COMPARATIVE STUDY OF DST INTERPOLATION APPROACH OF FRACTIONAL ORDERS

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Abstract. In this paper, comparative study of DST interpolation approach of various order by using different fractional derivatives are presented. First the definition of different fractional order derivatives like Grunwald-Letnikov, Weyl's and Conformable are reviewed. Next, Fractional derivative of a discrete signal is determined after applying the DST interpolation approach. Next, the DST-IV method approach transfer function are obtained with the help of index-mapping technique. Lastly, some computative problems are discussed for checking the effectiveness of digital fractional order differentiators for design of proposed method using the integral square error formula. Error values of various fractional order derivatives have been presented in the form of table.

Keywords: digital fractional order differentiator; Grunwald-Letnikov fractional derivative; Weyl's fractional derivative; conformable fractional derivative; hanning window; discrete sine transform.

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1. INTRODUCTION

The concept of fractional derivative is not entirely new, G.W. Leibniz mentioned about it in a correspondence (1695) with L'Hospital. In the 19th century it has been systematically studied at different periods by Liouville (1832), Reimann (1853), N. Sonin,A.V.Letinkov, Laurent and Holmgreen etc. Preceding them are Euler (1730) and Lagrange(1772).

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Fractional order of differentiation are more mysterious because they have no obvious geometric interpretation [1]-[4]. This subject started becoming more popular when it was realized that, compared to Frick's laws of diffusion, leads to the derivatives and integrals with half order for calculating the certain electrochemical problem is more convenient and economical.

Digital fractional order based differentiator applications is applicable in biomedical signal processing, digital signature verification, sharpness of images in digital image processing, neural networks, collection of real-time data using cloud computing etc [4]-[10]. $D^{\beta} f(x) = d^{\beta} f(x)/dx^{\beta}$ is a βth derivative order for a function $f(x)$. If β takes positive integral value then we get ordinary derivatives, otherwise it is known as fractional order derivative such that $Re(\beta) > 0$.

In this section, design approach of differentiator will be discussed of non-integer order derivatives definitions as Grunwald-Letnikov, Weyl's and Conformable derivative. In section 2, the different derivatives definitions are discussed. In section 3, transfer function of DST-IV is determined using various non-integer derivatives further, we also determine the transfer function of DST-I, DST-II and DST-III [11][12]. In section 4, computative problems and comparative analysis is discussed and at last, conclusion discussed based on DST interpolation approach.

2. DEFINITIONS OF VARIOUS FRACTIONAL ORDER DERIVATIVES

2.1. Grunwald-Letnikov fractional derivative. Fractional derivative of a function $f(t)$ of order Re $(\beta) > 0$ using Grunwald-Letinkov definition.

$$D^{\beta}(f(t)) = \lim_{h \to 0} \sum_{k=0}^{\infty} \frac{(-1)^k \binom{\beta}{k} f(t-kh)}{h^{\beta}} \tag{1}$$

Where

$$\binom{\beta}{k} = \frac{\beta!}{k!(\beta-k)!} = \frac{\Gamma(\beta+1)}{\Gamma(k+1)\Gamma(\beta-k+1)}$$

The symbol of gamma function is denoted $\Gamma(\cdot)$

$$\Gamma(x) = \int_0^{\infty} e^{-t} \ t^{x-1} dt$$

Theory of gamma function is generalizing the factorial function of natural numbers.

$$D^\beta e^{at} = a^\beta e^{at} \tag{2}$$

$$D^\beta A sin(\omega t + \phi) = A\omega^\beta \sin(\omega t + \phi + \frac{\pi}{2}\beta) \tag{3}$$

$$D^\beta A cos(\omega t + \phi) = A\omega^\beta \cos(\omega t + \phi + \frac{\pi}{2}\beta) \tag{4}$$

2.2. Weyl's Fractional Derivative.

$$W^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_t^\infty (\xi - t)^{\beta-1} f(\xi) d\xi, \quad Re(\beta) > 0, \quad t > 0$$

So that $W^{-\beta} f(t)$ exists for all $f \in S$ and all β with $Re(\beta) > 0$. Where S is the class of all functions f which are infinitely differentiable everywhere.

$$\beta = n - v$$

$v > 0$ and the integer with smallest value is denoted by n and n always greater than v. If f is a function, not necessarily of class S, for which $W^{-\beta} f(t)$ exists and has n continuous derivatives; then we define $W^\beta f(t)$

$$W^\beta f(t) = E^n [W^{-v} f(t)] \tag{5}$$

W^β is represented by Weyl's derivative.

Where

$$E^n = (-1)^n \frac{d^n}{dt^n}$$

$$W^\beta e^{at} = a^\beta e^{at} \tag{6}$$

$$W^\beta A sin(\omega t + \phi) = A\omega^\beta \sin(\omega t + \phi - \frac{\pi}{2}\beta) \tag{7}$$

$$W^\beta A cos(\omega t + \phi) = A\omega^\beta \cos(\omega t + \phi - \frac{\pi}{2}\beta) \tag{8}$$

2.3. Conformable fractional derivative.
If g is a function $g : [0, \infty) \to R$ then the fractional order β Conformable derivative definition is

$$D^\beta(g(t)) = \lim_{h \to 0} \frac{g(t + ht^{1-\beta}) - g(t)}{h} \tag{9}$$

$t>0$ i.e. for all values of t, $\beta \in (0,1)$. If an condition exists for a function g which is β-differentiable with in the range $(0,b)$, with condition $b > 0$ and $\lim_{t \to 0^+} D^\beta(g(t))$. Its expression can be define as

$$D^\beta(g(0)) = \lim_{t \to 0^+} D^\beta(g(t))$$

If function g is β-differentiable then the conformable fractional derivative of some elementary function is

$$D^\beta e^{at} = at^{1-\beta} e^{at}, \quad a \in R \tag{10}$$

$$D^\beta A sin(\omega t + \phi) = A\omega t^{1-\beta} \sin(\omega t + \phi + \frac{\pi}{2}), \tag{11}$$

$$D^\beta A cos(\omega t + \phi) = A\omega t^{1-\beta} \cos(\omega t + \phi + \frac{\pi}{2}), \tag{12}$$

3. DESIGN METHOD FOR VARIOUS FRACTIONAL ORDER DERIVATIVES USING DST-IV

3.1. Design method for Grunwald-Letnikov fractional order derivative using DST-IV.
Suppose we have a signal $f(t)$ in continuous-time domain and signal $f(t)$ are sampled and converted into $f(0), f(1), \cdots, f(P-1)$ i.e. finite-time sequence/discrete-time sequence. Then DST-IV function is defined as

$$F(k) = \sum_{m=0}^{P-1} \sqrt{\frac{2}{P}} f(m) sin\left(\frac{\pi(m+\frac{1}{2})(k+\frac{1}{2})}{P}\right) \tag{13}$$

$$f(m) = \sum_{k=0}^{P-1} \sqrt{\frac{2}{P}} F(k) sin\left(\frac{\pi(m+\frac{1}{2})(k+\frac{1}{2})}{P}\right) \tag{14}$$

After putting the value eq.(13) into eq.(14), we get

$$f(m) = \sum_{k=0}^{P-1} \sqrt{\frac{2}{P}} \left[\sqrt{\frac{2}{P}} \sum_{n=0}^{P-1} f(n) sin\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{P}\right)\right] sin\left(\frac{\pi(m+\frac{1}{2})(k+\frac{1}{2})}{P}\right) \tag{15}$$

$$f(m) = \sum_{n=0}^{P-1}\sum_{k=0}^{P-1}\left\{\frac{2}{P}f(n)sin\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{P}\right)sin\left(\frac{\pi(m+\frac{1}{2})(k+\frac{1}{2})}{P}\right)\right\}$$

Put t in place m in the previous equation, here t represent continuous-time and m represent discrete-time.

$$f(t) = \sum_{n=0}^{P-1} f(n)b(n,t) \tag{16}$$

$f(t)$ is a interpolated signal for the continuous-time domain and $b(n,t)$ is a basis interpolated function

$$b(n,t) = \frac{2}{P}\sum_{k=0}^{P-1} sin\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{P}\right)sin\left(\frac{\pi(t+\frac{1}{2})(k+\frac{1}{2})}{P}\right) \tag{17}$$

Apply Grunwald-Letnikov fractional derivative definition of βth order on equation (16)

$$D^{\beta}f(t) = \sum_{n=0}^{P-1} f(n)\left[D^{\beta}b(n,t)\right] \tag{18}$$

From eq. 18 fractional derivative of basis interpolated function is

$$D^{\beta}b(n,t) = \frac{2}{P}\sum_{k=0}^{P-1}\left(\frac{\pi(k+\frac{1}{2})}{P}\right)^{\beta} sin\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{P}\right)sin\left(\frac{\pi(t+\frac{1}{2})(k+\frac{1}{2})}{P}+\frac{\pi\beta}{2}\right) \tag{19}$$

