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CONCOMITANTS OF ORDER STATISTICS FOR BIVARIATE EXPONENTIATED INVERTED WEIBULL DISTRIBUTION

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Abstract: In this paper, bivariate exponentiated inverted Weibull distribution (BEIWD) has been proposed using the Morgenstern approach. We provide the *pdf* and *cdf* of exponentiated inverted Weibull distribution (EIWD) along with the order statistics and its important properties. We derive the Morgenstern type bivariate exponentiated inverted Weibull distribution (BEIWD) and discussed some of its properties. We obtain the concomitants of order statistics arising from BEIWD with its several distributional properties. The survival and hazard functions of concomitants of its order statistics are illustrated with graphical presentation and numerical computations for different sets of values of parameters.

Keywords: bivariate exponentiated inverted Weibull distribution; Morgenstern approach; concomitants of order statistics; survival and hazard functions.

2010 AMS Subject Classification: 62E10.

1. INTRODUCTION

The exponentiated inverted Weibull distribution (EIWD) is a life time probability distribution for

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modeling reliability data which is a generalization of inverted Weibull distribution (IWD). Flaih, Elsalloukh, Mendi and Milanova [1] proposed this distribution by raising the cumulative density function (*cdf*) of IWD to a non-negative parameter (say $\theta,\ \theta \in \Re$) by exponentiation. The exponentiated inverted Weibull distribution (EIWD) has been studied by many authors, for example; Aljuaid [3] estimated the parameters of EIWD under type-II censoring. Elbatal and Muhammed [9] proposed exponentiated generalized inverse Weibull (EGIW) distribution. Hassan, Marwa, Zaher and Elsherpieny [18] studied estimation of stress-strength reliability for EIWD based on lower record values. Ahmad, Ahmad and Ahmad [2] obtained bayesian estimators of the shape parameter of the EIWD. Saghir, Tazeem and Ahmad [4] innovated a new three parameter weighted exponentiated inverted Weibull distribution (WEIWD).

A bivariate distribution $F(x, y)$ for a pair of random variables (X, Y) expresses the dependence between X and Y in its functional form and parameters. The Morgenstern family of distribution provides a general technique by which a bivariate distribution can be constructed directly using its marginal distributions and the correlation between the variates, (Morgenstern [5]). A generalization of Morgenstern method was proposed by Farlie [6] which is known as Farlie-Gumbel-Morgenstern (FGM) family of distributions. The properties and extensions of Morgenstern family of distributions have been studied in details by Kotz Balakrishnan and Johnson [19]. Aleem [14] introduced a bivariate inverted Weibull distribution using the Farlie–Gumbel–Morgenstern idea. In this paper, an attempt has been taken to introduce bivariate EIW distribution using this system of distribution. The system is specified by the cumulative distribution function form $F_{X,Y}(x, y)$ of two dimensional continuous random variables (X, Y) with marginals $F_X(x)$ and $F_Y(y)$ which is,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)[1 + \rho\{1 - F_X(x)\}\{1 - F_Y(y)\}] \qquad -1 \le \rho \le 1 \qquad (1)$$

where $F_X(x)$ and $F_Y(y)$ denote the marginal *cdf* s and ρ denote the dependence parameter that indicate the degree of association between X and Y . It can be seen that when $\rho = 0$, X and Y are independent.

The corresponding probability density function is,

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)[1+\rho\{1-2F_X(x)\}\{1-2F_Y(y)\}] \qquad -1\le\rho\le 1 \quad (2)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal *pdf* s of X and Y.

Concomitant variables play a significant role to study order statistics for bivariate set up of distribution when the associated characteristic is not available or difficult to measure. It was first introduced by David [7] and also mentioned as the induced order statistic by Bhattacharya [17]. Concomitants of order statistics by using the concept of Morgenstern approach have been extensively used by several authors. Balasubramanian and Beg [11] studied concomitants of order statistics in Morgenstern type bivariate exponential distribution. Balasubramanian and Beg [12] studied concomitants of order statistics of Gumbel's bivariate exponential distribution. Chacko and Thomas [15] studied estimation of a parameter of Morgenstern type bivariate exponential distribution using ranked set sampling. Beg and Ahsanullah [16] studied concomitants of generalized order statistics from Farlie- Gumbel- Morgenstern distributions. Tahmasebi and Jafari [20] have studied concomitants of order statistics and record values from Morgenstern type bivariate generalized exponential distribution. Alawady, Barakat, Xiong and Elgawad [13] studied concomitants of generalized order statistics under the generalization of Farlie-Gumbel-Morgenstern type bivariate distributions. Following the increasing attention on concomitants, the present article deals with the concomitants of order statistics for Morgenstern type bivariate EIW distribution.

2. EXPONENTIATED INVERTED WEIBULL DISTRIBUTION (EIWD)

If X has EIWD with shape parameters β and θ and scale parameter λ then the distribution function is given by,

$$F(x;\beta,\lambda,\theta)=\left(e^{-(\lambda x)^{-\beta}}\right)^{\theta}=e^{-\theta\lambda^{-\beta}x^{-\beta}}; \qquad x,\beta,\theta,\lambda>0 \qquad (3)$$

The *pdf* is given by,

$$f(x;\beta,\lambda,\theta)=\frac{\theta\beta}{\lambda^{\beta}}x^{-(\beta+1)}\left(e^{-(\lambda x)^{-\beta}}\right)^{\theta}; \qquad x,\beta,\theta,\lambda>0 \qquad (4)$$

The k^{th} moments of the exponentiated inverted Weibull distribution is given as follows:

$$E(X^k) = \int_0^\infty x^k \frac{\theta\beta}{\lambda^\beta} x^{-(\beta+1)} \left(e^{-(\lambda x)^{-\beta}}\right)^\theta dx$$

$$= \frac{{\theta_2}^{k/\beta_2}\Gamma\left(1-\frac{k}{\beta_2}\right)}{{\lambda_2}^k} \qquad (5)$$

Hence, the mean and variance of the EIWD is given by

$$E(X) = \mu = \frac{{\theta_2}^{1/\beta_2}\Gamma\left(1-\frac{1}{\beta_2}\right)}{\lambda_2} \qquad (6)$$

And,

$$Var(X) = \frac{{\theta_2}^{2/\beta_2}}{{\lambda_2}^2}\left[\Gamma\left(1-\frac{2}{\beta_2}\right) - \left\{\Gamma\left(1-\frac{1}{\beta_2}\right)\right\}^2\right] \qquad (7)$$

2.1 Order Statistics of EIWD

Let us assume that $X_1, X_2, \ldots X_n$ is a random sample from an absolutely continuous population with probability density function *(pdf)* $f(x)$ and cumulative distribution function *(cdf)* $F(x)$. Let, $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics obtained by arranging the preceding random sample in increasing order of magnitude, then the density function of $X_{r:n}; (1 \le r \le n)$ be,

$$f_{r:n}(x) = \frac{1}{\beta(r, n-r+1)}[F_X(x)]^{r-1}[1-F_X(x)]^{n-r} f_X(x) \qquad (8)$$

Hence, using the *cdf* and *pdf* of EIWD as given by equations (3) and (4), we obtain the *pdf* of the r^{th} order statistic $X_{r:n}$; $(1 \le r \le n)$ as,

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}\left[e^{-\theta(\lambda x)^{-\beta}}\right]^r \left[1-e^{-\theta(\lambda x)^{-\beta}}\right]^{n-r} \frac{\beta\theta}{\lambda^\beta} x^{-(\beta+1)} \qquad (9)$$

putting $r = 1$ and $r = n$ in (9) we have the *pdf* s of the smallest and largest order statistics of EIWD as,

$$f_{1:n}(x) = n\left[1-e^{-\theta(\lambda x)^{-\beta}}\right]^{n-1} \frac{\beta\theta}{\lambda^\beta} x^{-(\beta+1)} \qquad (10)$$

$$f_{n:n}(x) = n\left[e^{-\theta(\lambda x)^{-\beta}}\right]^{n-1} \frac{\beta\theta}{\lambda^\beta} x^{-(\beta+1)} \qquad (11)$$

The *cdf* of $X_{r:n}$ may be obtained by integrating the *pdf* of $X_{r:n}$, (8) which gives,

$$F_{r:n}(x) = \sum_{i=r}^{n} \binom{n}{i} \{F(x)\}^{i}\{1 - F(x)\}^{n-i}; \quad -\infty < x < \infty \tag{12}$$

