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EXISTENCE OF SOLUTIONS OF INTEGRO-FRACTIONAL DIFFERENTIAL EQUATION WHEN $\alpha \in (2,3]$ THROUGH FIXED POINT THEOREM

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Abstract: In this work, the existence and uniqueness theorems for integro-differential equation involving the Caputo fractional derivative are established under some sufficient conditions and example for application the results of the theorems are presented.

Keywords: fractional differential equations; Caputo fractional derivative; fixed point theorems; integro-differential equations.

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1. INTRODUCTION

Fractional order Differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Applications can be found in fields of control, porous media, eletromagnetic, etc. (see [9] [11] [12]). Some theoretical aspects on the existence and uniqueness of solutions of differential equations involving the fractional

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derivative have been considered by some authors [2,5 ,6, 8, 10, 13]. In [14] the authors study the boundary value problem for the nonlinear fractional differential equations involving Caputo fractional derivative by means Banach fixed point theorem and the Schauder fixed point theorem. M. Benchohra et al [1] deals with the existence of solutions for fractional differential equations with integral integral boundary conditions. For instance, the authors explored the existence and uniqueness of solution to the fractional differential equations in the frame of generalized Caputo fractional derivatives [3]. In this article, we are interesting in the solutions for integro-differential equations

$$^{c}D^{\alpha}y(t) = f\left(t, y(t), \int_0^t k\big(t, s, y(s)\big)ds\right), \quad t \in J = [0, T], \ 2 < \alpha \leq 3 \ , \qquad (1.1)$$

With boundary conditions

$$\beta y(0) + \gamma y(T) = \delta$$
$$\lambda y'(0) + \nu y'(T) = \mu \qquad (1.2)$$
$$a\, y''(0) + b y''(T) = d$$

where D^{α} is Caputo fractional derivative with $2 < \alpha \leq 3$, $\beta, \gamma, \delta, \lambda, \nu, \mu, a, b$ and d are constants and $f: J \times X \times X \to X$ and $k: \Delta \times X \to X$, $\Delta = \{(t, s): 0 \leq s \leq t \leq T\}$. Here $C = C(J, X)$ denotes the Banach space of continuous functions from J into X equipped the norm $\|.\|_{\infty}$, where $\|y\|_{\infty} = sup\{|y|: t \in J\}$. By the Banach fixed point theorem and Schauder fixed point theorem, we state and prove some existence and uniqueness results of the solution. An example is given to demonstrate the application of our main results.

2. PRELIMINARIES

The basic definitions, lemmas and theorem are given in the following

Definition 2.1 [12] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f: (0, \infty) \to R$ is defined

$$I_{0+}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds$$

Provided that this integral exists, where Γ is gamma function.

Definition 2.2 [7] The Caputo derivative of order $\alpha > 0$ for a function $f: (0, \infty) \to R$ is written as

$$^{c}D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds$$

where n=$[\alpha]$+1 and $[\alpha]$ denotes the integer part of α.

Lemma 2.3[1] Let $\alpha > 0$. Then the differential equation $^{c}D^{\alpha} f(t) = 0$ has solution

$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$,$c_i \in R, i = 0,1,2,..n-1,$

and $I^{\alpha}D^{\alpha}f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$

for some $c_i \in R, i = 0,1,2,..n-1,$ n=$[\alpha]$+1.

Theorem 2.4[4] (Schauder's Fixed point theorem). If K is closed , bounded and convex subset of a Banach space and the mapping $T: K \to K$ Is completely continuous, then T has a fixed point in K.

3. MAIN RESULTS

This section is devoted to study the existence of solutions for the boundary value problem (1.1)-(2.1).

Lemma 3.1 Let $2 < \alpha \leq 3$, $y(t) \in C(J, X)$. Then the boundary problem (1.1)-(2.1) has a unique solution, given by

$$\begin{aligned} y(t) = {} & c + pt + \frac{d\,t^2}{2(a+b)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\left(\tau, y(\tau), \int_0^\tau k\left(\tau, s, y(s)\right)ds\right)d\tau \\ & + \frac{1}{\Gamma(\alpha-2)} (w_1 T^2 + w_2(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2})) \int_0^T (T-\tau)^{\alpha-3} f\left(\tau, y(\tau), \int_0^\tau k\left(\tau, s, y(s)\right)ds\right)d\tau \\ & + \left(\frac{\gamma T}{(\beta+\gamma)} - t\right) \frac{w_3}{\Gamma(\alpha-1)} \int_0^T (T-\tau)^{\alpha-2} f\left(\tau, y(\tau), \int_0^\tau k\left(\tau, s, y(s)\right)ds\right)d\tau \\ & - \frac{\gamma}{(\beta+\gamma)\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1} f\left(\tau, y(\tau), \int_0^\tau k\left(\tau, s, y(s)\right)ds\right)d\tau \qquad (3.1) \end{aligned}$$