Putting the value of $D^{\beta}b(n,t)$ into the eq.(18)

$$D^{\beta}f(t) = \sum_{n=0}^{P-1} f(n)G_n(t) \tag{20}$$

$$G_n(t) = \frac{2}{P}\sum_{k=0}^{P-1}\left(\frac{\pi(k+\frac{1}{2})}{P}\right)^{\beta} sin\left(\frac{\pi(n+\frac{1}{2})(k+\frac{1}{2})}{P}\right)sin\left(\frac{\pi(t+\frac{1}{2})(k+\frac{1}{2})}{P}+\frac{\pi\beta}{2}\right) \tag{21}$$

The ideal frequency response of digital differentiator and its transfer function approximates equal

$$H_d(\omega) = (j\omega)^{\beta}e^{-j\omega I} \tag{22}$$

Delay value denotes by I and the equation of the FIR filters system function is

$$H(z) = \sum_{\upsilon=0}^{P-1} h(\upsilon)z^{-\upsilon} \tag{23}$$

When an input signal $u(m)$ is passed through a system FIR filter then its output generate $u(m-1), u(m-2), ..., u(m-P+1)$ samples with equal amount of delay in each of the input signal. The output of the FIR filter is

$$y(m) = \sum_{v=0}^{P-1} h(v)u(m-v) \tag{24}$$

The filter coefficients $h(v)$ is determined from the eq.(20), when $y(m)$ approximately equal to the $D^\beta u(m-I)$.

$$y(m) \approx D^\beta u(m-I) \tag{25}$$

For solving this problem an index mapping technique is used

$$\begin{bmatrix} u(m) = f(P-1) \\ u(m-1) = f(P-2) \\ \vdots \\ u(m-P+1) = f(0) \end{bmatrix} \tag{26}$$

The eq.(26) can be simplified after linking eq.(20) and eq.(24)

$$f(n) = u(m-(P-1)+n) \quad 0 \leqslant n \leqslant P-1 \tag{27}$$

Equate $f(t) = u(m-(P-1)+t)$ into eq.(20)

$$D^\beta u(m-(P-1)+t) = \sum_{n=0}^{P-1} u(m-(P-1)+n)G_n(t) \tag{28}$$

$$h(v) = G_{P-1-v}(P-1-I) \tag{29}$$

FIR filter coefficients is determined after equating eq.(21) into (29),

$$h(v) = \frac{2}{P} \sum_{k=0}^{P-1} \left(\frac{(k+\frac{1}{2})\pi}{P} \right)^\beta sin \left(\frac{\pi(P-v-\frac{1}{2})(k+\frac{1}{2})}{P} \right) sin \left(\frac{\pi(P-I-\frac{1}{2})(k+\frac{1}{2})}{P} + \frac{\pi\beta}{2} \right) \tag{30}$$

With the help of window techniques, we can modify the coefficients of FIR filter. So, in this paper we are using Hanning window and it's transfer function is defined below as

$$w(v) = 0.5 - 0.5 \ cos \left(\frac{2\pi v}{P-1} \right) \tag{31}$$

Modified coefficients of FIR filter using window techniques is

$$h_w(v) = h(v)w(v) \tag{32}$$

The system performance of the digital fractional order differentiator can be evaluated for DST-IV method with the help of integral error squares formula in frequency domain.

$$E = \sqrt{\int_0^{\lambda_1 \pi} |H(e^{j\omega}) - H_d(\omega)|^2 d\omega} \tag{33}$$

Above expression in term of E is used for checking performance of designing approach of digital fractional order differentiator.

3.2. Design method for Weyl's fractional derivative using DST-IV.

Similarly for DST-IV method the system transfer function using Weyl's fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} \left(\frac{(k+\frac{1}{2})\pi}{P} \right)^{\beta} sin\left(\frac{\pi(P-\upsilon-\frac{1}{2})(k+\frac{1}{2})}{P} \right) sin\left(\frac{\pi(P-I-\frac{1}{2})(k+\frac{1}{2})}{P} - \frac{\pi\beta}{2} \right) \tag{34}$$

With the help of window techniques, we can modify the coefficients of FIR filter. In this paper we are using Hanning window

$$w(\upsilon) = 0.5 - 0.5 \ cos\left(\frac{2\pi\upsilon}{P-1} \right) \tag{35}$$

Modified coefficients of FIR filter using window techniques is

$$h_w(\upsilon) = h(\upsilon)w(\upsilon) \tag{36}$$

3.3. Design method For Conformable fractional derivative using DST-IV.

Similarly for DST-IV method the system transfer function using Conformable fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} \left(\frac{\pi(P-1-I)(k+\frac{1}{2})}{P} \right)^{1-\beta} sin\left(\frac{\pi(P-\upsilon-\frac{1}{2})(k+\frac{1}{2})}{P} \right) sin\left(\frac{\pi(P-I-\frac{1}{2})(k+\frac{1}{2})}{P} + \frac{\pi}{2} \right) \tag{37}$$

With the help of window techniques, we can modify the coefficients of FIR filter.In this paper we are using Hanning window and it's transfer function is defined below as

$$w(\upsilon) = 0.5 - 0.5 \ cos\left(\frac{2\pi\upsilon}{P-1} \right) \tag{38}$$

Modified coefficients of FIR filter using window techniques is

$$h_w(\upsilon) = h(\upsilon)w(\upsilon) \tag{39}$$

4. DESIGN EXAMPLES

Example 1: For the proposed design method error calculated for differentiators with varying fractional order using DST-IV method are given in the below table.The digital fractional order differentiator performance evaluated with the help of eq. (33). It is given in terms of frequency response of DST-IV approach and measured for various differentiators with the help of error size E.Minimum value of E means that the performance of proposed design method using DST-IV DFOD perform well. The optimum design values are selected as $P = 100$, $I = 50$, $\lambda_1 = 0.9$, orders $\beta = 0.3, 0.5, 0.7, 0.9$ for digital fractional order differentiator (DFOD).

Orders	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
$E_{Grunwald}$	0.0159	0.0087	0.0034	8.0863×10^{-4}
$E_{Weyl's}$	0.1245	0.2378	0.3688	0.5032
$E_{Conformable}$	3.0838	1.0645	0.2742	0.1049

With the help of above given error table, we can determine which digital fractional order differentiator (DFOD) will be suited for our proposed mehtod.In DST-IV case, the order from onward $\beta = 0.3$ to $\beta = 0.9$ the size of error for Grunwald-Letinkov based digital fractional order differentiator are smaller than Conformable and Weyl's based digital fractional order differentiator.Size of error for fractional order onward $\beta = 0.7$ to $\beta = 0.9$ Conformable based DFOD is smaller than the Weyl's based DFOD.