From equation (12) by using the *cdf* of EIWD as given by equation (3) we have the distribution function of $X_{r:n}$ as

$$F_{r:n}(x) = \sum_{i=r}^{n} \binom{n}{i} \left\{e^{-\theta(\lambda x)^{-\beta}}\right\}^{i} \left\{1 - e^{-\theta(\lambda x)^{-\beta}}\right\}^{n-i} \tag{13}$$

By putting $r = 1$ and $r = n$ in equation (13) we get the distribution functions of the smallest and largest order statistics of EIWD as,

$$F_{1:n}(x) = 1 - \left\{1 - e^{-\theta(\lambda x)^{-\beta}}\right\}^{n}; 0 < x < \infty \tag{14}$$

$$F_{n:n}(x) = \left\{e^{-\theta(\lambda x)^{-\beta}}\right\}^{n}; \quad 0 < x < \infty \tag{15}$$

The k^{th} moment of $X_{r:n}$ ($1 \leq r \leq n$) of EIWD is,

$$\mu_{r:n}{}^{(k)} = E\left(X_{r:n}{}^{k}\right) = \frac{1}{\beta(r, n - r + 1)} \int_{-\infty}^{\infty} x^{k} f_{r:n}(x) dx$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \frac{\beta\theta}{\lambda^{\beta}} \int_{0}^{\infty} x^{k-\beta-1} \left\{e^{-\theta(\lambda x)^{-\beta}}\right\}^{r} \left\{1 - e^{-\theta(\lambda x)^{-\beta}}\right\}^{n-r} dx \tag{16}$$

We obtain the joint density function of two order statistics $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ of EIWD as

$$f_{r,s:n}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \left(\frac{\beta\theta}{\lambda^{\beta}}\right)^{2} x_r{}^{-(\beta+1)} x_s{}^{-(\beta+1)} \left\{e^{-\theta(\lambda x_r)^{-\beta}}\right\}^{r} \left\{e^{-\theta(\lambda x_s)^{-\beta}} - e^{-\theta(\lambda x_r)^{-\beta}}\right\}^{s-r-1} \left\{1 - e^{-\theta(\lambda x_s)^{-\beta}}\right\}^{n-s} e^{-\theta(\lambda x_r)^{-\beta}} e^{-\theta(\lambda x_s)^{-\beta}}; \quad 0 < x_r < x_s < \infty \tag{17}$$

The joint cumulative distribution function of the order statistics $X_{r:n}$ and $X_{s:n}$ be,

$$F_{r,s:n}(x_r, x_s) = P(X_{r:n} \leq x_r, X_{s:n} \leq x_s); \quad -\infty < x_r < x_s < \infty$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!\,(j-i)!\,(n-j)!} \{F(x_r)\}^{i}\{F(x_s) - F(x_r)\}^{j-i}\{1 - F(x_s)\}^{n-j}$$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!\,(j-i)!\,(n-j)!} \left\{e^{-\theta(\lambda x_r)^{-\beta}}\right\}^{i} \left\{e^{-\theta(\lambda x_s)^{-\beta}} - e^{-\theta(\lambda x_r)^{-\beta}}\right\}^{j-i} \left\{1 - e^{-\theta(\lambda x_s)^{-\beta}}\right\}^{n-j} \quad ; \qquad 0 < x_r < x_s < \infty \tag{18}$$

The $(k_r, k_s)^{th}$ product moment of $(X_{r:n}, X_{s:n})$ of EIWD to be

$$\mu_{r,s:n}{}^{(k_r,k_s)}$$

$$= \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} \left(\frac{\beta\theta}{\lambda^{\beta}}\right)^{2} \iint x_r{}^{k_r} x_s{}^{k_s} \left\{e^{-\theta(\lambda x_r)^{-\beta}}\right\}^{r-1} \left\{e^{-\theta(\lambda x_s)^{-\beta}} - e^{-\theta(\lambda x_r)^{-\beta}}\right\}^{s-r-1} \left\{1 - e^{-\theta(\lambda x_s)^{-\beta}}\right\}^{n-s} (x_r x_s)^{-(\beta+1)} e^{-\theta(\lambda x_r)^{-\beta}} e^{-\theta(\lambda x_s)^{-\beta}}\, dx_r dx_s$$

$$1 \le r < s \le n;\ (k_r, k_s) \ge 1 \tag{19}$$

3. BIVARIATE EXPONENTIATED INVERTED WEIBULL DISTRIBUTION (BEIWD)

In this section, **w**e develop the bivariate exponentiated inverted Weibull distribution (BEIWD) using the Morgenstern approach. This system provides a very general expression of a bivariate distribution from which it can be derived by substituting any desired set of marginal distributions. Since both the bivariate distribution function and density are given in terms of marginals, it is easy to generate a random sample from a Morgenstern distribution.

Let Y be the life time of a very expensive component of a two component system and X be an inexpensive variable (directly measurable or observable) which is correlated with Y.

We assume $X \sim EIWD(\beta_1, \theta_1, \lambda_1)$ and $Y \sim EIWD(\beta_2, \theta_2, \lambda_2)$ where, $\beta_1, \theta_1, \lambda_1 > 0$ and $\beta_2, \theta_2, \lambda_2 > 0$. Hence, using the expression of marginal *cdf* s in the expression (1) we get the *cdf* of the Morgenstern type bivariate exponentiated inverted Weibull distribution, which is given by,

$$F_{X,Y}(x,y) = e^{-\theta_1(\lambda_1 x)^{-\beta_1}} e^{-\theta_2(\lambda_2 y)^{-\beta_2}} \left[+\rho\left\{1 - e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\left\{1 - e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\right];$$

$$(X,Y) > 0,\ (\beta_1, \theta_1, \lambda_1, \beta_2, \theta_2, \lambda_2) > 0,\ -1 \le \rho \le 1 \tag{20}$$

Again, using the marginal probability density functions of X and Y in the expression (2) we get the *pdf* of the Morgenstern type bivariate exponentiated inverted Weibull distribution as,

$$f_{X,Y}(x,y)=\frac{\beta_1\theta_1}{{\lambda_1}^{\beta_1}}\frac{\beta_2\theta_2}{{\lambda_2}^{\beta_2}}x^{-(\beta_1+1)}e^{-\theta_1(\lambda_1 x)^{-\beta_1}}y^{-(\beta_2+1)}e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\Big[1+\rho\left\{1-2e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\left\{1-2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\Big];$$

$$(X,Y)>0,\ (\beta_1,\theta_1,\lambda_1,\beta_2,\theta_2,\lambda_2)>0\,,\ -1\le\rho\le 1 \qquad (21)$$

The plots of the density function expressed in (21) are displayed below. From the plots, it is seen that the joint density has a long right tail as compared to its left tail.