Where $w_1 = \frac{\nu b((\lambda+\nu)-2\gamma)}{2(a+b)(\beta+\gamma)(\lambda+\nu)}$, $w_2 = \frac{b}{(a+b)}$ and $w_3 = \frac{\nu}{(\lambda+\nu)}$,

$c = \frac{\delta}{(\beta+\gamma)} - \frac{\gamma\mu T}{(\beta+\gamma)(\lambda+\nu)} + \frac{\gamma\nu T^2 d}{(a+b)(\beta+\gamma)(\lambda+\nu)} - \frac{\gamma d T^2}{2(a+b)(\beta+\gamma)}$ and $p = \frac{\mu}{(\lambda+\nu)} - \frac{\nu T d}{(a+b)(\lambda+\nu)}$.

Proof: Assume that $y(t)$ is a solution of boundary value problem (1.1)-(1.2), then by using lemma 2.3, we get

$$y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau+c_0+c_1t+c_2t^2 \quad (3.2)$$

From first boundary condition of (1.2), we have

$$c_0=\frac{\delta}{(\beta+\gamma)}-\frac{\gamma}{(\beta+\gamma)}c_1T-\frac{\gamma}{(\beta+\gamma)}c_2T^2-$$
$$\gamma\frac{1}{\Gamma(\alpha)}\int_0^T(T-\tau)^{\alpha-1}f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau \quad (3.3)$$

Applying the second boundary condition of (1.2), we get

$$c_1=\frac{\mu}{(\lambda+\nu)}-\frac{2\nu c_2T}{(\lambda+\nu)}-\frac{\nu}{(\lambda+\nu)\Gamma(\alpha-1)}\int_0^T(T-\tau)^{\alpha-2}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau \quad (3.4)$$

Then from the last boundary condition of (1.2), we obtain

$$c_2=\frac{d}{2(a+b)}-\frac{b}{2(a+b)\Gamma(\alpha-2)}\int_0^T(T-\tau)^{\alpha-3}f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau \quad (3.5)$$

By substituting equation (3.5) into (3.4), we get

$$c_1=\frac{\mu}{(\lambda+\nu)}-\frac{2\nu Td}{2(a+b)(\lambda+\nu)}+\frac{2\nu Tb}{2(a+b)(\lambda+\nu)\Gamma(\alpha-2)}\int_0^T(T-\tau)^{\alpha-3}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$
$$-\frac{\nu}{(\lambda+\nu)\Gamma(\alpha=1)}\ \int_0^T(T-\tau)^{\alpha-2}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau \quad (3.6)$$

By substituting equation (3.5) and (3.6) into (3.3), we get

$$c_0=\frac{\delta}{(\beta+\gamma)}-\frac{\gamma\mu T}{(\beta+\gamma)(\lambda+\nu)}+\frac{\gamma\nu T^2d}{(a+b)(\beta+\gamma)(\lambda+\nu)}-\frac{\gamma\nu T^2b}{(a+b)(\beta+\gamma)(\lambda+\nu)\Gamma(\alpha-2)}\times$$
$$\int_0^T(T-\tau)^{\alpha-3}\ f\left(s,y(s),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$
$$+\frac{\gamma\nu T}{(\beta+\gamma)(\lambda+\nu)\Gamma(\alpha-1)}\int_0^T(T-\tau)^{\alpha-2}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau-\frac{\gamma dT^2}{2(a+b)(\beta+\gamma)}$$
$$+\frac{\gamma T^2b}{2(a+b)(\beta+\gamma)\Gamma(\alpha-2)}\times\ \int_0^T(T-\tau)^{\alpha-3}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau)$$
$$-\ \frac{\gamma}{(\beta+\gamma)\Gamma(\alpha)}\ \int_0^T(T-\tau)^{\alpha-1}f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau \quad (3.7)$$

Then from (3.2), the solution is

$$y(t)=c+pt+\frac{d\,t^2}{2(a+b)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$
$$+\ \frac{1}{\Gamma(\alpha-2)}(w_1T^2+w_2(\frac{\nu Tt}{(\lambda+\nu)}-\frac{t^2}{2}))\int_0^T(T-\tau)^{\alpha-3}\ \ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$

$$+\left(\frac{\gamma T}{(\beta+\gamma)}-t\right)\frac{w_3}{\Gamma(\alpha-1)}\int_0^T(T-\tau)^{\alpha-2}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$

$$-\frac{\gamma}{(\beta+\gamma)\Gamma(\alpha)}\int_0^T(T-\tau)^{\alpha-1}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$

Which complete the proof.

