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# BIFURCATION AND CHAOS MEASURE IN SOME DISCRETE DYNAMICAL SYSTEMS

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**Abstract.** Almost all natural systems have certain nonlinear properties and display ergodic and chaotic behavior during evolution when the set of parameters of such systems assume a critical set of values. So, while studying nonlinear systems with proper justification, mathematical analysis and computational skills are needed to identify the nature of chaos and the evolutionary property of any such system.

In the present work, some discrete nonlinear models have been considered and computational techniques such as bifurcation diagrams, Lyapunov exponents, correlation dimension, topological entropy etc. have been used to identify regular and chaotic motion. The results obtained are displayed through various interesting graphics. The work also incorporates the concept of fractals and the properties of fractals. A correlation between fractals and chaos have also been discussed with proper justification.

**Keywords:** bifurcation, chaos, Lyapunov exponents, correlation dimension, topological entropy.

**2000 AMS Subject Classification:** 34C23, 34F10, 34H20, 34C28, 34D08, 37B40

## 1. Introduction:

Appearance of chaos is now a well accepted phenomenon and is observed in numerous nonlinear systems. It was discovered by Poincaré (1913), while studying the motion of a particle in Sun-Earth-Moon system. Poincaré observed the system's sensitivity to initial condition which is now termed as chaos. Chaos can occur only in nonlinear systems during evolution either due to

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sensitivity to initial conditions or due to sensitivity to a parameter of the system. Sensitivity to initial conditions implies two trajectories originating at nearby states diverge rapidly after a short amount of time.

Study of chaos in nonlinear dynamical systems is a subject of applied mathematics. The chaos theory is now applied to several disciplines including physics, economics, atmospheric science, biology and medical sciences, philosophy etc. Chaos in a system is a state when the system shows sensitivity to the initial conditions i.e. *a very small difference in the initial conditions produce a divergence in behavior*. In such a situation, the deterministic nature of such system does not make them predictable. The unpredictable or chaotic behavior of the system can be displayed through graphics like time series graph, phase plot, Poincaré map, power spectrum etc. Other powerful indicators which efficiently provide the measures of regular and chaotic motion are Lyapunov exponents and topological entropy.

The natural systems are mostly nonlinear and chaotic behavior can be observed in many of them viz. chaos in market, chaotic population explosion, epidemics, chaos in weather, social chaos etc. In the state of chaos, prediction becomes impossible, and we have fortuitous phenomenon. Since Poincaré, numerous articles are written on chaos and chaos control. Some of the pioneer articles in this direction are those of Lorenz (1963), Sharkovskii (1964), Smale (1967), May (1976), Feigenbaum (1978), Devany (1989 ), Chirikov (1979), Grassberger and Procaccia (1983), Moon (1987), Gleick (1987), Stewart (1989), Mandelbrot (1983), Hao Bai-Lin (1984), Henon (1976) and many others.

The objective of the present work is to investigate the evolutional properties of certain real systems represented by one dimensional discrete mathematical models. The regular and chaotic motion observed through bifurcation phenomena by varying certain parameter of the system. The graphics are presented for bifurcations, Lyapunov exponents, topological entropies and plots of correlation integrals data for each system. Certain Mathematica codes are generated and used for numerical calculations.

**2.Descriptions of Lyapunov exponents, Correlation Dimensions and Topological Entropies:**

(a) ***Lyapunov Exponents***:

The Lyapunov exponent, (or Lyapunov characteristic exponent LCE), provides an average measure of exponential divergence of two orbits initiated with infinitesimal separation. The largest eigenvalue of a complex dynamical system is an indicator of chaos ( Saha and Budhraja ,2007).

Consider two orbits initiated at $x_0$ and $y_0$ with $x_0, y_0 \in [0, 1]$, of a one dimension map

$$f : [0, 1] \to [0,1]$$

such that $| x_0 - y_0 | << 1$ , then assuming $| x_n - y_n | << 1$, where $x_n = f^n(x_0)$, $y_n = f^n(y_0)$, are respectively the $n^{th}$ iterations of $x_0$ and $y_0$ under f, by Taylor's theorem, one can obtains

$$| x_n - y_n | \approx \prod_{t=0}^{n-1} |f'(t)||x_0 - y_0|. \qquad (2.1)$$

Then, the exponential separation rate $\log | f'(x) |$ of two nearby initial conditions, averaged over the entire trajectory, can be given by