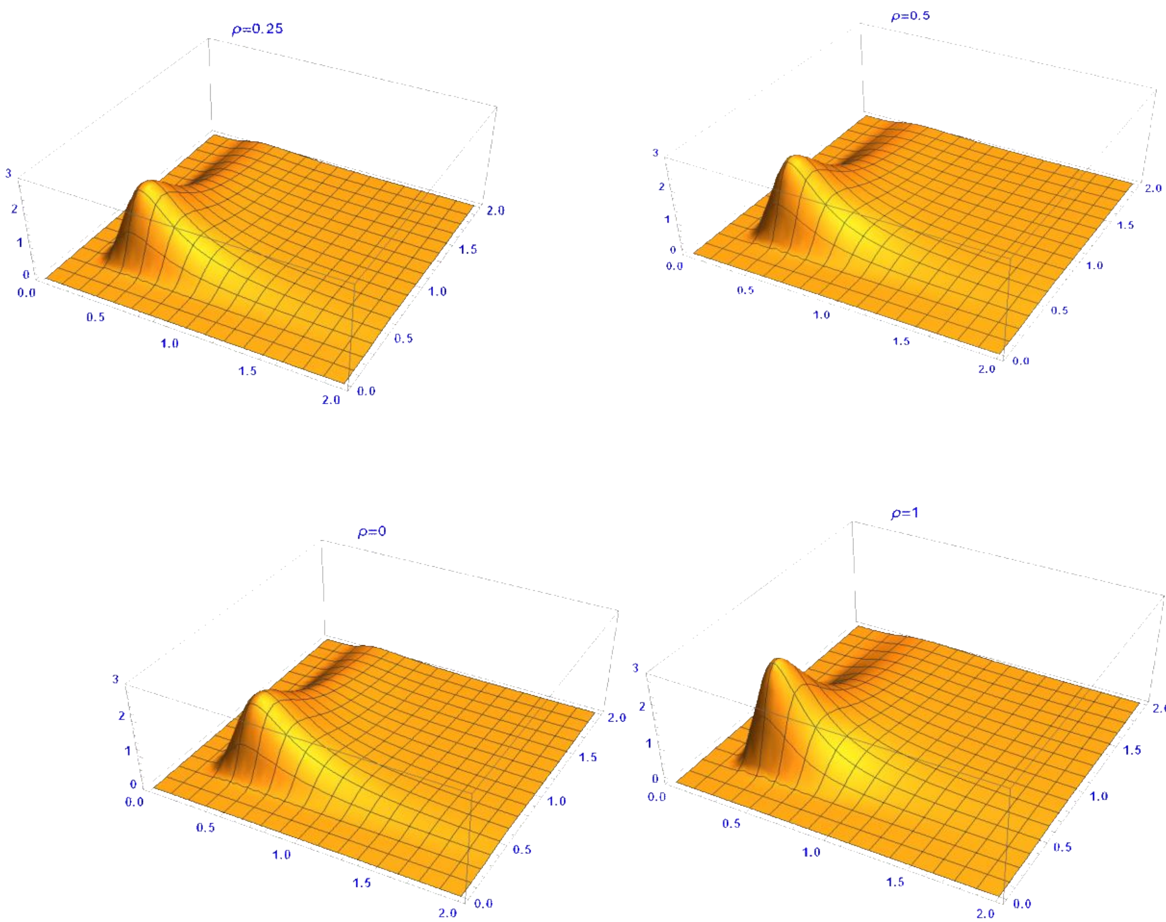


Fig.1. *pdf* of BEIWD for different values of parameters

The plots of cumulative density function are shown below

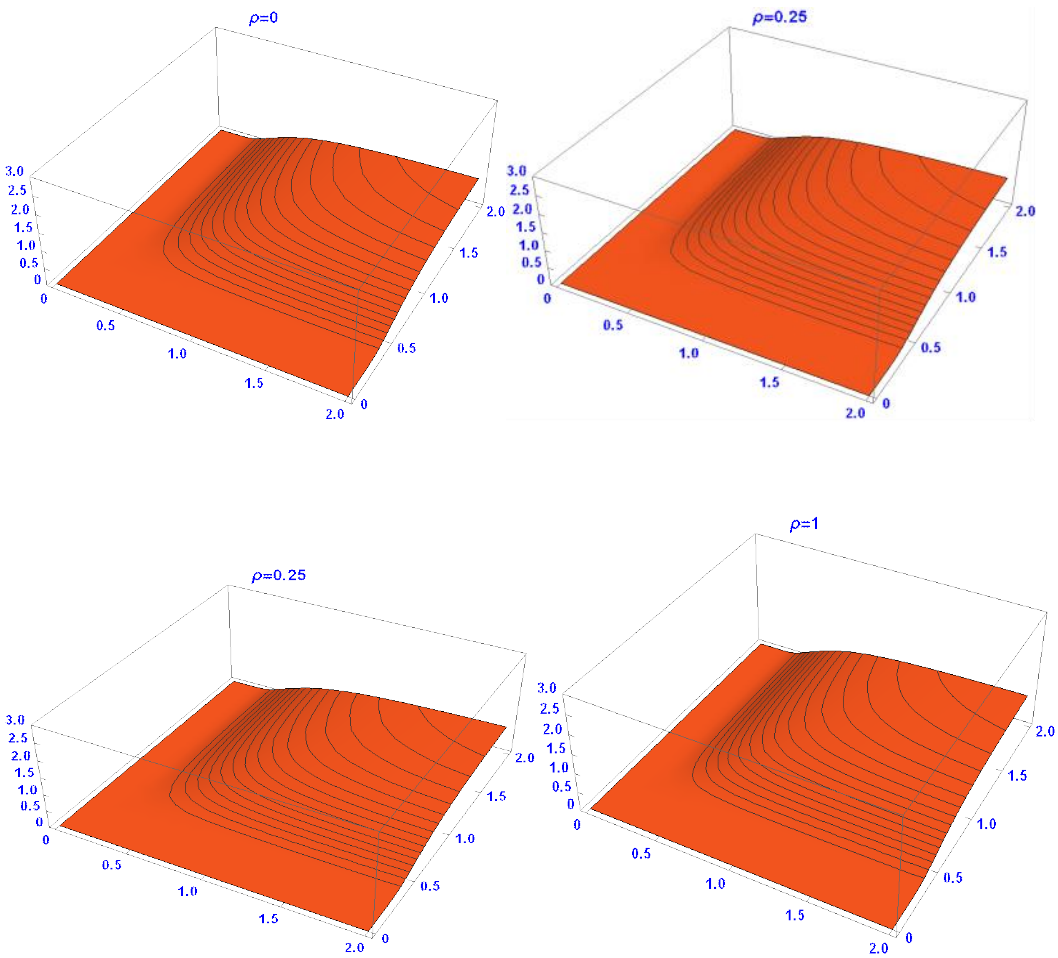


Fig.2. *cdf* of BEIWD for different values of parameters

For Morgenstern distribution, the regression curve of Y given $X = x$ is

$$E(Y|X=x) = E(Y) + \rho(1 - 2F_X(x)) \int y\big(1 - 2F_Y(y)\big) f_Y(y) dy \tag{22}$$

From equation (22), we can derive the regression curve of Y given $X = x$ for MTBEIWD as

$$E(Y|X=x) = \frac{\theta_2{}^{1/\beta_2}\Gamma\left(1 - \frac{1}{\beta_2}\right)}{\lambda_2} + \rho\left(1 - 2e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right)$$

$$\int y\left(1 - 2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right) \frac{\beta_2\theta_2}{\lambda_2{}^{\beta_2}} y^{-(\beta_2+1)} e^{-\theta_2(\lambda_2 y)^{-\beta_2}} dy$$

$$= \frac{\theta_2{}^{1/\beta_2}\Gamma\left(1 - \frac{1}{\beta_2}\right)}{\lambda_2}\left[1 + \rho\left(1 - 2e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right)\left(1 - 2^{1/\beta_2}\right)\right] \tag{23}$$

Thus, it is seen that the conditional expectation is non-linear with respect to X.

The product moment for the joint *pdf* where $1 \leq r < s \leq n$ is given by

$$\acute{\mu}_r \acute{\mu}_s = E(X^r Y^s) = \int_0^\infty \int_0^\infty x^r y^s f_{X,Y}(x,y) dxdy$$

$$= \frac{\theta_1^{r/\beta_1} \theta_2^{s/\beta_2}}{\lambda_1^r \lambda_2^s} \Gamma\left(1 - \frac{r}{\beta_1}\right) \Gamma\left(1 - \frac{s}{\beta_2}\right) \left[1 + \rho\left(1 - 2^{r/\beta_1}\right)\left(1 - 2^{s/\beta_2}\right)\right] \quad (24)$$

Hence, the first order product moment between X and Y is,

$$E(X,Y) = \frac{\theta_1^{1/\beta_1} \theta_2^{1/\beta_2}}{\lambda_1 \lambda_2} \Gamma\left(1 - \frac{1}{\beta_1}\right) \Gamma\left(1 - \frac{1}{\beta_2}\right) \left[1 + \rho\left(1 - 2^{1/\beta_1}\right)\left(1 - 2^{1/\beta_2}\right)\right] \quad (25)$$

So, the covariance term is given by,

$$cov(X,Y) = \frac{\theta_1^{1/\beta_1} \theta_2^{1/\beta_2}}{\lambda_1 \lambda_2} \Gamma\left(1 - \frac{1}{\beta_1}\right) \Gamma\left(1 - \frac{1}{\beta_2}\right) \rho\left(1 - 2^{1/\beta_1}\right)\left(1 - 2^{1/\beta_2}\right) \quad (26)$$

Therefore, the Karl- Pearson's co-efficient of correlation is obtained as,

$$\rho_0 = \frac{\Gamma\left(1 - \frac{1}{\beta_1}\right) \Gamma\left(1 - \frac{1}{\beta_2}\right) \rho\left(1 - 2^{1/\beta_1}\right)\left(1 - 2^{1/\beta_2}\right)}{\left[\Gamma\left(1 - \frac{2}{\beta_1}\right) - \left\{\Gamma\left(1 - \frac{1}{\beta_1}\right)\right\}^2\right]^{1/2} \left[\Gamma\left(1 - \frac{2}{\beta_2}\right) - \left\{\Gamma\left(1 - \frac{1}{\beta_2}\right)\right\}^2\right]^{1/2}} \quad (27)$$

For Morgenstern family of distribution the Pearson's correlation coefficient lies between $-1/3$ and $1/3$ for desired values of parameters.