Lemma 3.2 The solution y(t) in equation (3.1) is satisfies the fractional differential equations(1.1) and the boundary conditions (1.2).

Theorem 3.3 Assume that

(H1) The function $f: J\times X\to X$ is a Lipschitz continuous function and there is a constant $L_1>0$ such that

$$|f(t,y_1,\overline{y_1})-f(t,y_2,\overline{y_2})|\le L_1(|y_1-y_2|+|\overline{y_1}-\overline{y_2}|)$$

for $t\in J$ and every $y_1,\overline{y_1},y_2,\overline{y_2}\in X$.

(H2) The function $k: D\times X\to X$ where $D=\{(t,s)\in J\times J: t\ge s\}$ is continuous mapping and there exists a positive constant L_2 such that

$$\left|\int_0^t[k(t,s,y_1)-k(t,s,y_2)]\,ds\right|\le L_2|y_1-y_2|\ ,\quad \text{for } t\in J \text{ and every } y_1,y_2\in X.$$

Then there exist a unique solution for the boundary value problem (1.1) and (1.2).

Proof. Consider the operator $H: C(J,X)\to C(J,X)$, defined by

$$Hy(t)=c+pt+\frac{d\,t^2}{2(a+b)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$

$$+\frac{1}{\Gamma(\alpha-2)}(w_1T^2+w_2(\frac{\nu Tt}{(\lambda+\nu)}-\frac{t^2}{2}))\int_0^T(T-\tau)^{\alpha-3}f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$

$$+\left(\frac{\gamma T}{(\beta+\gamma)}-t\right)\frac{w_3}{\Gamma(\alpha-1)}\int_0^T(T-\tau)^{\alpha-2}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau$$

$$-\frac{\gamma}{(\beta+\gamma)\Gamma(\alpha)}\int_0^T(T-\tau)^{\alpha-1}\ f\left(\tau,y(\tau),\int_0^\tau k\big(\tau,s,y(s)\big)ds\right)d\tau \qquad (3.8)$$

Clearly from Lemma 3.1 the fixed point of H is a solution to (1.1)-(1.2), Now we shall show that H is a contraction.

Let $x,y\in C(J,X)$, then for each $t\in J$, we have

$$|H(x)(t)-H(y)(t)|\le\Big[\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}+\frac{1}{\Gamma(\alpha-2)}\Big|w_1T^2+w_2(\frac{\nu Tt}{(\lambda+\nu)}-\frac{t^2}{2})\Big|\int_0^T(T-\tau)^{\alpha-3}$$

$$+\frac{\left|(\frac{\gamma T}{(\beta+\gamma)}-t)w_3\right|}{\Gamma(\alpha-1)}\int_0^T(T-\tau)^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\left|\frac{\gamma T}{(\beta+\gamma)}\right|\int_0^T(T-\tau)^{\alpha-1}\Big]\times$$

$$\left| f\left(\tau, x(\tau), \int_0^\tau k\left(\tau, s, x(s)\right)ds\right) - f\left(\tau, y(\tau), \int_0^\tau k\left(\tau, s, y(s)\right)ds\right)\right| d\tau$$

By the condition (H1), we have

$$|H(x)(t) - H(y)(t)| \leq \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} + \frac{1}{\Gamma(\alpha-2)}\left|w_1 T^2 + w_2\left(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\right)\right| \times \right.$$

$$\left.\int_0^T (T-\tau)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\left|\left(\frac{\gamma T}{(\beta+\gamma)} - t\right) w_3\right| \int_0^T (T-\tau)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\left|\frac{\gamma T}{(\beta+\gamma)}\right| \int_0^T (T-\tau)^{\alpha-1}\right] \times$$

$$L_1(|x(\tau) - y(\tau)| + \left|\int_0^\tau \left[k\left(\tau, s, x(s)\right) - k\left(\tau, s, y(s)\right)\right] ds\right|) d\tau$$