Fig. 1,2,3,4 indicates the frequency response i.e. the magnitude and phase response of $H(z)$ for order $\beta = 0.3, 0.5, 0.7, 0.9$. For the ideal case the magnitude response is w^{β} and phase response is 90β. The normalized phase response for the conventional method is $90 * [angle(H(e^{j\omega})) + \omega I]/0.5\pi$. Error determine with the help of eq.(33).
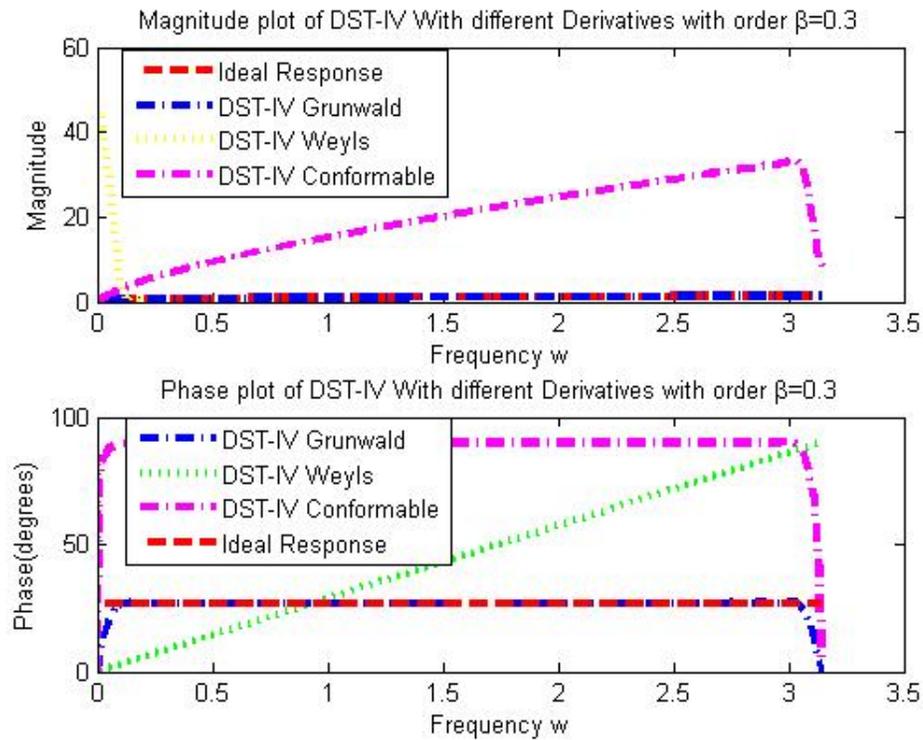
Fig. 1: The result of the proposed design method using window technique for DST-IV based DFOD with order $\beta = 0.3$.The error is shown with the help of magnitude and phase response.For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response.In the Phase response graph, it shows that the Ideal response and proposed design method response i.e. DST-IV Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
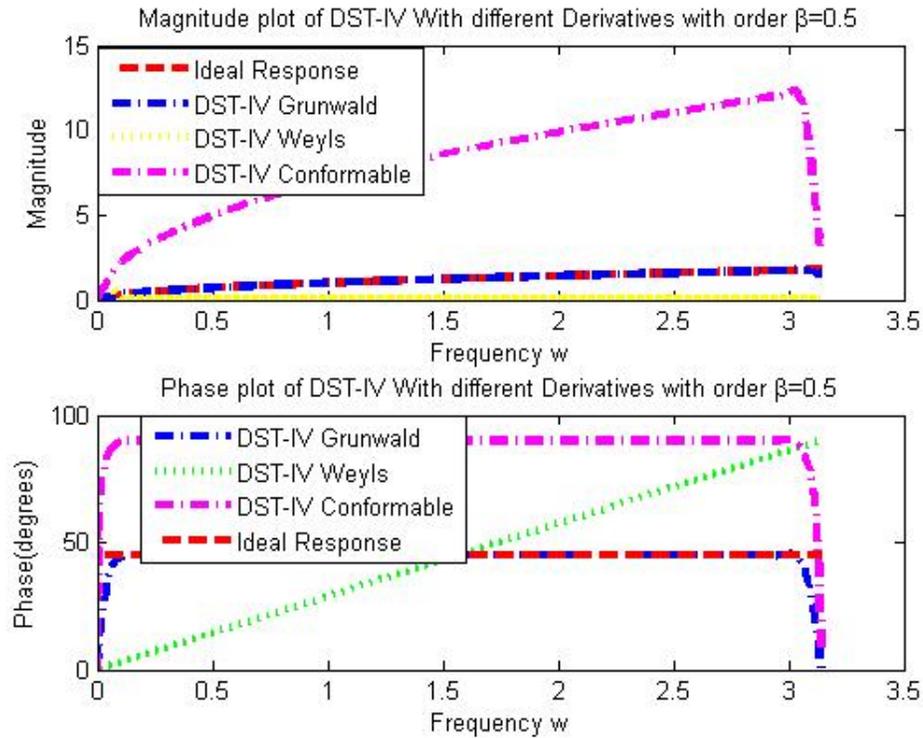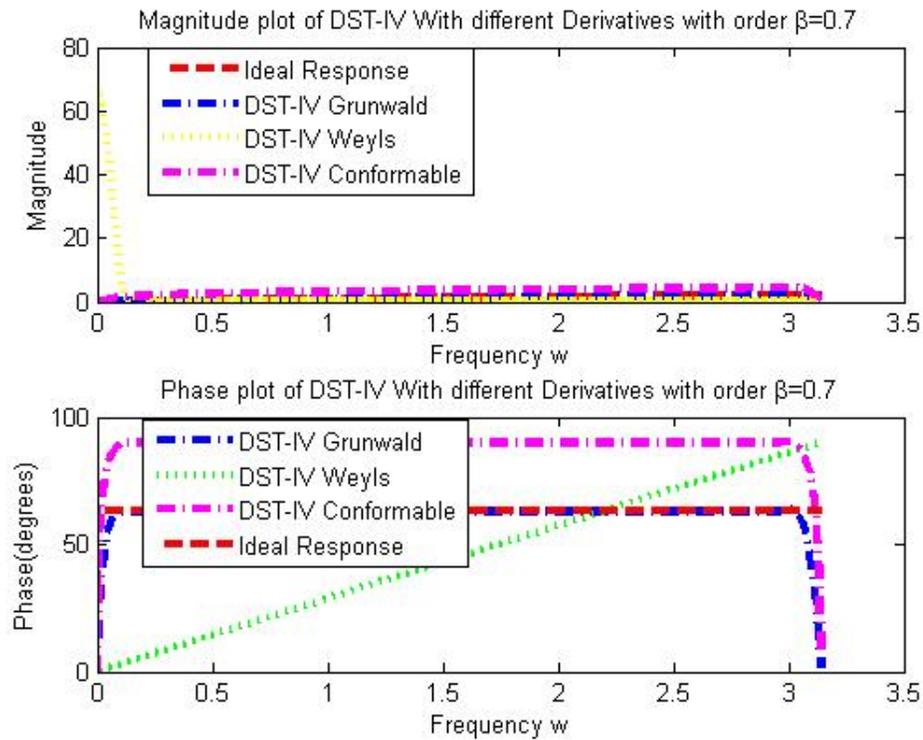
Fig. 2: The result of the proposed design methods using window technique for DST-IV based DFOD with order $\beta = 0.5$.For magnitude-frequency graph Grunwald based DFOD aproximately same as the ideal response.In the Phase response graph,it shows that the Ideal response and proposed design method response i.e. DST-IV Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.

Fig. 3: The result of the proposed design methods using window technique, DST-IV based DFOD with order $\beta = 0.7$. In the magnitude response graph Grunwald DFOD aproximately same as the ideal response. While Conformable and Weyl's DFOD littel bit closer to the ideal response.In the Phase response graph,it shows that the Ideal response and proposed design method response i.e. DST-IV Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
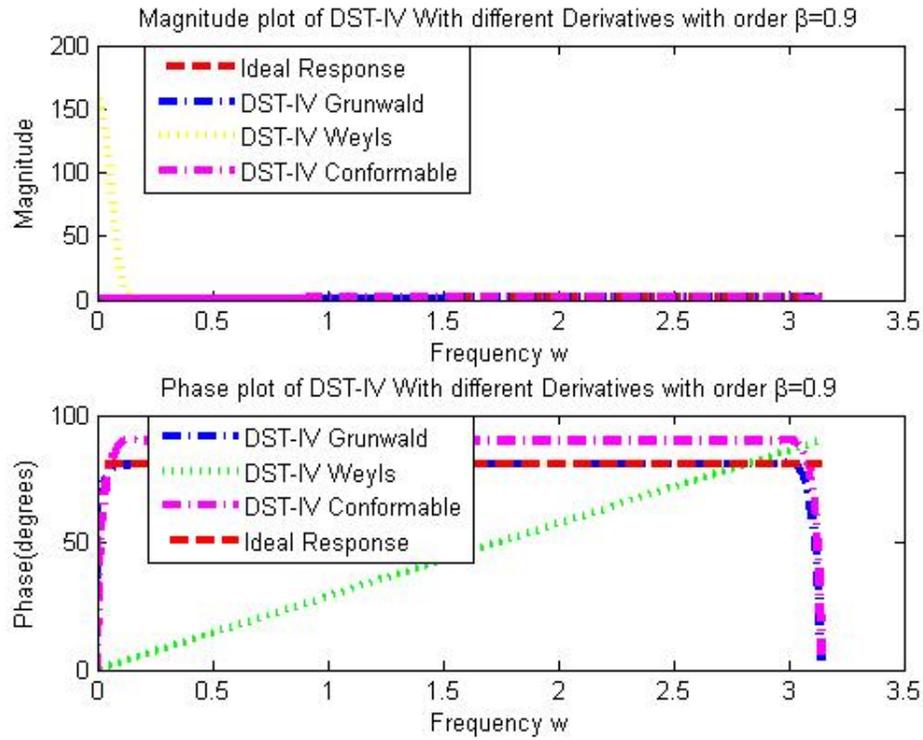
Fig. 4: The result of the proposed design methods using window technique, DST-IV based DFOD for order $\beta = 0.9$. In the magnitude response graph all the DFOD are close to the ideal response. In the Phase response graph,it shows that the Ideal response and proposed design method response i.e. DST-IV Grunwald DFOD approximately the same at near region $\omega = \pi$ and Conformable DFOD littel bit closer to Ideal response.

Example 2: The frequency response of the proposed design approach for various DFOD using DST-III method are shown in this example.Performance of fractional order differentiator evaluated using DST-III method with the help of integral square error formula. It is given in terms of frequency response.