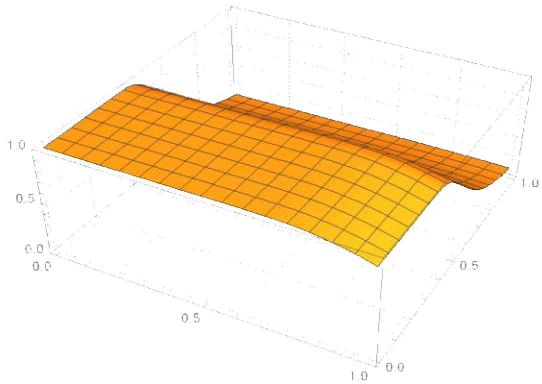
Survival function of Morgenstern family is of the form

$$S(x,y) = (1 - F_X(x))(1 - F_Y(y))[1 + \rho F_X(x) F_Y(y)] \quad -1 \leq \rho \leq 1 \quad (28)$$

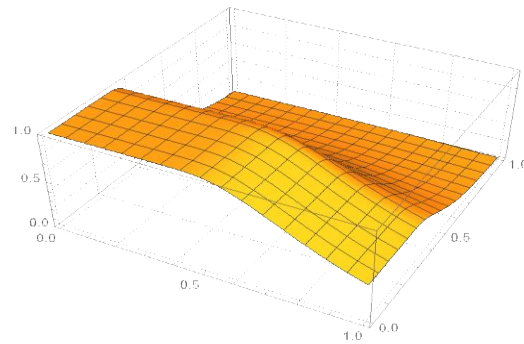
From equation (28) the survival function of BEIWD is given by,

$$S_{X,Y}(x,y) = \left\{1 - e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\left\{1 - e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left[1 + \rho e^{-\theta_1(\lambda_1 x)^{-\beta_1}} e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right] \quad (29)$$

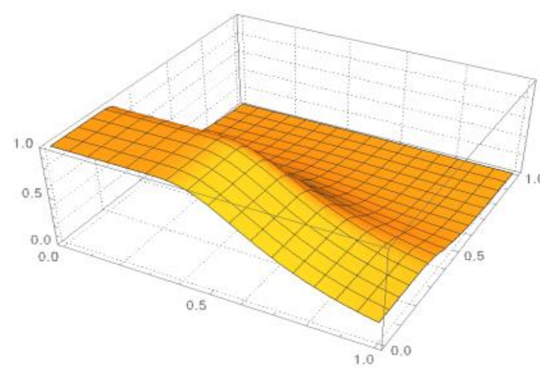
The plots of the survival function of BEIWD are displayed below. The plots indicate that the survival function decreases faster as the values of θ and λ increase.



$\theta_1 = 2, \theta_2 = 2, \lambda_1 = 1, \lambda_2 = 2, \beta_1 = 3, \beta_2 = 4, \rho = 0.5$



$\theta_1 = 2, \theta_2 = 2, \lambda_1 = 2, \lambda_2 = 3, \beta_1 = 3, \beta_2 = 4, \rho = 0.5$



$\theta_1 = 4, \theta_2 = 4, \lambda_1 = 2, \lambda_2 = 3, \beta_1 = 3, \beta_2 = 4, \rho = 0.5$

Fig.3 Survival function for different values of parameters

4. Distribution of Concomitants of Bivariate Exponentiated Inverted Weibull Distribution (BEIWD)

4.1 Distribution of r^{th} Concomitants for BTEGD

In this section, we obtain the distribution of the concomitants of the r^{th} order statistics for the Bivariate exponentiated inverted Weibull distribution. Suppose, random variables (X_i, Y_i); $i = 1,2, \dots n$; are i.i.d and follows $EIWD(\beta_1, \theta_1, \lambda_1)$ and $EIWD(\beta_2, \theta_2, \lambda_2)$ respectively.Then as given by David and Nagaraja [8], if $F_{Y|X}(y|x)$ be the conditional distribution function of Y

given $X=x$, $f_{Y|X}(y|x)$ be the conditional density of Y given $X=x$ and $f_{r:n}(x)$ is the *pdf* of r^{th} order statistic $X_{r:n}$ then the distribution function and density function of r^{th} concomitant $Y_{r:n}$ is given respectively as,

$$F_{Y_{[r:n]}}(y)=\int_0^\infty F_{Y|X}(y|x)f_{r:n}(x)dx \quad (30)$$

$$f_{Y_{[r:n]}}(y)=\int_0^\infty f_{Y|X}(y|x)f_{r:n}(x)dx \quad (31)$$

The conditional distribution function of Y given $X=x$ for BEIWD is obtained as

$$F_{Y|X}(y|x)=e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\left\{1-e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\right]$$
$$-1\le\rho\le 1 \quad (32)$$

The conditional density function of Y given $X=x$ for Morgenstern family is defined as,

$$f_{Y|X}(y|x)=\frac{\beta_2\theta_2}{{\lambda_2}^{\beta_2}}y^{-\beta_2-1}e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-2e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\left\{1-2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\right]$$
$$-1\le\rho\le 1 \quad (33)$$

Now, by using equations (32) and (9) we get the *cdf* of the concomitant of r^{th} order statistic $X_{r:n}$ from equation (30)

$$F_{Y_{[r:n]}}(y)=\int_0^\infty F_{Y|X}(y|x)f_{r:n}(x)dx$$

$$=\int_0^\infty e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\Big[1$$
$$+\rho\left\{1-e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\left\{1-e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\Big]\frac{n!}{(r-1)!\,(n-r)!}\left\{e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}^r\Big\{1$$
$$-e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\Big\}^{n-r}\frac{\beta_1\theta_1}{{\lambda_1}^{\beta_1}}x^{-(\beta_1+1)}dx$$

$$=e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\frac{n-r+1}{n+1}\right] \quad (34)$$

Hence, putting $r=n$ in equation (34) we have the *cdf* of the concomitant of largest order statistic $X_{n:n}$ is,

$$F_{Y_{[n:n]}}(y)=e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\frac{1}{n+1}\right] \quad (35)$$

Similarly, the *cdf* of the concomitant of smallest order statistic $X_{1:n}$ is,

$$F_{Y_{[1:n]}}(y) = e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\frac{n}{n+1}\right] \quad (36)$$

Similarly, using equations (33) and (9) in equation (31) we have the *pdf* of the concomitant of r^{th} order statistic $X_{r:n}$ as

$$f_{Y_{[r:n]}}(y) = \int_0^\infty \frac{\beta_2\theta_2}{\lambda_2{}^{\beta_2}} y^{-(\beta_2+1)} e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\Big[1 + \rho\left\{1-2e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}\Big\{1 - 2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\Big\}\Big]\frac{n!}{(r-1)!\,(n-r)!}\left\{e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\right\}^r\Big\{1 - e^{-\theta_1(\lambda_1 x)^{-\beta_1}}\Big\}^{n-r}\frac{\beta_1\theta_1}{\lambda_1{}^{\beta_1}} x^{-(\beta_1+1)}dx$$

$$= \frac{\beta_2\theta_2}{\lambda_2{}^{\beta_2}} y^{-(\beta_2+1)} e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left(\frac{n-2r+1}{n+1}\right)\right] \quad (37)$$

By putting $r = n$, in equation (37) we get the *pdf* of the concomitant of largest order statistic $X_{n:n}$ which is,

$$f_{Y_{[n:n]}}(y) = \frac{\beta_2\theta_2}{\lambda_2{}^{\beta_2}} y^{-(\beta_2+1)} e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left(\frac{-n+1}{n+1}\right)\right] \quad (38)$$