By the condition (H2), we get

$$|H(x)(t) - H(y)(t)| \leq \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} + \frac{1}{\Gamma(\alpha-2)}\left|w_1 T^2 + w_2\left(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\right)\right| \times \right.$$

$$\left.\int_0^T (T-\tau)^{\alpha-3} \left|\left(\frac{\gamma T}{(\beta+\gamma)} - t\right) w_3\right| \int_0^T (T-\tau)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\left|\frac{\gamma T}{(\beta+\gamma)}\right| \int_0^T (T-\tau)^{\alpha-1}\right] \times$$

$$L_1\,(1 + L_2)|x(\tau) - y(\tau)| d\tau$$

Therefore, we have

$$\|H(x)(t) - H(y)(t)\|_\infty \leq L_1\,(1 + L_2)\left[\frac{T^\alpha}{\Gamma(\alpha-1)}\left(|w|_1 + |w_2|\left(\frac{|\nu|}{|\lambda+\nu|} + \frac{1}{2}\right)\right) + \right.$$

$$\left.\frac{|w_3| T^\alpha}{\Gamma(\alpha)}\left(\frac{|\gamma|}{|\beta+\gamma|} + 1\right) + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|\gamma|}{|\beta+\gamma|}\right)\right] \quad \|x(t) - y(t)\|_\infty$$

$$\|H(x)(t) - H(y)(t)\|_\infty \leq Q\|x(t) - y(t)\|_\infty \qquad (3.9)$$

Where $Q < 1$, H is a contraction operator. As a consequence of the Banach Fixed point theorem, H has a fixed point which is the unique solution of the problem (1.1)-(1.2).

The proof is completed.

Now, we give an existence result based on the Schauder's fixed point theorem.

Theorem 3.4. Assume that

(H3) The function $f: J \times X \times X \to X$ is continuous.

(H4) There exist constants $M_1, M_2 > 0$, such that $M_1 = sup|f(s, 0, 0)|$ and $M_2 = sup|k(t, s, 0)|$, for each $t \in J$. Then the BVP (1.1)-(1.2) has at least one solution on J.

Proof. Schauder's Fixed point theorem is used to prove that H defined by (3.8) has a fixed point. The proof will be given in several steps.

Step 1: H is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,T],\mathbb{R})$. Then for each $t \in [0,T]$, we have

$$|H(y_n)(t) - H(y)(t)| \le \Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} + \frac{1}{\Gamma(\alpha-2)}\Big|w_1 T^2 + w_2\big(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\big)\Big|\int_0^T (T-\tau)^{\alpha-3}$$
$$+ \frac{1}{\Gamma(\alpha-1)}\Big|\Big(\frac{\gamma T}{(\beta+\gamma)} - t\Big) w_3\Big|\int_0^T (T-\tau)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\Big|\frac{\gamma T}{(\beta+\gamma)}\Big|\int_0^T (T-\tau)^{\alpha-1}\Big] \times$$
$$L_1\,(1+L_2)|y_n(\tau) - y(\tau)|d\tau$$

So that, we have

$$\|H(y_n)(t) - H(y)(t)\|_\infty \le \Big[\frac{T^\alpha}{\Gamma(\alpha-1)}\Big(|w|_1 + |w_2|\Big(\frac{|\nu|}{|\lambda+\nu|} + \frac{1}{2}\Big)\Big) + \frac{|w_3|T^\alpha}{\Gamma(\alpha)}\Big(\frac{|\gamma|}{|\beta+\gamma|} + 1\Big)$$
$$+ \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(1 + \frac{|\gamma|}{|\beta+\gamma|}\Big)\Big]\ \|y_n(t) - y(t)\|_\infty.$$

Therefore

$$\|H(y_n)(t) - H(y)(t)\|_\infty \le W\|y_n(t) - y(t)\|_\infty \qquad (3.10)$$

Then by Lebesgue dominated convergence theorem implies that $\|H(y_n)(t) - H(y)(t)\|_\infty \to 0$ as $n \to \infty$.