DST-III method system transfer function is given below:

$$F(k) = \sum_{m=0}^{P-1} \sqrt{\frac{2}{P}} c_m f(m) sin\left(\frac{\pi(k+\frac{1}{2})(m+1)}{P}\right) \tag{40}$$

$$f(m) = \sum_{k=0}^{P-1} \sqrt{\frac{2}{P}} c_m F(k) sin\left(\frac{\pi(k+\frac{1}{2})(m+1)}{P}\right) \tag{41}$$

$$c_m = \begin{cases} \dfrac{1}{\sqrt{2}} & m = P - 1 \\ \\ 1 & otherwise \end{cases}$$

The transfer function of DST-III using Grunwald-Letnikov fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} c_{P-1-\upsilon} c_{P-1-I} \left(\frac{\pi(k+\frac{1}{2})}{P} \right)^{\beta} sin \left(\frac{\pi(k+\frac{1}{2})(P-\upsilon)}{P} \right) sin \left(\frac{\pi(k+\frac{1}{2})(P-I)}{P} + \frac{\pi\beta}{2} \right) \quad (42)$$

The transfer function of DST-III using Weyl's fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} c_{P-1-\upsilon} c_{P-1-I} \left(\frac{\pi(k+\frac{1}{2})}{P} \right)^{\beta} sin \left(\frac{\pi(k+\frac{1}{2})(P-\upsilon)}{P} \right) sin \left(\frac{\pi(k+\frac{1}{2})(P-I)}{P} - \frac{\pi\beta}{2} \right) \quad (43)$$

The transfer function of DST-III using Conformable fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} c_{P-1-\upsilon} c_{P-1-I} \left(\frac{\pi(k+\frac{1}{2})(P-1-I)}{P} \right)^{1-\beta} sin \left(\frac{\pi(k+\frac{1}{2})(P-\upsilon)}{P} \right) sin \left(\frac{\pi(k+\frac{1}{2})(P-I)}{P} + \frac{\pi}{2} \right) \quad (44)$$

Performance of DST-III based DFOD evaluated with the help of integral square error formula of eq.(33) and error measured for DST-III based approach. Effectiveness of DST-III approach can be measured for various differentiators with the help of size of error E. The Optimum design values are selected as $P = 100, I = 50, \lambda_1 = 0.9$, orders $\beta = 0.3, 0.5, 0.7, 0.9$ for the different fractional derivatives.

Orders	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
$E_{Grunwald}$	0.0159	0.0087	0.0034	8.0090×10^{-4}
$E_{Weyl's}$	0.1247	0.2379	0.3688	0.5032
$E_{Conformable}$	3.0927	1.1645	0.2742	0.1050

With the help of above given error table, we can determine which digital fractional order differentiator (DFOD) will be suited for our proposed mehtod. In DST-III case, the order from onward $\beta = 0.3$ to $\beta = 0.9$ the size of error for Grunwald-Letinkov based digital fractional order differentiator are smaller than Conformable and Weyl's based digital fractional order differentiator. Size of error for fractional order onward $\beta = 0.7$ to $\beta = 0.9$ Conformable based DFOD is smaller than the weyl's based DFOD. Fig. 5,6,7,8 indicates the frequency response

i.e. the magnitude and phase response of $H(z)$. For the ideal case the magnitude response is w^β and phase response is 90β. The normalized phase response for the conventional method is $90 * [angle(H(e^{j\omega})) + \omega I]/0.5\pi$.
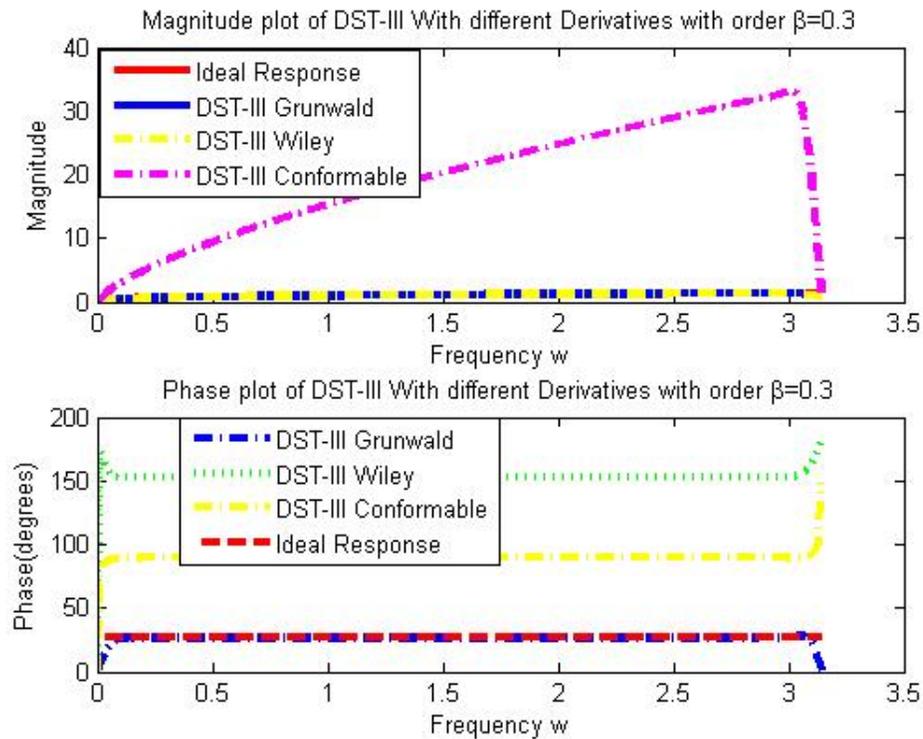