By putting $r = 1$, we get the *pdf* of the concomitant of smallest order statistic $X_{1:n}$ which is,

$$f_{Y_{[1:n]}} = \frac{\beta_2\theta_2}{\lambda_2{}^{\beta_2}} y^{-(\beta_2+1)} e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\left[1+\rho\left\{1-2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left(\frac{n-1}{n+1}\right)\right] \quad (39)$$

4.2. Joint density of Concomitants of Two Order Statistics

To study the behavior of one concomitant to others we need the joint probability distribution of any two concomitants. In this section, we derive the joint distribution of concomitants of two order statistics of BEIWD. Suppose, random variables (X_i, Y_i); $i = 1,2, \ldots n$; are i.i.d and follows $EIWD(\beta_1, \theta_1, \lambda_1)$ and $EIWD(\beta_2, \theta_2, \lambda_2)$ respectively. Let, $(X_{r:n}, X_{s:n})$, $(r < s)$ be the pair of r^{th} and s^{th} order statistics of the inexpensive variable X and $(Y_{[r:n]}, Y_{[s:n]})$ be the associated Y measurements or concomitants. Then, we get the joint density of $Y_{[r:n]}$ and $Y_{[s:n]}$, where $1 \leq r < s \leq n$

$$f_{[r,s;n]}(y_1, y_2) = \int_0^\infty \int_0^\infty f_{Y|X}(y_1|x_1)\, f_{Y|X}(y_2|x_2) f_{r,s:n}(x_1, x_2) dx_1 dx_2$$

$$= \frac{{\beta_2}^2{\theta_2}^2}{{\lambda_2}^{2\beta_2}} e^{-\theta_2[(\lambda_2 y_1)^{-\beta_2}+(\lambda_2 y_2)^{-\beta_2}]}(y_1 y_2)^{-(\beta_2+1)} \left[1 + \rho\left\{1 - 2e^{-\theta_2(\lambda_2 y_1)^{-\beta_2}}\right\}\frac{n-2r+1}{n+1} \right.$$
$$+ \rho\left\{1 - 2e^{-\theta_2(\lambda_2 y_2)^{-\beta_2}}\right\}\frac{n-2s+1}{n+1}$$
$$\left. + \rho^2\left\{1 - 2e^{-\theta_2(\lambda_2 y_1)^{-\beta_2}}\right\}\left\{1 - 2e^{-\theta_2(\lambda_2 y_2)^{-\beta_2}}\right\}\left\{\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right\}\right] \tag{40}$$

From equation (40), we obtain,

$$E\left[Y_{[r:n]}, Y_{[s:n]}\right] = \left(\frac{{\theta_2}^{1/\beta_2}}{\lambda_2}\right)^2 \left(\Gamma\left(1 - \frac{1}{\beta_2}\right)\right)^2 \left(1 - 2^{1/\beta_2}\right)\left[\left(1 - 2^{1/\beta_2}\right)^{-1} \right.$$
$$+ \rho\left(\frac{n-2r+1}{n+1} + \frac{n-2s+1}{n+1}\right)$$
$$\left. + \rho^2\left\{\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right\}\left(1 - 2^{1/\beta_2}\right)\right] \tag{41}$$

In equation (40), putting $y_1 = Y_{[1:n]}$ and $y_2 = Y_{[n:n]}$, we get the joint density of the lowest and highest order statistics.

$$f_{[1,n:n]}(y_1, y_2) = \frac{{\beta_2}^2{\theta_2}^2}{{\lambda_2}^{2\beta_2}} e^{-\theta_2[(\lambda_2 y_1)^{-\beta_2}+(\lambda_2 y_2)^{-\beta_2}]}(y_1 y_2)^{-(\beta_2+1)} \left[1 \right.$$
$$+ \rho\left\{1 - 2e^{-\theta_2(\lambda_2 y_1)^{-\beta_2}}\right\}\frac{n-1}{n+1} + \rho\left\{1 - 2e^{-\theta_2(\lambda_2 y_2)^{-\beta_2}}\right\}\frac{-n+1}{n+1}$$
$$\left. + \rho^2\left\{1 - 2e^{-\theta_2(\lambda_2 y_1)^{-\beta_2}}\right\}\left\{1 - 2e^{-\theta_2(\lambda_2 y_2)^{-\beta_2}}\right\}\left\{\frac{-n+1}{n+1} + \frac{2n}{(n+1)(n+2)}\right\}\right] \tag{42}$$

4.3. Some properties of Concomitants of Order Statistics

In this section, the explicit expression for the moment of concomitant of r^{th} order statistics is deduced when random variables; $(X_i, Y_i); i = 1,2, \dots n$ are i.i.d and follows $EIWD(\beta_i, \theta_i, \lambda_i)$. Using the result, we can compute mean and variance of the concomitant of r^{th} order statistic.

According to Scaria and Nair [10], the density of the concomitant $Y_{[r:n]}$ can be also written as,

$$f_{y[r:n]}(y) = f_Y(y) + \frac{\rho}{2}\frac{n-r+1}{n+1}[f_{1:2}(y) - f_{2:2}(y)] \tag{43}$$

This reveals that the distribution of the r^{th} concomitant depends only on the marginal distribution of Y and the distribution of the order statistics $Y_{1:2}$ and $Y_{2:2}$.

Resulting from (43), the k^{th} moment of $Y_{[r:n]}$ can be derived as,

$$E\left({Y_{[r:n]}}^k\right) = {\mu_{[r:n]}}^{(k)} = \mu^{(k)} + \frac{\rho}{2}\frac{n-2r+1}{n+1}\left[{\mu_{1:2}}^{(k)} - {\mu_{2:2}}^{(k)}\right] \tag{44}$$

where, $\mu^{(k)} = E(Y^k) \;\; and \;\; {\mu_{r:2}}^{(k)} = E\left[{Y_{r:2}}^k\right]$

Hence, using equations (5) and (16), the k^{th} moment of concomitant of order statistic $Y_{[r:n]}$ is obtained as,

$$E\left({Y_{[r:n]}}^k\right) = {\mu_{[r:n]}}^{(k)}$$

$$= \frac{{\theta_2}^{k/\beta_2}}{{\lambda_2}^k}\Gamma\left(1 - \frac{k}{\beta_2}\right) + \frac{\rho}{2}\frac{n-2r+1}{n+1}\left[\frac{2\Gamma\left(1 - {k}/{\beta_2}\right)}{{\lambda_2}^{\kappa}}{\theta_2}^{k/\beta_2}\left(1 - 2^{k/\beta_2}\right)\right]$$

$$= \frac{{\theta_2}^{k/\beta_2}}{{\lambda_2}^k}\Gamma\left(1 - \frac{k}{\beta_2}\right)\left[1 + \rho\frac{n-2r+1}{n+1}\left(1 - 2^{k/\beta_2}\right)\right] \tag{45}$$

In particular,

$$E\left(Y_{[r:n]}\right) = \mu_{[r:n]} = \frac{{\theta_2}^{1/\beta_2}}{\lambda_2}\Gamma\left(1 - \frac{1}{\beta_2}\right)\left[1 + \rho\frac{n-2r+1}{n+1}\left(1 - 2^{1/\beta_2}\right)\right] \tag{46}$$

And,

$$V\left(Y_{[r:n]}\right) = \frac{{\theta_2}^{2/\beta_2}}{{\lambda_2}^2}\left[\Gamma\left(1 - \frac{2}{\beta_2}\right)\left\{1 + \rho\frac{n-2r+1}{n+1}\left(1 - 2^{2/\beta_2}\right)\right\} - \left(\Gamma\left(1 - \frac{1}{\beta_2}\right)\right)^2\left\{1 + \rho\frac{n-2r+1}{n+1}\left(1 - 2^{1/\beta_2}\right)\right\}^2\right] \tag{47}$$

5. SURVIVAL AND HAZARD FUNCTIONS OF CONCOMITANTS OF ORDER STATISTICS

When the random vector (X, Y) is defined as the lifetimes of two units or lives at a certain time point with observable ages, the remaining lifetime probabilities can be computed by using the general definition of a survival function, $S(t) = 1 - F(t)$ for each of (X, Y).

