-\frac{\mu}{p}[\eta(U_2)(\nabla_{U_1}\eta)U_3+\eta(U_3)(\nabla_{U_1}\eta)U_2]. \tag{6.3}$$

In view of (2.4) and (2.8), (6.3) becomes

$$\begin{aligned}(\nabla_{U_1}S)(U_2,U_3)=-\tfrac{\mu}{p}[\eta(U_2)\Phi(U_1,U_3)+\eta(U_3)\Phi(U_1,U_2)]\\ -\tfrac{1}{p}[\eta(U_2)g(U_1,U_3)+\eta(U_3)g(U_1,U_2)]-\tfrac{2}{p}\eta(U_1)\eta(U_2)\eta(U_3).\end{aligned} \tag{6.4}$$

Then

$$\begin{aligned}&(\nabla_{U_1}S)(U_2,U_3)+(\nabla_{U_2}S)(U_3,U_1)+(\nabla_{U_3}S)(U_1,U_2)=\\ &-2[\frac{\mu}{p}[\eta(U_1)\Phi(U_2,U_3)+\eta(U_2)\Phi(U_1,U_3)+\eta(U_3)\Phi(U_1,U_2)]\\ &+\frac{1}{p}[\eta(U_1)g(U_2,U_3)+\eta(U_2)g(U_1,U_3)+\eta(U_3)g(U_1,U_2)]\\ &+\frac{3}{p}\eta(U_1)\eta(U_2)\eta(U_3)].\end{aligned} \tag{6.5}$$

Taking $U_3 = \zeta$ and in view of (2.1), (2.8) and (6.1), equation (6.5) implies

$$\frac{1}{p}[\mu\Phi(U_1,U_2)+g(\phi U_1,\phi U_2)] = 0. \tag{6.6}$$

Use $U_2 = \phi U_2$ in (6.6), one has

$$\frac{1}{p}[\mu g(\phi U_1,\phi U_2)+\Phi(U_1,U_2)] = 0. \tag{6.7}$$

On adding (6.6) and (6.7), we obtain

$$\left(\frac{\mu+1}{p}\right)[g(\phi U_1,\phi U_2)+\Phi(U_1,U_2)] = 0,$$

for any $U_1,U_2 \in TM$ and follows $\mu = -1$. From the relation (2.18) we have $\lambda = \frac{qr}{2}+n$.

Theorem 6.7. *Let (ϕ,ζ,η,g) be a LP-Sasakian structure on the manifold M admits η-Ricci Yamabe soliton with cyclic Ricci tensor, then $\mu = -1$ and $\lambda = \frac{qr}{2}+n$.*

If $q = 0$ then we have $\lambda = n$.

Thus we can state the result as

Corollary 6.7. *If LP-Sasakian manifold admits η-Ricci Yamabe soliton with cyclic Ricci tensor, then $p-\eta$-Ricci soliton is shrinking if $n>0$, expanding if $n<0$ or steady if $n=0$.*

By virtue of (2.15), (6.5) and (6.2), we yield

$$\begin{aligned}
&\left(\frac{1-2\mu}{p}\right)\eta(U_1)\Phi(U_2,U_3)+\eta(U_2)\Phi(U_1,U_3)+\eta(U_3)\Phi(U_1,U_2)]\\
&+\left(\frac{2\lambda-qr-4}{2p}\right)[\eta(U_1)g(U_2,U_3)+\eta(U_2)g(U_1,U_3)+\eta(U_3)g(U_1,U_2)]\\
&+\left(\frac{3\mu-6}{p}\right)\eta(U_1)\eta(U_2)\eta(U_3) = 0.
\end{aligned} \tag{6.8}$$

Putting $U_2 = \zeta$, $U_3 = \zeta$, we obtain

$$\left(\frac{2\mu-2\lambda+qr}{2p}\right)\eta(U_1) = 0, \tag{6.9}$$

for any $U_1 \in TM$ and follows $2\mu-2\lambda+qr = 0$. From the relation (2.18) we have $\lambda = \frac{qr-p(n-1)}{2}$ and $\mu = -\frac{p(n-1)}{2}$.

Theorem 6.8. *Let (ϕ, ζ, η, g) be a LP-Sasakian structure on the manifold M admits η-Ricci Yamabe soliton with cyclic η-recurrent Ricci tensor, then $\lambda = \frac{qr-p(n-1)}{2}$ and $\mu = -\frac{p(n-1)}{2}$.*

If $q = 0$ then we have $\lambda = -\frac{p(n-1)}{2}$.
Thus we can state the result as

Corollary 6.8. *If LP-Sasakian manifold admits η-Ricci Yamabe soliton with cyclic η-recurrent Ricci tensor, then $p-\eta$-Ricci soliton is shrinking if $p(n-1) < 0$, expanding if $p(n-1) > 0$ or steady if $p = 0$ or $n = 1$.*

If $p = 0$ then we have $\lambda = \frac{qr}{2}$.
Thus we can state the result as

Corollary 6.9. *If LP-Sasakian manifold admits η-Ricci Yamabe soliton with cyclic η-recurrent Ricci tensor, then $q-\eta$-Yamabe soliton is shrinking if $qr > 0$, expanding if $qr < 0$ or steady if $q = 0$ or $r = 0$.*

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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