Fig. 5: The result of the proposed design method using window technique for DST-III based DFOD with order $\beta = 0.3$.The error is shown with the help of magnitude and phase response.For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response.In the Phase response graph, it shows that the Ideal response and proposed design method response i.e. DST-III Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
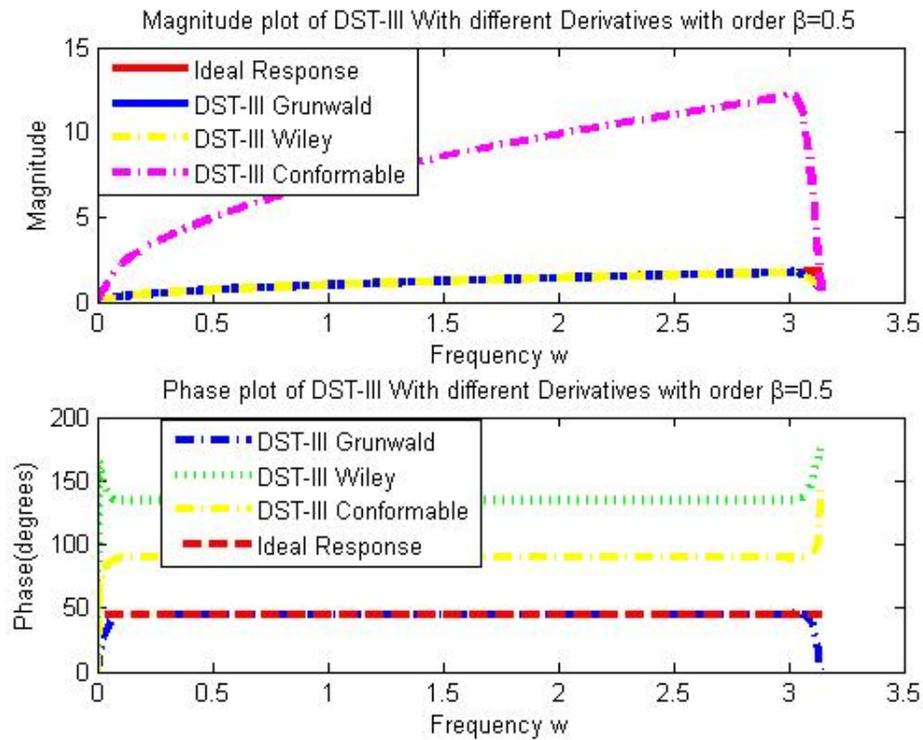
Fig. 6 : The result of the proposed design methods using window technique for DST-III based DFOD with order $\beta = 0.5$.For magnitude-frequency graph Grunwald based DFOD aproximately same as the ideal response and Weyl's DFOD also close to Ideal response.In the Phase response graph the Ideal response and proposed design method response i.e. DST-III Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
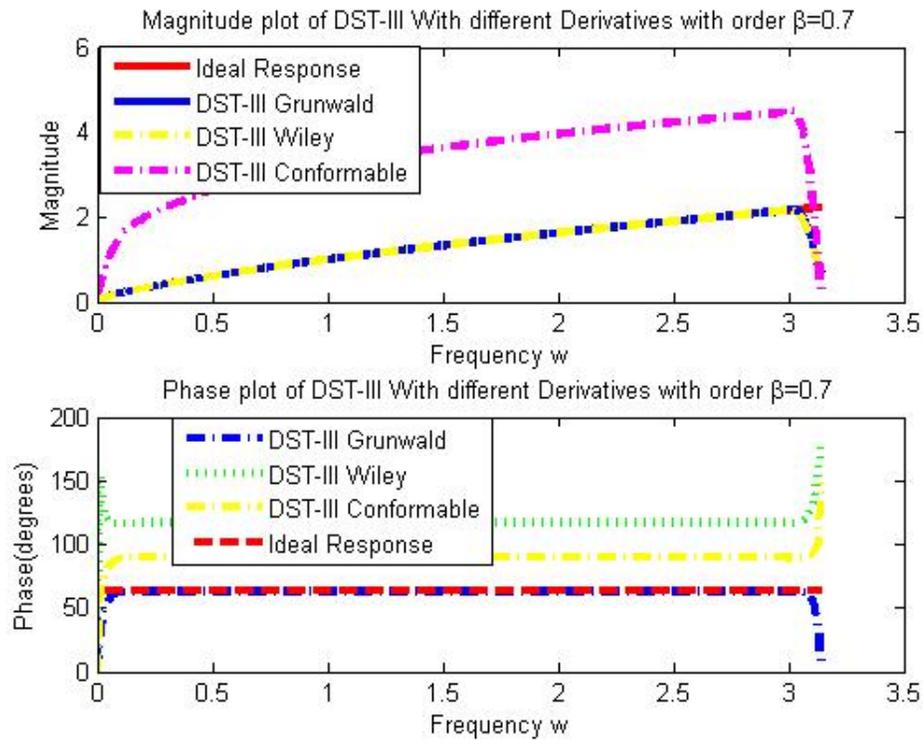
Fig. 7 : The result of the proposed design methods using window technique for DST-III based DFOD with order $\beta = 0.7$. For magnitude-frequency graph Grunwald based DFOD aproximately same as the ideal response and Weyl's DFOD also close to Ideal response. In the Phase response graph the Ideal response and proposed design method response i.e. DST-III Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
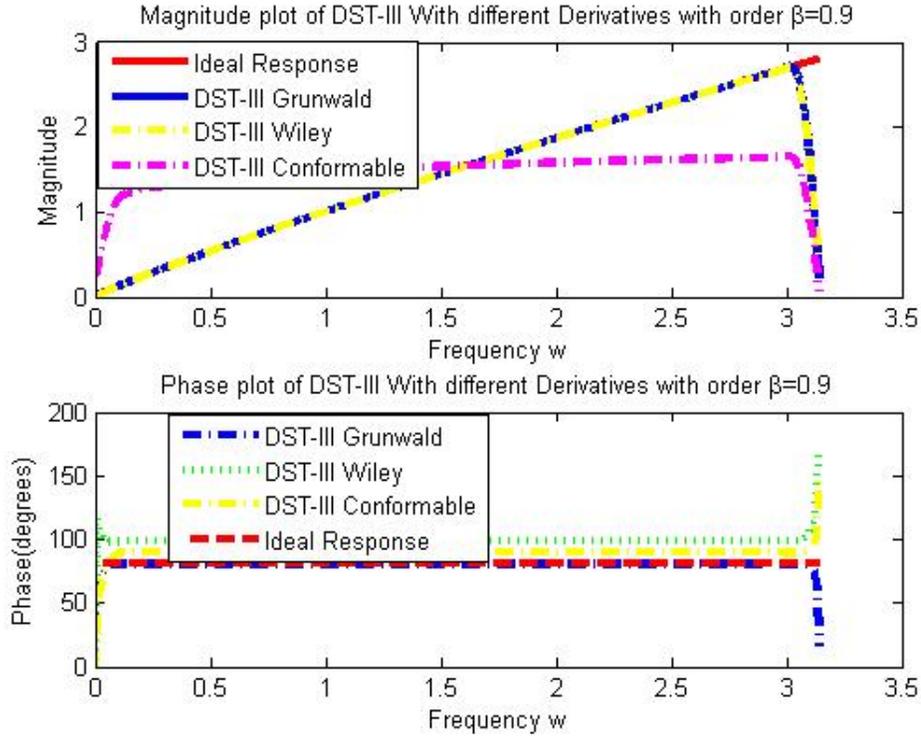
Fig. 8: The result of the proposed design methods using window technique for DST-III based DFOD with order $\beta = 0.9$. For magnitude-frequency graph Grunwald based DFOD aproximately same as the ideal response and Weyl's DFOD also close to Ideal response.In the Phase response graph the Ideal response and proposed design method response i.e. DST-III Grunwald DFOD approximately the same at near region $\omega = \pi$ and other DFOD littel bit closer to ideal response.

Example 3: The frequency response of the proposed design approach for various DFOD using DST-II method are shown in this example.Performance of fractional order differentiator evaluated using DST-II method with the help of integral square error formula. It is given in terms of frequency response.

DST-II method system transfer function is given below

$$F(k) = \sum_{m=0}^{P-1} \sqrt{\frac{2}{P}} c_m f(m) sin\left(\frac{\pi(m+\frac{1}{2})(k+1)}{P}\right) \qquad (45)$$

$$f(m) = \sum_{k=0}^{P-1} \sqrt{\frac{2}{P}} c_m F(k) sin \left(\frac{\pi(m+\frac{1}{2})(k+1)}{P} \right) \qquad (46)$$

$$c_m = \begin{cases} \dfrac{1}{\sqrt{2}} & m = P-1 \\[2mm] 1 & otherwise \end{cases}$$

The transfer function of DST-II using Grunwald-Letnikov fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} c_m^2 \left(\frac{\pi(k+1)}{P} \right)^\beta sin \left(\frac{\pi(P-\upsilon-\frac{1}{2})(k+1)}{P} \right) sin \left(\frac{\pi(P-I-\frac{1}{2})(k+1)}{P} + \frac{\pi\beta}{2} \right) \quad (47)$$

The transfer function of DST-II using Weyl's fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} c_m^2 \left(\frac{\pi(k+1)}{P} \right)^\beta sin \left(\frac{\pi(P-\upsilon-\frac{1}{2})(k+1)}{P} \right) sin \left(\frac{\pi(P-I-\frac{1}{2})(k+1)}{P} - \frac{\pi\beta}{2} \right) \quad (48)$$

The transfer function of DST-II using Conformable fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} c_m^2 \left(\frac{\pi(P-1-I)(k+1)}{P} \right)^{1-\beta} sin \left(\frac{\pi(P-\upsilon-\frac{1}{2})(k+1)}{P} \right) sin \left(\frac{\pi(P-I-\frac{1}{2})(k+1)}{P} + \frac{\pi}{2} \right) \quad (49)$$

Performance of DST-II based DFOD evaluated with the help of integral square error formula and error measured for DST-II based approach. Effectiveness of DST-II approach can be measured for various differentiators with the help of size of error E.The Optimum design values are selected as $P = 100, I = 50, \lambda_1 = 0.9$, orders $\beta = 0.3, 0.5, 0.7, 0.9$ for the different fractional derivatives.

Orders	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
$E_{Grunwald}$	0.0148	0.0081	0.0033	8.5137×10^{-4}
$E_{Weyl's}$	0.1246	0.2377	0.3686	0.5029
$E_{Conformable}$	3.0899	1.0630	0.6305	0.0989