By means of the concept of order statistics, system reliability models can be developed without any consideration on whether the component failures are independent of each other or not. The survival analysis in terms of concomitants gives a fair insight to the analysis of the systems with pairs of units as components where each unit deals as reserve unit for the other. In this section, we derive the survival and hazard functions for the concomitants $Y_{[r:n]}$.

The survival function of concomitant of order statistic $Y_{[r:n]}$ is given by,

$$\begin{aligned} S_{Y_{[r:n]}(y)} &= 1 - F_{Y_{[r:n\}}}(y) \\ &= 1 - e^{-\theta_2(\lambda_2 y)^{-\beta_2}} \left[1 + \rho\left\{1 - e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\frac{n-r+1}{n+1}\right] \\ &= \left\{1 - e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left\{1 - \rho. e^{-\theta_2(\lambda_2 y)^{-\beta_2}}.\frac{n-r+1}{n+1}\right\} \end{aligned} \tag{48}$$

The hazard function of concomitant of order statistic $Y_{[r:n]}$ is given by,

$$\begin{aligned} H_{Y_{[r:n]}}(y) &= \frac{f_{Y_{[r:n]}}(y)}{S_{Y_{[r:n]}}(y)} \\ &= \frac{\frac{\beta_2\theta_2}{{\lambda_2}^{\beta_2}} y^{-(\beta_2+1)} e^{-\theta_2(\lambda_2 y)^{-\beta_2}} \left[1 + \rho\left\{1 - 2e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left(\frac{n-2r+1}{n+1}\right)\right]}{\left\{1 - e^{-\theta_2(\lambda_2 y)^{-\beta_2}}\right\}\left\{1 - \rho. e^{-\theta_2(\lambda_2 y)^{-\beta_2}}.\frac{n-r+1}{n+1}\right\}} \end{aligned} \tag{49}$$

The characteristic behaviors of the survival and hazard functions for the concomitant $Y_{[r:n]}$ are tabled below for $n = 10$; $r = 1,2, \ldots 10$ and $0.1 \le y \le 1$ for the selected values of the parameters β_2, θ_2 and λ_2 of the bivariate exponentiated inverted Weibull distribution.

Table1. Survival function for the concomitant $Y_{[r:n]}$.

y	r									
	1	2	3	4	5	6	7	8	9	10
	$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 0.5, \rho = 0.25$									
0.1	1.0744	1.0563	1.0381	1.0200	1.0019	0.9838	0.9656	0.9475	0.9294	0.9113
0.2	0.9376	0.9233	0.9089	0.8946	0.8803	0.8659	0.8516	0.8373	0.8229	0.8086
0.3	0.8445	0.8325	0.8206	0.8086	0.7967	0.7847	0.7728	0.7608	0.7489	0.7369
0.4	0.7760	0.7657	0.7554	0.7451	0.7348	0.7245	0.7143	0.7040	0.6937	0.6833
0.5	0.7229	0.7138	0.7048	0.6957	0.6866	0.6775	0.6684	0.6594	0.6503	0.6412
0.6	0.6801	0.6719	0.6638	0.6556	0.6475	0.6393	0.6312	0.6231	0.6149	0.6067
0.7	0.6445	0.6371	0.6297	0.6223	0.6149	0.6075	0.6001	0.5927	0.5853	0.5780
0.8	0.6143	0.6075	0.6007	0.5939	0.5871	0.5803	0.5735	0.5668	0.5600	0.5532
0.9	0.5882	0.5820	0.5756	0.5693	0.5631	0.5568	0.5505	0.5443	0.5380	0.5317
1.0	0.5653	0.5595	0.5536	0.5478	0.5420	0.5361	0.5303	0.5245	0.5186	0.5128
$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 0.75, \rho = 0.25$										
0.1	0.9988	0.9828	0.9668	0.9509	0.9349	0.9188	0.9029	0.8869	0.8709	0.8549
0.2	0.8445	0.8325	0.8206	0.8086	0.7967	0.7847	0.7728	0.7608	0.7489	0.7369
0.3	0.7479	0.7383	0.7287	0.7190	0.7094	0.6997	0.6901	0.6804	0.6708	0.6611
0.4	0.6801	0.6719	0.6638	0.6556	0.6475	0.6393	0.6312	0.6231	0.6150	0.6068
0.5	0.6288	0.6217	0.6146	0.6076	0.6005	0.5934	0.5863	0.5792	0.5722	0.5651
0.6	0.5882	0.5819	0.5756	0.5693	0.5631	0.5568	0.5505	0.5442	0.5380	0.5317
0.7	0.5549	0.5493	0.5436	0.5380	0.5323	0.5267	0.5210	0.5154	0.5097	0.5041
0.8	0.5270	0.5219	0.5167	0.5116	0.5064	0.5013	0.4962	0.4910	0.4859	0.4807
0.9	0.5031	0.4984	0.4937	0.4889	0.4842	0.4795	0.4748	0.4700	0.4653	0.4606
1.0	0.4823	0.4780	0.4736	0.4692	0.4648	0.4605	0.4561	0.4517	0.4474	0.4430

$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 1.0, \rho = 0.25$										
0.1	0.9376	0.9233	0.9089	0.8946	0.8803	0.8659	0.8516	0.8373	0.8229	0.8086
0.2	0.7760	0.7657	0.7554	0.7451	0.7349	0.7245	0.7143	0.7040	0.6937	0.6834
0.3	0.6801	0.6719	0.6638	0.6556	0.6475	0.6393	0.6312	0.6231	0.6149	0.6068
0.4	0.6143	0.6075	0.6007	0.5939	0.5871	0.5803	0.5736	0.5668	0.5600	0.5532
0.5	0.5653	0.5595	0.5537	0.5478	0.5420	0.5361	0.5303	0.5244	0.5186	0.5128
0.6	0.5270	0.5219	0.5167	0.5116	0.5064	0.5013	0.4961	0.4910	0.4859	0.4807
0.7	0.4959	0.4913	0.4867	0.4821	0.4775	0.4729	0.4683	0.4637	0.4591	0.4545
0.8	0.4699	0.4657	0.4616	0.4574	0.4532	0.4491	0.4449	0.4407	0.4365	0.4324
0.9	0.4478	0.4440	0.4402	0.4364	0.4325	0.4287	0.4249	0.4211	0.4173	0.4135
1.0	0.4286	0.4251	0.4216	0.4181	0.4146	0.4111	0.4075	0.4040	0.4005	0.3970

$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 1.5, \rho = 0.25$										
0.1	0.8445	0.8325	0.8206	0.8086	0.7967	0.7847	0.7728	0.7608	0.7489	0.7369
0.2	0.6801	0.6719	0.6638	0.6556	0.6475	0.6393	0.6312	0.6230	0.6149	0.6068
0.3	0.5882	0.5819	0.5756	0.5693	0.5631	0.5568	0.5505	0.5442	0.5380	0.5317
0.4	0.5270	0.5219	0.5167	0.5116	0.5064	0.5013	0.4961	0.4910	0.4859	0.4807
0.5	0.4823	0.4780	0.4736	0.4692	0.4648	0.4605	0.4561	0.4517	0.4474	0.4430
0.6	0.4478	0.4439	0.4402	0.4363	0.4325	0.4287	0.4249	0.4211	0.4173	0.4135
0.7	0.4200	0.4166	0.4132	0.4098	0.4064	0.4030	0.3997	0.3963	0.3929	0.3895
0.8	0.3970	0.3939	0.3908	0.3878	0.3848	0.3817	0.3787	0.3756	0.3726	0.3695
0.9	0.3775	0.3747	0.3719	0.3692	0.3664	0.3636	0.3608	0.3580	0.3552	0.3525
1.0	0.3607	0.3582	0.3556	0.3530	0.3505	0.3479	0.3454	0.3428	0.3403	0.3377

Table 2: Hazard function for the concomitant $Y_{[r:n]}$.