Step 2: H maps the bounded sets into the bounded sets in $C(J,X)$. For any $r > 0$, such that $B_r = \{y \in X: \|y\|_\infty \le r\}$. It is clear that B_r is a closed, convex subset of $C(J,X)$. Let $y \in C$, then for each $t \in [0,T]$, we have

$$|Hy(t)| \le |c| + |pt| + \Big|\frac{d\,t^2}{2(a+b)}\Big| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Big|f\Big(\tau, y(\tau), \int_0^\tau k\big(\tau,s,y(s)\big)ds\Big)\Big|\,d\tau +$$
$$\frac{1}{\Gamma(\alpha-2)}\Big|w_1 T^2 + w_2\Big(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\Big)\Big|\int_0^T (T-\tau)^{\alpha-3}\ \Big|f\Big(\tau, y(\tau), \int_0^\tau k\big(\tau,s,y(s)\big)ds\Big)\Big|\,d\tau +$$
$$\frac{1}{\Gamma(\alpha-1)}\Big|\big(\frac{\gamma T}{(\beta+\gamma)} - t\big)w_3\Big|\int_0^T (T-\tau)^{\alpha-2}\,\Big|f\Big(\tau, y(\tau), \int_0^\tau k\big(\tau,s,y(s)\big)ds\Big)\Big|d\tau +$$
$$\frac{1}{\Gamma(\alpha)}\Big|\frac{\gamma T}{(\beta+\gamma)}\Big|\int_0^T (T-\tau)^{\alpha-1}\Big|f\Big(\tau, y(\tau), \int_0^\tau k\big(\tau,s,y(s)\big)ds\Big)\Big|\,d\tau$$
$$\le |c| + |pt| + \Big|\frac{d\,t^2}{2(a+b)}\Big| + \Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} + \frac{1}{\Gamma(\alpha-2)}\Big|w_1 T^2 + w_2\Big(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\Big)\Big| \times$$
$$\int_0^T (T-\tau)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\Big|\Big(\frac{\gamma T}{(\beta+\gamma)} - t\Big)w_3\Big|\int_0^T (T-\tau)^{\alpha-2} + \frac{\Big|\frac{\gamma T}{(\beta+\gamma)}\Big|}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\Big] \times$$
$$\Big|f\Big(\tau, y(\tau), \int_0^\tau k\big(\tau,s,y(s)\big)ds\Big) - f(s,0,0) + f(s,0,0)\Big|\,d\tau$$

$$\leq \ |c| + |pt| + \left|\frac{d\,t^2}{2(a+b)}\right| + \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} + \frac{1}{\Gamma(\alpha-2)}\left|w_1 T^2 + w_2\left(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\right)\right| \times \right.$$

$$\left. \int_0^T (T-\tau)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\left|(\frac{\gamma T}{(\beta+\gamma)} - t)w_3\right| \int_0^T (T-\tau)^{\alpha-2} + \frac{\left|\frac{\gamma T}{(\beta+\gamma)}\right|}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\right] \times$$

$$\left(L_1|y| + L_1\left(\left|\int_0^\tau \left(k(\tau,s,y(s)) - k(\tau,0,0)\right) ds\right| + \int_0^\tau |k(\tau,0,0)| ds\right) + M_1\right)\ d\tau$$

Therefore, we get

$$\|Hy(t)\|_\infty \leq \ |c| + |p|T + \left|\frac{d}{2(a+b)}\right| T + (L_1(r + L_2 r + M_2 T) + M_1\,) \times$$

$$\left[\frac{T^\alpha}{\Gamma(\alpha-1)}\left(|w|_1 + |w_2|\left(\frac{|\nu|}{|\lambda+\nu|} + \frac{1}{2}\right)\right) + \frac{|w_3|T^\alpha}{\Gamma(\alpha)}\left(\frac{|\gamma|}{|\beta+\gamma|} + 1\right) + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|\gamma|}{|\beta+\gamma|}\right)\right] = \ell$$

Thus, $\|Hy(t)\|_\infty \leq \ell$, for some constant ℓ .