On the basis of above given error table, we can determine which DFOD will be suited for our proposed design mehtod.In DST-II case, the order from onward $\beta = 0.3$ to $\beta = 0.9$ the size of error for Grunwald-Letinkov based DFOD are smaller than other DFOD.For order $\beta = 0.7$ size of error for weyl's DFOD is smaaler than Conformanble DFOD, while for order $\beta = 0.9$ size of error for Conformable based DFOD is smaller than the weyl's based DFOD.Fig. 9,10,11 and 12 shows the frequency response.
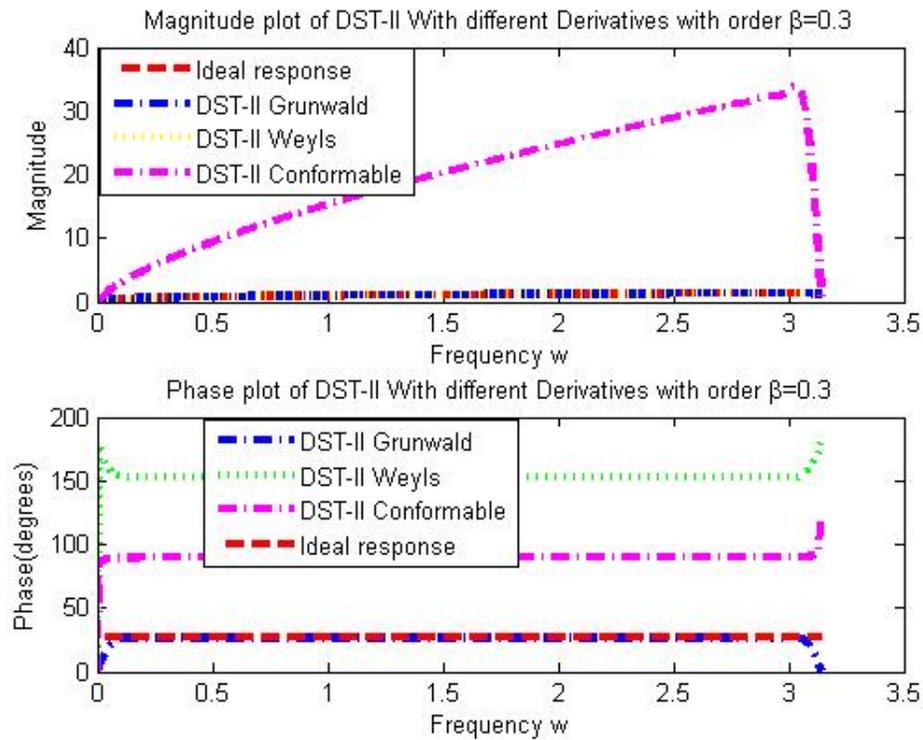
Fig. 9 : The result of the proposed design method using window technique for DST-II based DFOD with order $\beta = 0.3$. For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response. In the Phase response graph, the Ideal response and proposed design method response i.e. DST-II Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
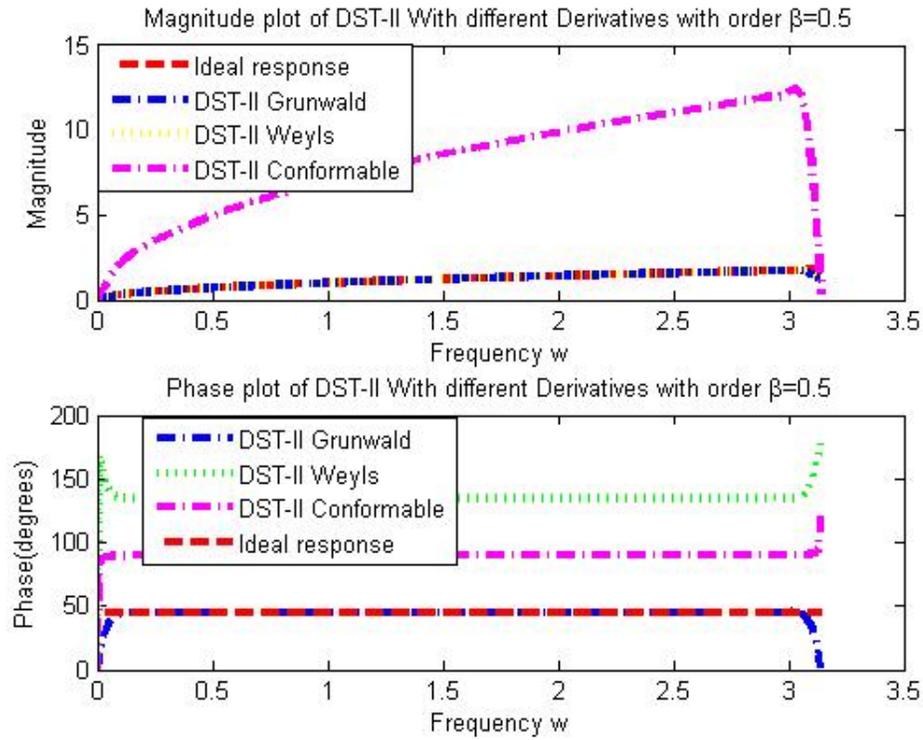
Fig. 10 : The result of the proposed design method using window technique for DST-II based DFOD with order $\beta = 0.5$.For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response.In the Phase response graph, the Ideal response and proposed design method response i.e. DST-II Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
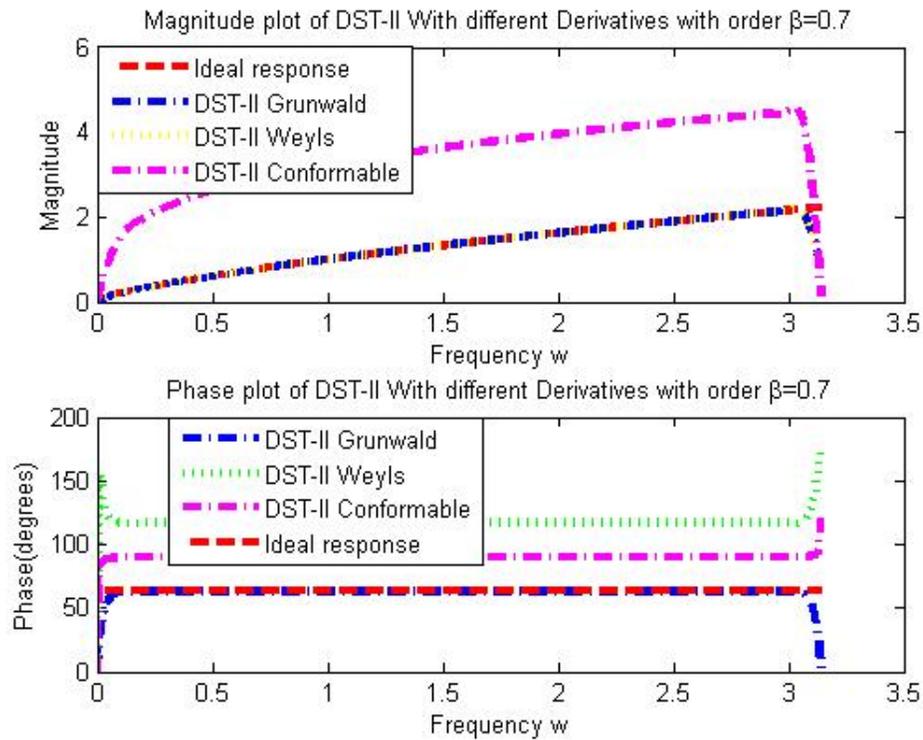
Fig. 11 :The result of the proposed design method using window technique for DST-II based DFOD with order $\beta = 0.7$.For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response.In the Phase response graph,the Ideal response and proposed design method response i.e. DST-II Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it show huge error.
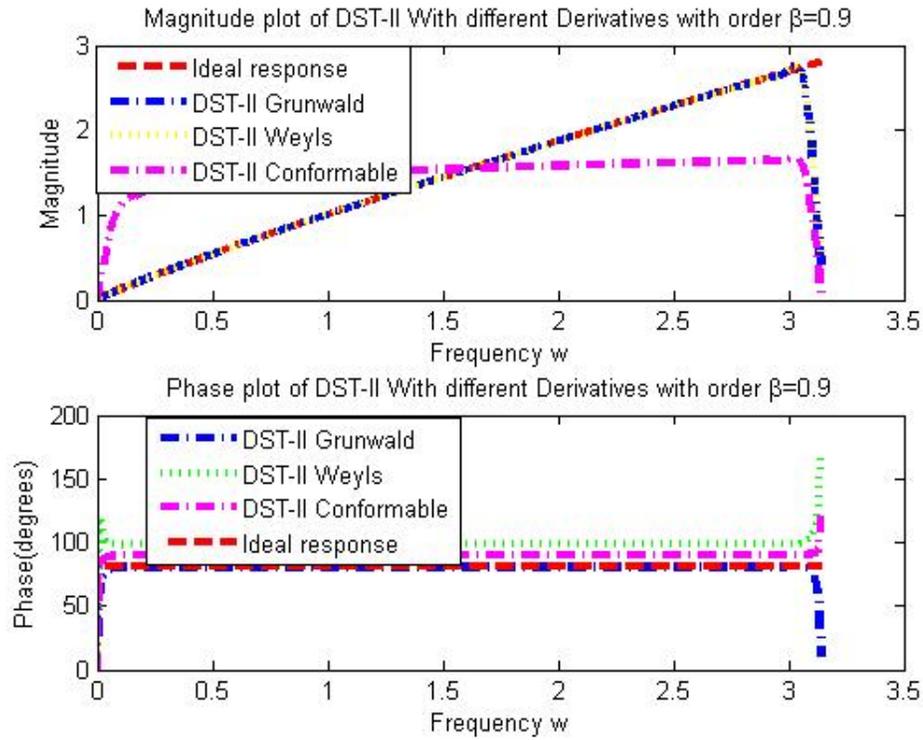
Fig. 12 : The result of the proposed design methods using window technique for DST-II based DFOD with order $\beta = 0.9$. For magnitude-frequency graph Grunwald based DFOD aproximately same as the ideal response and Weyl's DFOD also close to Ideal response.In the Phase response graph the Ideal response and proposed design method response i.e. DST-II Grunwald DFOD approximately the same at near region $\omega = \pi$ and other DFOD littel bit closer to ideal response.