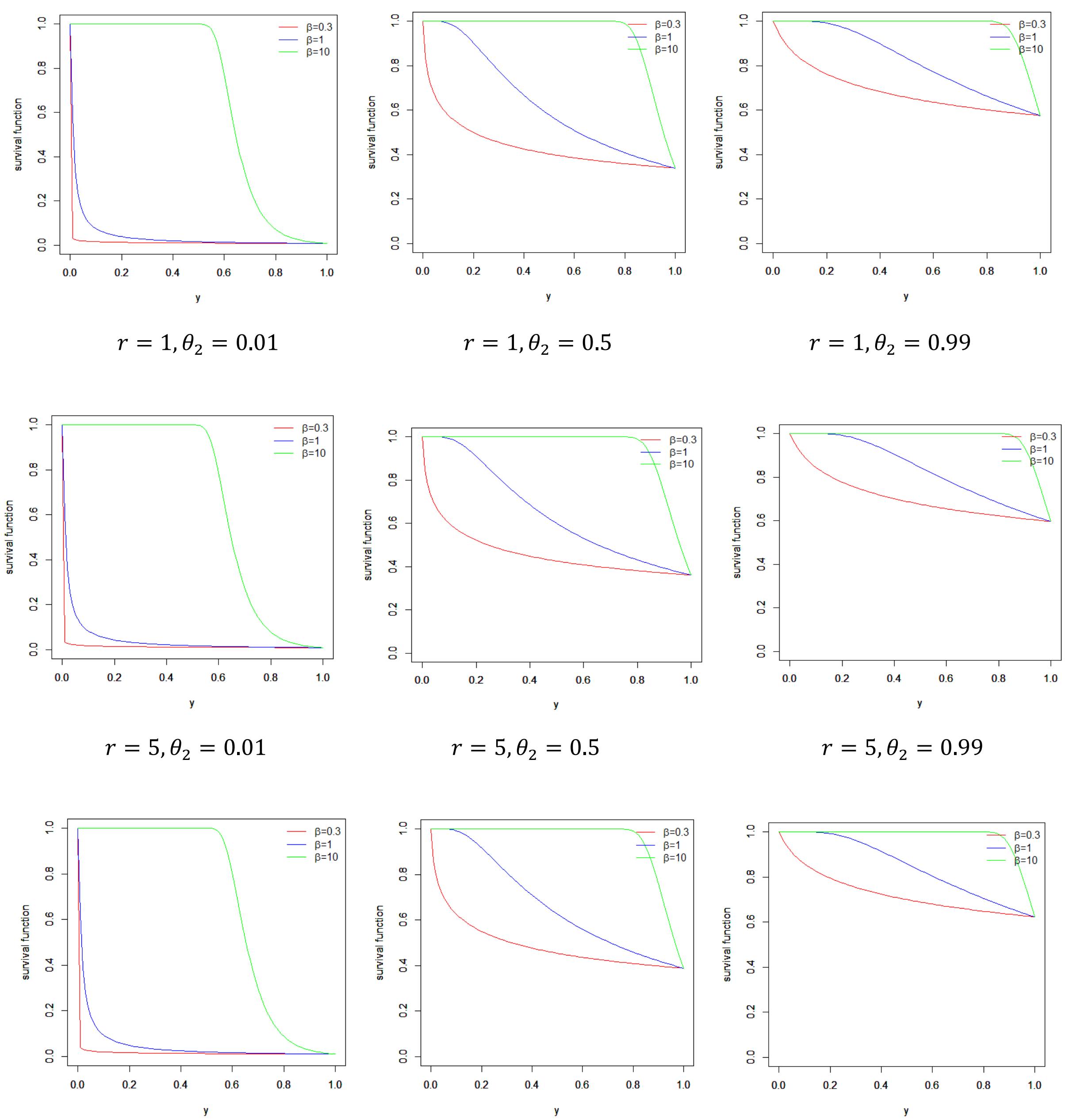
y	r									
	1	2	3	4	5	6	7	8	9	10
	$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 0.5, \rho = 0.25$									
0.1	1.8683	1.7802	1.6941	1.6099	1.5277	1.4474	1.3690	1.2926	1.2181	1.1455
0.2	1.3543	1.3017	1.2502	1.1996	1.1500	1.1014	1.0538	1.0072	0.9616	0.9169
0.3	1.0381	1.0042	0.9709	0.9382	0.9060	0.8743	0.8432	0.8127	0.7827	0.7532
0.4	0.8381	0.8148	0.7918	0.7692	0.7469	0.7250	0.7034	0.6820	0.6610	0.6403
0.5	0.7016	0.6848	0.6683	0.6520	0.6359	0.6200	0.6043	0.5888	0.5735	0.5584
0.6	0.6028	0.5905	0.5782	0.5661	0.5542	0.5423	0.5306	0.5190	0.5075	0.4961
0.7	0.5282	0.5188	0.5096	0.5005	0.4914	0.4824	0.4734	0.4646	0.4558	0.4471
0.8	0.4698	0.4627	0.4557	0.4486	0.4416	0.4347	0.4278	0.4209	0.4141	0.4074
0.9	0.4230	0.4175	0.4121	0.4067	0.4012	0.3959	0.3905	0.3851	0.3798	0.3745
1.0	0.3846	0.3804	0.3762	0.3720	0.3678	0.3636	0.3594	0.3552	0.3510	0.3469
$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 0.75, \rho = 0.25$										
0.1	2.3766	2.2753	2.1760	2.0788	1.9836	1.8906	1.7995	1.7105	1.6236	1.5388
0.2	1.5573	1.5064	1.4564	1.4073	1.3590	1.3115	1.2648	1.2190	1.1740	1.1298
0.3	1.1458	1.1164	1.0873	1.0586	1.0303	1.0024	0.9749	0.9477	0.9210	0.8946
0.4	0.9042	0.8857	0.8674	0.8492	0.8312	0.8135	0.7959	0.7784	0.7612	0.7442
0.5	0.7460	0.7338	0.7217	0.7097	0.6977	0.6859	0.6741	0.6624	0.6509	0.6393
0.6	0.6345	0.6263	0.6181	0.6100	0.6019	0.5938	0.5857	0.5777	0.5697	0.5618
0.7	0.5518	0.5462	0.5407	0.5352	0.5296	0.5241	0.5185	0.5130	0.5074	0.5019
0.8	0.4880	0.4843	0.4806	0.4769	0.4731	0.4694	0.4656	0.4618	0.4580	0.4541
0.9	0.4373	0.4350	0.4326	0.4302	0.4278	0.4253	0.4228	0.4203	0.4177	0.4151
1.0	0.3961	0.3947	0.3934	0.3919	0.3905	0.3890	0.3874	0.3858	0.3842	0.3825

$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 1.0, \rho = 0.25$										
0.1	2.7085	2.6034	2.5003	2.3991	2.3000	2.2028	2.1076	2.0144	1.9231	1.8338
0.2	1.6761	1.6296	1.5837	1.5384	1.4938	1.4499	1.4067	1.3640	1.3221	1.2808
0.3	1.2056	1.1809	1.1565	1.1322	1.1083	1.0846	1.0611	1.0379	1.0150	0.9923
0.4	0.9397	0.9255	0.9113	0.8973	0.8833	0.8694	0.8556	0.8419	0.8283	0.8147
0.5	0.7692	0.7607	0.7523	0.7439	0.7355	0.7272	0.7188	0.7104	0.7021	0.6937
0.6	0.6507	0.6457	0.6408	0.6359	0.6309	0.6259	0.6208	0.6157	0.61063	0.6055
0.7	0.5636	0.5609	0.5582	0.5555	0.5527	0.5499	0.5470	0.5441	0.5411	0.5381
0.8	0.4969	0.4958	0.4946	0.4934	0.4921	0.4908	0.4894	0.4879	0.4864	0.4848
0.9	0.4442	0.4441	0.4440	0.4439	0.4437	0.4434	0.4431	0.4426	0.4422	0.4416
1.0	0.4015	0.4023	0.4020	0.4035	0.4041	0.4045	0.4049	0.4053	0.4056	0.4058
$\theta_2 = 0.5, \beta_2 = 0.5, \lambda_2 = 1.5, \rho = 0.25$										
0.1	3.1143	3.0128	2.9128	2.8145	2.7180	2.6230	2.5297	2.4380	2.3480	2.2597
0.2	1.8085	1.7714	1.7347	1.6984	1.6625	1.6269	1.5917	1.5569	1.5225	1.4884
0.3	1.2690	1.2526	1.2363	1.2200	1.2038	1.1876	1.1715	1.1555	1.1395	1.1236
0.4	0.9760	0.9686	0.9612	0.9538	0.9463	0.9388	0.9312	0.9236	0.9159	0.9083
0.5	0.7922	0.7895	0.7867	0.7839	0.7810	0.7779	0.7748	0.7717	0.7684	0.7651
0.6	0.6663	0.6662	0.6661	0.6659	0.6655	0.6651	0.6646	0.6640	0.6632	0.6624
0.7	0.5747	0.5762	0.5777	0.5790	0.5803	0.5814	0.5825	0.5834	0.5843	0.5850
0.8	0.5051	0.5076	0.5100	0.5124	0.5147	0.5168	0.5189	0.5208	0.5227	0.5244
0.9	0.4504	0.4536	0.4567	0.4597	0.4626	0.4654	0.4681	0.4707	0.4732	0.4756
1.0	0.4063	0.4099	0.4134	0.4169	0.4202	0.4234	0.4266	0.4296	0.4326	0.4354