Step 3: H maps C into an equicontinuous set of $C(J,X)$. Let $y \in C$, then for each $t_1, t_2 \in [0,T]$, $t_1 < t_2$, then

$$|Hy(t_2) - Hy(t_1)| \leq |p|(t_2 - t_1) + \frac{d(\,t_2^2 - t_1^2)}{2(a+b)} + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1}] \times$$

$$\left|f\left(\tau, y(\tau), \int_0^\tau k(\tau,s,y(s))ds\right)\right| d\tau + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha-1}\left|f\left(\tau, y(\tau), \int_0^\tau k(\tau,s,y(s))ds\right)\right| d\tau$$

$$+ \frac{w_2}{\Gamma(\alpha-2)}\left(\frac{\nu T(t_2-t_1)}{(\lambda+\nu)} + \frac{(\,t_2^2 - t_1^2)}{2}\right)\int_0^T (T-\tau)^{\alpha-3}\left|\,f\left(\tau, y(\tau), \int_0^\tau k(\tau,s,y(s))ds\right)\right| d\tau$$

$$+ \frac{w_3(t_2-t_1)}{\Gamma(\alpha-1)}\int_0^T (T-\tau)^{\alpha-2}\left|\,f\left(\tau, y(\tau), \int_0^\tau k(\tau,s,y(s))ds\right)\right| d\tau$$

Since

$$\left|\,f\left(\tau, y(\tau), \int_0^\tau k(\tau,s,y(s))ds\right)\right| \leq L_1(|y(\tau)| + \left|\int_0^\tau \left(k(\tau,s,y(s)) - k(\tau,s,0) + k(\tau,s,0)\right) ds\right| + M_1$$

$$\leq L_1(|y(\tau)| + L_2|y(\tau)| + M_2 T) + M_1$$

Then, we get

$$\left\|f\left(\tau, y(\tau), \int_0^\tau k(\tau,s,y(s))ds\right)\right\|_\infty \leq L_1(r + L_2 r + M_2 T)\ + M_1$$

Therefore, we have

$$\|Hy(t_2) - Hy(t_1)\|_\infty \leq |p|(t_2 - t_1) + \left[\frac{|d|(\,t_2^2 - t_1^2)}{2|a+b|} + \frac{1}{\Gamma(\alpha+1)}(\,t_2^\alpha - \,t_2^\alpha) + \right.$$

$$\left. \frac{|w_2|T^{\alpha-2}}{\Gamma(\alpha-1)}\left(\frac{|\nu|T(t_2-t_1)}{|\lambda+\nu|} + \frac{(\,t_2^2 - t_1^2)}{2}\right) + \frac{|w_3|T^{\alpha-1}(t_2-t_1)}{\Gamma(\alpha)}\right](L_1(r + L_2 r + M_2 T) + M_1\,) \qquad (3.12)$$

At $t_1 \to t_2$, the right hand side of the above inequality tends to zero. By Arzela – Ascoli theorem H is completely continuous.

Step 4: A priori bound.

Let $\varepsilon = \{y \in C(J,X): y = \theta H y \; for\; some\; 0 < \theta < 1\}$, it shall be shown that the set is bounded.

Let $y \in \varepsilon$, then $y = \theta H y$ for some $0 < \rho < 1$. Thus for each $t \in J$, $y = \theta H y$

$$|Hy(t)| \leq \theta|c| + \theta|pt| + \theta\left|\frac{d\,t^2}{2(a+b)}\right| + \frac{\theta}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left| f\left(\tau, y(\tau), \int_0^\tau k\left(\tau,s,y(s)\right)ds\right)\right| d\tau +$$
$$\frac{\theta}{\Gamma(\alpha-2)}\left|w_1T^2 + w_2(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2})\right| \int_0^T (T-\tau)^{\alpha-3} \left| f\left(\tau, y(\tau), \int_0^\tau k\left(\tau,s,y(s)\right)ds\right)\right| d\tau +$$
$$\frac{\theta}{\Gamma(\alpha-1)}\left|\left(\frac{\gamma T}{(\beta+\gamma)} - t\right) w_3\right| \int_0^T (T-\tau)^{\alpha-2} \left| f\left(\tau, y(\tau), \int_0^\tau k\left(\tau,s,y(s)\right)ds\right)\right| d\tau +$$
$$\frac{\theta}{\Gamma(\alpha)}\left|\frac{\gamma T}{(\beta+\gamma)}\right| \int_0^T (T-\tau)^{\alpha-1}\left| f\left(\tau, y(\tau), \int_0^\tau k\left(\tau,s,y(s)\right)ds\right)\right| d\tau$$