Example 4: The frequency response of the proposed design approach for DFOD using DST-I method are shown in this example. The performance of differentiators evaluated using integral square formula in term of frequency.

DST-I method system transfer function is given below

$$F(k) = \sum_{m=0}^{P-1} \sqrt{\frac{2}{P+1}} f(m) sin\left(\frac{\pi(m+1)(k+1)}{P+1}\right) \tag{50}$$

$$f(m) = \sum_{k=0}^{P-1} \sqrt{\frac{2}{P+1}} F(k) sin\left(\frac{\pi(m+1)(k+1)}{P+1}\right) \tag{51}$$

The transfer function of DST-I using Grunwald-Letnikov fractional order derivative is

$$h(\upsilon) = \frac{2}{P+1} \sum_{k=0}^{P-1} \left(\frac{\pi(k+1)}{P+1} \right)^{\beta} sin \left(\frac{\pi(P-\upsilon)(k+1)}{P+1} \right) sin \left(\frac{\pi(P-I)(k+1)}{P+1} + \frac{\pi\beta}{2} \right) \quad (52)$$

The transfer function of DST-I using Weyl's fractional order derivative is

$$h(\upsilon) = \frac{2}{P+1} \sum_{k=0}^{P-1} \left(\frac{\pi(k+1)}{P+1} \right)^{\beta} sin \left(\frac{\pi(P-\upsilon)(k+1)}{P+1} \right) sin \left(\frac{\pi(P-I)(k+1)}{P+1} - \frac{\pi\beta}{2} \right) \quad (53)$$

The transfer function of DST-I using Conformable fractional order derivative is

$$h(\upsilon) = \frac{2}{P} \sum_{k=0}^{P-1} \left(\frac{\pi(P-1-I)(k+1)}{P+1} \right)^{1-\beta} sin \left(\frac{\pi(P-\upsilon)(k+1)}{P+1} \right) sin \left(\frac{\pi(P-I)(k+1)}{P+1} + \frac{\pi}{2} \right) \quad (54)$$

Effectiveness of DST-I approach can be measured for various differentiators with the help of size of error E. The Optimum design values are selected as $P = 100, I = 50, \lambda_1 = 0.9$, orders $\beta = 0.3, 0.5, 0.7, 0.9$ for DFOD.

Orders	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
$E_{Grunwald}$	0.0147	0.0131	0.0033	7.426×10^{-4}
$E_{Weyl's}$	0.1247	0.2382	0.3689	0.4183
$E_{Conformable}$	3.950	1.0650	0.0965	0.0985

On the basis of above given error table, we can determine which DFOD will be suited for our proposed design mehtod. In DST-I case, the order from onward $\beta = 0.3$ to $\beta = 0.9$ the size of error for Grunwald-Letinkov based DFOD are smaller than other DFOD. Onwards $\beta = 0.7$ to $\beta = 0.9$ size of error for Conformable based DFOD is smaller than the weyl's based DFOD. Fig. 13,14,15,16 shows the frequency response.
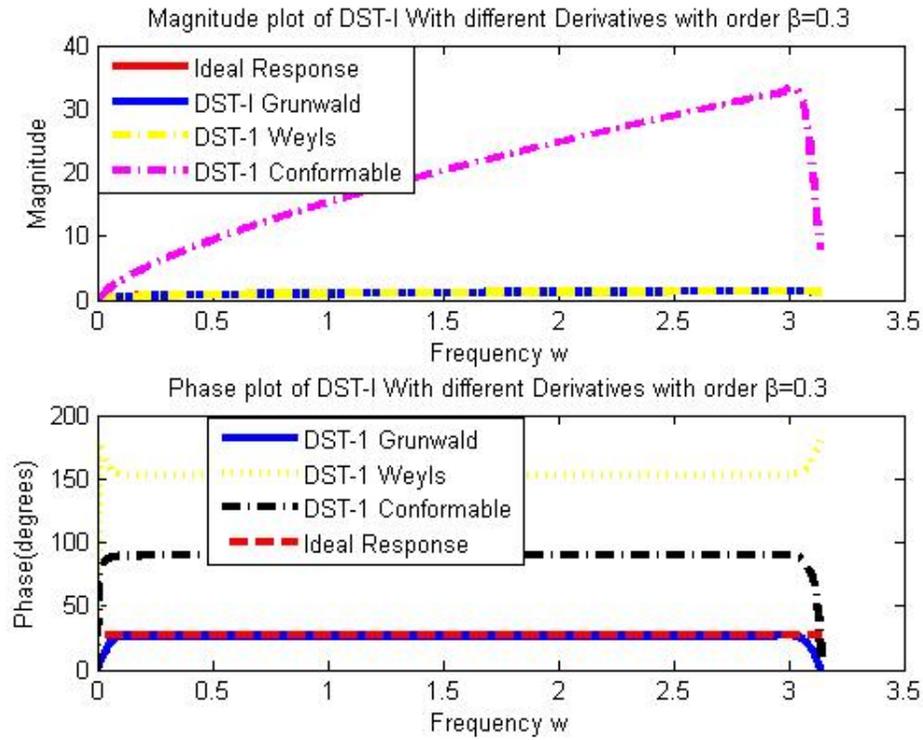
Fig. 13 :The result of the proposed design method using window technique for DST-I based DFOD for order $\beta = 0.3$.For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response. In the Phase response graph,the Ideal response and proposed design method response i.e. DST-I Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it is showing huge error.
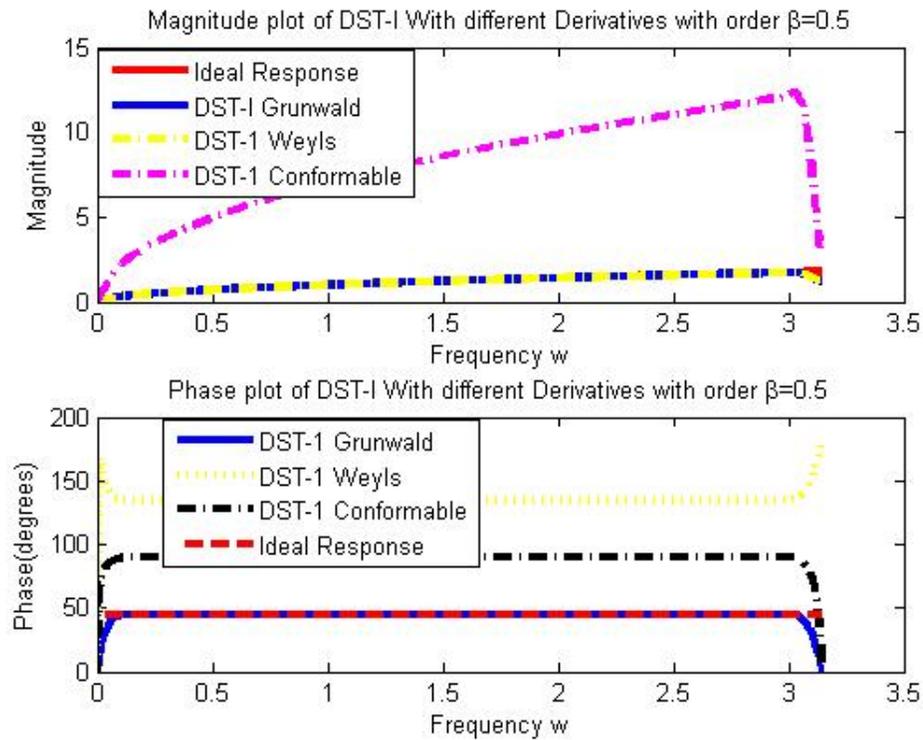
Fig. 14 : The result of the proposed design method using window technique for DST-I based DFOD for order $\beta = 0.5$.For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response. In the Phase response graph,the Ideal response and proposed design method response i.e. DST-I Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it is showing huge error.
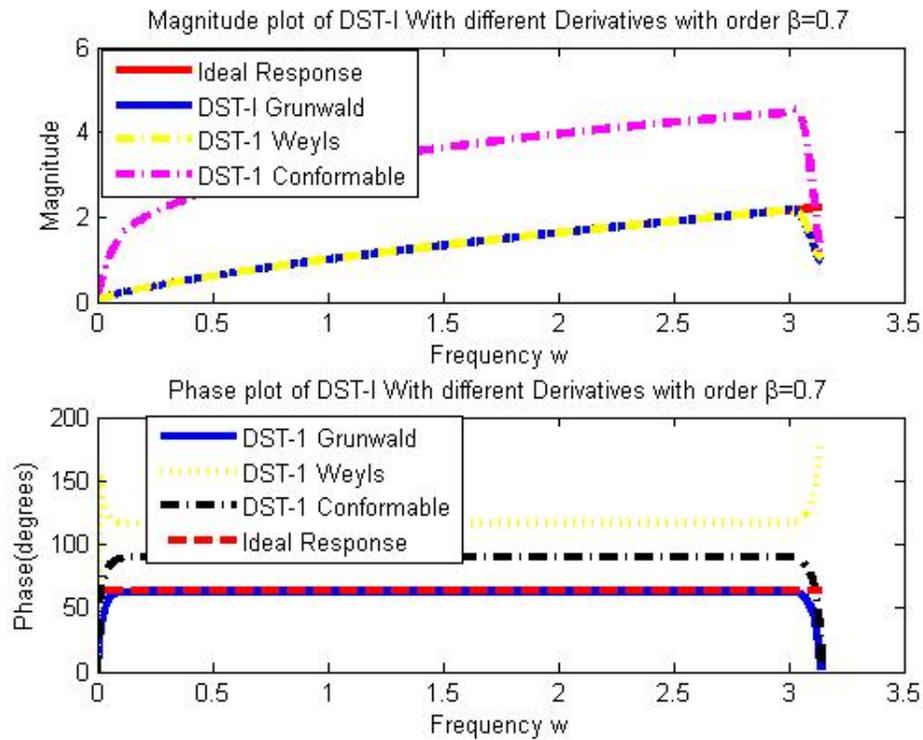
Fig. 15 : The result of the proposed design method using window technique for DST-I based DFOD for order $\beta = 0.7$. For magnitude-frequency graph Grunwald and Weyl's DFOD aproximately same as the ideal response. In the Phase response graph, the Ideal response and proposed design method response i.e. DST-I Grunwald DFOD approximately the same at near region $\omega = \pi$ for other DFOD it is showing huge error.
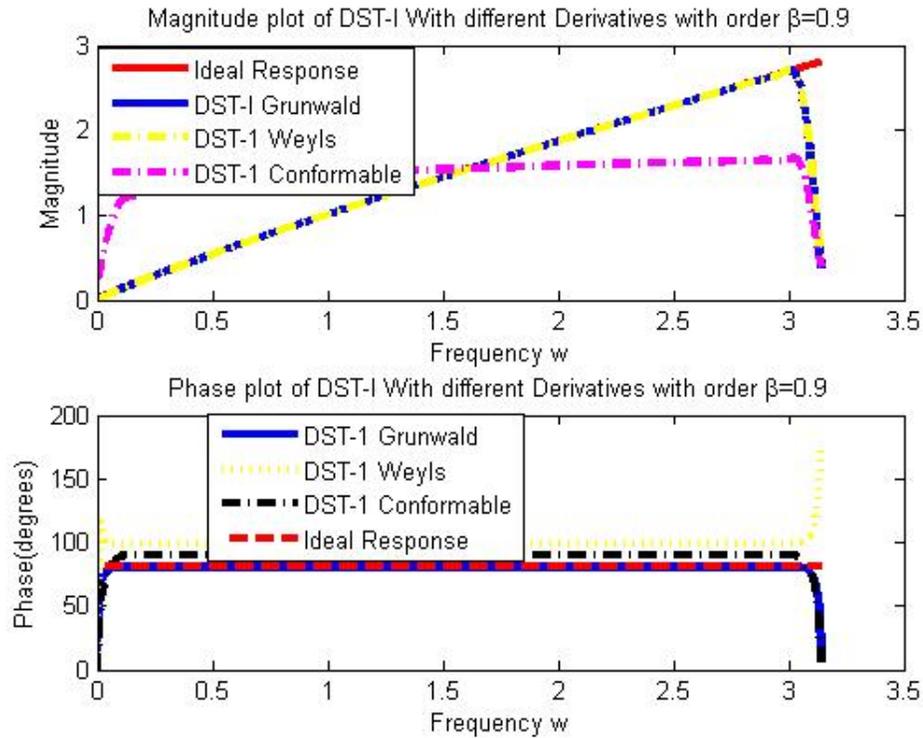
Fig. 16 :The result of the proposed design methods using window technique for DST-I based DFOD with order $\beta = 0.9$.For magnitude-frequency graph Grunwald based DFOD aproximately same as the ideal response and Weyl's DFOD also close to Ideal response.In the Phase response graph the Ideal response and proposed design method response i.e. DST-I Grunwald DFOD approximately the same at near region $\omega = \pi$ and other DFOD littel bit closer to ideal response.