Table 1 contains the values of the survival function (48). Looking at the table we can see that the survival probability of the concomitant decreases as the value of λ_2 increases, holding y, r, β_2 and θ_2 at a fixed level. For fixed $\lambda_2, y, \beta_2, \theta_2$ the survival probability decreases while the rank r of the order statistic increases. The table also shows that for the fixed $\lambda_2, r, \beta_2, \theta_2$ values the survival probability decreases as the value y of the concomitant $Y_{[r:n]}$ increases.

Table 2 contains the values of the hazard function in equation (49). It is seen that the hazard rate of the concomitant decreases as the value of λ_2 increases, while holding y, r, β_2 and θ_2 at fixed values. Further, for the fixed values of $\lambda_2, y, \beta_2, \theta_2$, the hazard function decreases as the rank order r increases. Again, for the fixed $\lambda_2, r, \beta_2, \theta_2$ values the hazard rate decreases as the value y of the concomitant $Y_{[r:n]}$ increases.

The behavior of the survival function $S_{Y_{[r:n]}}(y)$ of the concomitant $Y_{[r:n]}$ is presented in the graphics below.



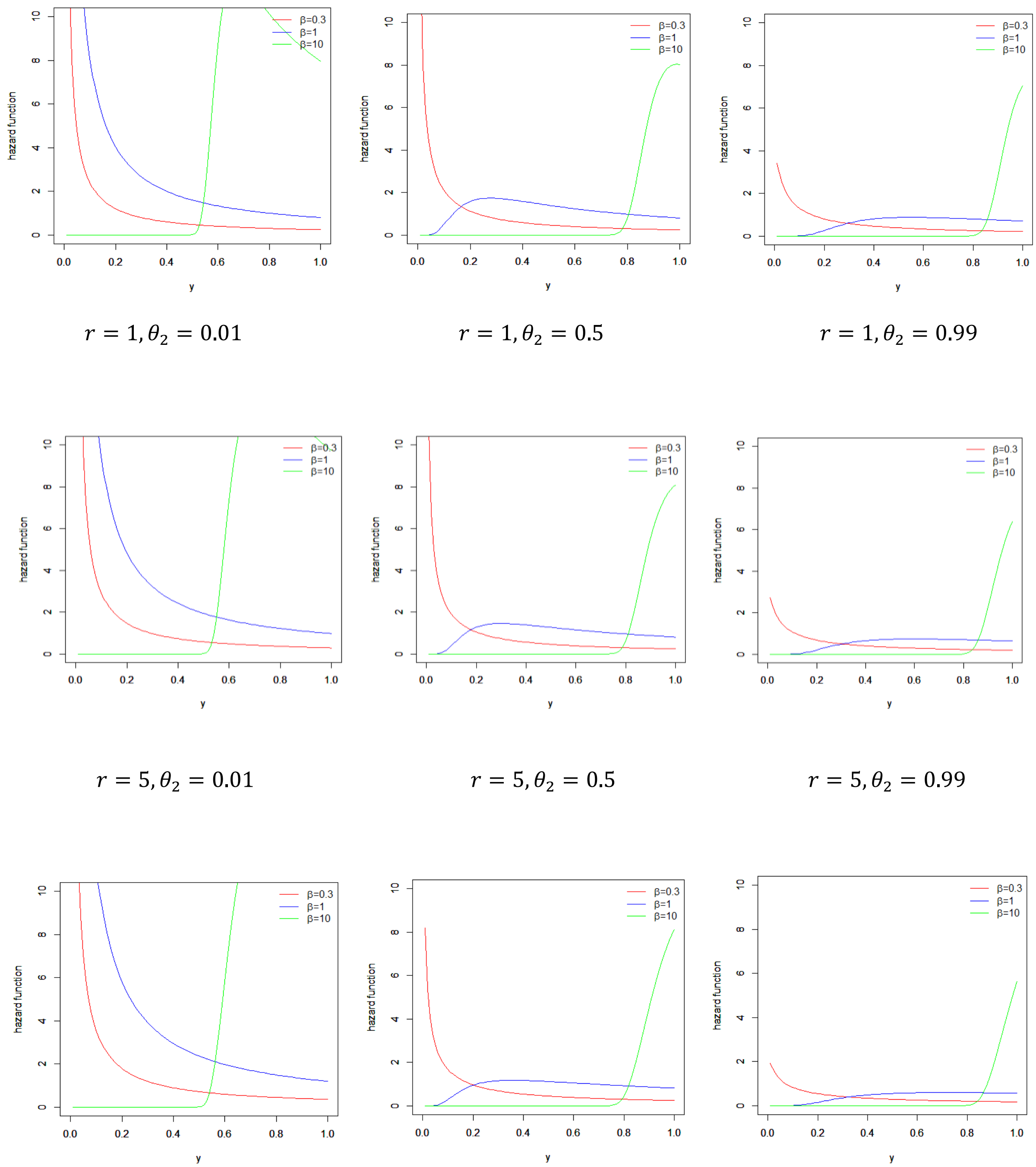
$r = 1, \theta_2 = 0.01$ $r = 1, \theta_2 = 0.5$ $r = 1, \theta_2 = 0.99$

$r = 5, \theta_2 = 0.01$ $r = 5, \theta_2 = 0.5$ $r = 5, \theta_2 = 0.99$

$r = 10, \theta_2 = 0.01$ $r = 10, \theta_2 = 0.5$ $r = 10, \theta_2 = 0.99$

Fig. 4.1. *Survival plot of concomitants of order statistics* $Y_{[r:n]}$

The behavior of the hazard function $H_{Y_{[r:n]}}(y)$ of the concomitant of order statistic $Y_{[r:n]}$ is shown below:



$r = 1, \theta_2 = 0.01$ $r = 1, \theta_2 = 0.5$ $r = 1, \theta_2 = 0.99$

$r = 5, \theta_2 = 0.01$ $r = 5, \theta_2 = 0.5$ $r = 5, \theta_2 = 0.99$

$r = 10, \theta_2 = 0.01$ $r = 10, \theta_2 = 0.5$ $r = 10, \ \theta_2 = 0.9$

Fig. 4.2. *Hazard plot of concomitants of order statistics* $Y_{[r:n]}$

Fig. 4.1 displays the survival function for the ranks of order $r = 1, 5$ and 10. As seen in the figure, the survival function declines for the increasing values of lifetime y, and the rate of the decline becomes slower as r and θ values get larger. But for larger value of β survival rate declines faster after a certain point of constant rate.

As noticed from Fig 4.2 the hazard function has a declining shape for the increasing y values of the concomitant $Y_{[r:n]}$. But, for larger values of β_2 and θ_2 the hazard function increases at a constant rate , then takes an upright move The hazard rate decreases as we increase the rank of order r , but decreasing rate gets slower with larger values of r .

6. CONCLUSION

In this paper, we have developed bivariate exponentiated inverted Weibull distribution using Morgenstern system and studied some of its important properties. We obtained the distribution of concomitants of order statistics with it's various properties such as joint density of the concomitants of two order statistics, distribution of k^{th} moments of concomitants of order statistics, survival and hazard functions. The survival and hazard functions are provided in the tables for some selected parameters and values of the concerned variables. These functions are graphically presented according to different rank order values which showed a decreasing manner. The decreasing hazard rates have been found in many situations such as infant mortality, immunity, life of integrated circuit modules etc. Hence, this probability distribution has the potentiality of applications in life time modeling in reliability and survival analysis.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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