By the step 2, we have

$$|Hy(t)| \leq |c| + |pt| + \left|\frac{d\,t^2}{2(a+b)}\right| + \left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} + \frac{1}{\Gamma(\alpha-2)}\left|w_1T^2 + w_2\left(\frac{\nu T t}{(\lambda+\nu)} - \frac{t^2}{2}\right)\right| \times\right.$$
$$\left.\int_0^T (T-\tau)^{\alpha-3} + \frac{1}{\Gamma(\alpha-1)}\left|\left(\frac{\gamma T}{(\beta+\gamma)} - t\right) w_3\right| \int_0^T (T-\tau)^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\left|\frac{\gamma T}{(\beta+\gamma)}\right| \int_0^T (T-\tau)^{\alpha-1}\right] \times$$
$$\left(L_1|y| + L_1\left(\left|\int_0^\tau \left(k\left(\tau,s,y(s)\right) - k(\tau,0,0)\right) ds\right| + \int_0^\tau |k(\tau,0,0)| ds\right) + M_1\right) d\tau$$

Thus for every $t \in J$,

$$\|Hy(t)\|_\infty \leq |c| + |p|T + \left|\frac{d}{2(a+b)}\right| T + (L_1(r + L_2 r + M_2 T) + M_1) \times$$
$$\left[\frac{T^\alpha}{\Gamma(\alpha-1)}\left(|w|_1 + |w_2|\left(\frac{|\nu|}{|\lambda+\nu|} + \frac{1}{2}\right)\right) + \frac{|w_3|T^\alpha}{\Gamma(\alpha)}\left(\frac{|\gamma|}{|\beta+\gamma|} + 1\right) + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|\gamma|}{|\beta+\gamma|}\right)\right] = G$$

This shows that the set ε is bounded. As a consequence of Schauder's fixed point theorem, H has a fixed point which is a solution of the problem (1.1)-(1.2).

4. EXAMPLE

Consider the fractional differential equation

$$D^{2.5}y(t) = \frac{y(t)}{3e^t} - \frac{1}{3} + \int_0^t \left(\frac{y(s)}{(t+1)(3+e^{-s}+t)} - \frac{1}{2}\right) ds, \quad t \in [0,1] \quad , \quad \alpha \in (2,3] \qquad (4.1)$$

With boundary conditions

$$\left.\begin{aligned} y(0)+0.2y(1) &= 1 \\ 2y'(0)+y'(1) &= 0 \\ y''(0)+0.1y''(1) &= 0 \end{aligned}\right\} \qquad (4.2)$$

$f\left(t, y(t), \int_0^t k\big(t,s,y(s)\big)ds\right) = \frac{y(t)}{3e^t} - \frac{1}{3} + \int_0^t \left(\frac{y(s)}{(t+1)(3+e^{-s}+t)} - \frac{1}{2}\right) ds$ *and*

$$\left|k\big(t,s,x(s)\big) - k\big(t,s,y(s)\big)\right| = \left|\frac{x(s)}{(t+1)(3+e^{-s}+t)} - \frac{y(s)}{(t+1)(3+e^{-s}+t)}\right| \le \frac{1}{4}|x-y|$$

$$\left|f\left(t, y(t), \int_0^t k\big(t,s,y(s)\big)ds\right) - f\left(t, y(t), \int_0^t k\big(t,s,y(s)\big)ds\right)\right| \le \frac{1}{3}|x-y| + \int_0^t \left|k\big(t,s,x(s)\big) - k\big(t,s,y(s)\big)\right| ds \le \frac{7}{12}|x-y|$$

From equation (4.1) and (4.2), we find

$\alpha = 2.5$, $T = 1$, $\beta = 1$, $\gamma = 0.2$, $\delta = 1$, $\lambda = 2$, $\nu = 1$, $\mu = 0$, $a = 1$, $b = 0.1$, $d = 0$

then we calculate

$L_1 = 0.5833$, $L_2 = 0.25$, $M_1 = 0.3333$, $M_2 = 0.5$, $w_1 = 0.0884$,

$w_2 = 0.0909$, $w_3 = 0.3333$,

$r = 1.9262$, $c = 0.8333$, $p = 0$, then we have

$$Q = \left[\frac{T^\alpha}{\Gamma(\alpha-1)}\left(|w|_1 + |w_2|\left(\frac{|\nu|}{|\lambda+\nu|} + \frac{1}{2}\right)\right) + \frac{|w_3|T^\alpha}{\Gamma(\alpha)}\left(\frac{|\gamma|}{|\beta+\gamma|} + 1\right) + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|\gamma|}{|\beta+\gamma|}\right)\right] \times L_1(1 + L_2) = 0.4215 < 1.$$

Then by Theorem 3.3 and Theorem 3.4, there exist a unique solution.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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