5. CONCLUSION

In this paper, comparative analysis of DSTs interpolation approach of Grunwald-Letinkov, Weyl's and Conformable DFOD with respect to ideal response are presented. On the basis of computative problems we conclude that, Grunwald-Letinkov FOD with DST-IV is well suited for optimal design values $P = 100$, $I = 50$, $\lambda_1 = 0.9$. Weyl's DFOD perform better than Conformable DFOD in case of DST-I,DST-II, DST-III and DST-IV.For order $\beta = 0.9$, the size of error for Conformables DFOD is smaller than Weyls DFOD in case of all DST aprroach. In future our interest is to design a digital fractional order differentiators for other fractional

derivatives with DCTs/DSTs. We are also interested in extending interpolation approach to multidimensional DCTs/DSTs.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

[2] K. S. Miller and R. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[3] I. Podlubny, Fractional Calculus An Introduction For Physicists, 2nd Edition, World Scientific, 2014.

[4] M.D. Ortigueira, Fractional Calculus for Scientists and Engineers, Springer, 2011.

[5] R. Herrmann, Fractional Calculus: An Introduction for Physicists, World Scientific, 2011.

[6] B. Mathieu, P. Melchior, A. oustaloup, C. Ceyral, Fractional differentiation for edge detection, Signal Process. 83 (11) (2003) 2421-2432.

[7] J. Bai, X.C. Feng, Fractional-order anisotropic diffusion for image denoising, IEEE Trans. Image Process. 16 (10) (2007) 2492-2502

[8] C. C. Tseng, Design of variable and adaptive fractional order differentiators, Signal Process. 86 (10) (2006) 1317-1327.

[9] H.O. Peitgen, H. Jurgens, D. Saupe, Chaos and Fractals, New Frontiers of science, 2nd ed. Springer, 2004.

[10] T.I. Laakso, V. Valimaki, M. Karjalainen, U.K. Laine, Splitting the unit delay [FIR/all pass filters design], IEEE Signal Process. Mag. 13 (1996) 30–60.

[11] H. Pratap, D. Singh, N. Kumar, A. Singh, Designing approach of digital fractional order hilbert transformer using DST-II method, 2nd IEEE International Conference on Power Electronics, Intelligent Control and Energy systems, ICPEICES-2018.

[12] D. Singh, N. Kumar, H. Pratap, H. Kushwah, Discrete cosine transform interpolation approach to design an fractional order hilbert transformer, IEEE International Conference on Advance in Computing, Communication Control and Networking, ICACCCN-2018.

[13] J. Lee, C. Un, Performance of transform-domain LMS adaptive digital filters, IEEE Trans. Acoust. Speech Signal Process. 34 (1986) 499–510.

[14] Z. Wang, G.A. Jullien, W.C. Miller, Interpolation using the discrete sine transform with increased accuracy, Electron. Lett. 29 (1993) 1918-1920.

[15] E.G. Pelaes, Y. Iano, Image coding using discrete sine transform with axis rotation, IEEE Trans. Consumer Electron. 44 (1998) 1284–1290.

[16] X. Li, H. Xie, B. Cheng, Noisy Speech Enhancement Based on Discrete Sine Transform, in: First International Multi-Symposiums on Computer and Computational Sciences (IMSCCS'06), IEEE, Hangzhou, Zhejiang, China, 2006: pp. 199–202.

[17] H. Zhao, Q. Ran, G. Ge, J. Ma, L. Tan, Image Encryption Based on Random Fractional Discrete Cosine and Sine Transforms, in: 2009 First International Workshop on Education Technology and Computer Science, IEEE, Wuhan, Hubei, China, 2009: pp. 804–808.

[18] R. Khalil, M. AL Horani, A. Yousef, M. Sababheh, A new Definitions of Fractional Derivative, J. Comput. Appl. Math. 264 (2014), 65-70.

[19] U.N. Katugampola, A New Fractional Derivative with Classical Properties, ArXiv:1410.6535 [Math]. (2014).

[20] I. Podlubny, Fractional Calculus An Introduction For Physicists, 2nd Edition, World Scientific, 2014.

[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.