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log(\prod_{t=0}^{n-1} |f'(x_t)|, \qquad (2.2)$$

where $\prod_{t=0}^{n-1} |f'(x_t)| \approx e^{\lambda(x_0)n}$, for n >> 1

and this implies

$$|x_n - y_n| \approx e^{\lambda(x_0)n}|x_0 - y_0| \qquad (2.3)$$

We can generalize the above one dimensional case to higher dimensions and obtain

$$\lambda(X_0, U_0) = \lim_{n\to\infty} \frac{1}{n} log \parallel \prod_{t=0}^{n-1} J(X_t)U_0 \parallel, \qquad (2.4)$$

and

$$\parallel X_n - Y_n \parallel \approx e^{\lambda(X_0,U_0)n},$$

where $X \in \mathbb{R}^n$, $F: \mathbb{R}^n \to \mathbb{R}^n$ , $U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F.

Quantitatively, two trajectories in phase space with initial separation δx(0) diverges as:

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \tag{2.5}$$

,where $\lambda > 0$ is the Lyapunov exponent.

The system described by the map $f$ be *regular* as long as $\lambda \leq 0$ and *chaotic* when $\lambda > 0$.

**(b) *Topological Entropy*:**

The usefulness of Lyapunov exponents are limited because of the following important features, Gribble (1995):

- Lyapunov exponents are local in nature and are not necessarily constant throughout the evolution and so ergodicity is also required to characterize chaos.
- As per their definitions, Lyapunov exponents are time dependent and this leads to a serious drawback for systems arising from relativistic considerations.

A chaotic attractor is composed of a complex pattern. To investigate chaotic behavior in a wide variety of systems evolving with time, an alternate replacement of Lyapunov exponents which could be more reliable and acceptable as indicator is the topological entropy (Balmforth et. Al., (1964), Adler et. Al., (1965), Bowen (1970), Boyarsky et. Al., (1991) and Iwai (1998)). Topological entropy describes the *rate of mixing* of a dynamical system. It has a relationship to both Lyapunov exponents, through the dependence of rate, and to the ergodicity, because of the association of mixing. For a system having non-zero topological entropy, the rate of mixing must be exponential which is reminiscent of Lyapunov exponent. But such exponentiality of mixing is not relative to time, but rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. A mathematical definition of topological entropy can be obtained from the book by Nagashima and Baba, (2005).

Topological entropy $h(f)$ for a map $f$ defined in a close interval **I** = [a, b], is closely related to Li - Yorke chaos[ Nagashima and Baba (2005)], and measures the complexity of the map $f$.

If $f$ be a continuous map from **I** to **I** and if α be an open initial cover of **I**, then the topological entropy $h(f)$ can be described by the supremum, $\sup h(\alpha, f)$, , for all the covers of interval **I** such that

$$h(\alpha, f) = \lim_{n\to\infty} \frac{1}{n} logN(\vee_{i=0}^{n-1} f^{-1}\alpha) \quad . \qquad (2.6)$$

The topological entropy $h(f)$ is thus given by

$$h(f) = \sup h(\alpha, f) \ . \qquad (2.7)$$

When the map $f$ is piecewise-monotonic over I, the topological entropy can be determined by the lap number, lap($f^n$) of the iterated map $f^n$ as follows :

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log lap(f^n). \qquad (2.8)$$

The lap number of $f$ grows with n in general. If the growth obeys the power law,

$$lap(f^n) \sim k\, n^{\alpha},$$

then by (2.8),

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(kn^{\alpha}) = \lim_{n\to\infty} \frac{\alpha}{n} \log n = 0 \qquad (2.9)$$

However, if it grow exponentially,$lap(f^n) \sim k\alpha^n$ , ($\alpha > 1$), then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(k\alpha^n) = \log \alpha \qquad (2.10)$$

This shows that $h(f)$ is determined by the way $lap\ (f^n)$ increases.

In case of superstable periodic orbits, the method of structure matrix can be employed. For take the case of logistic map $f(x) = \mu x(1-x)$ when $\mu \approx 3.960270$, one can obtain the structure matrix **M** (cf.Nagashima and Baba,2005) and then find out the largest eigenvalue, $\lambda_{max}$ of **M**. Then, the topological entropy can be obtained as

$$h = \log(\lambda_{max}) \qquad (2.11)$$

(c) ***Correlation Dimensions***:

As stated, chaos may exist in nonlinear systems during evolution and that can be seen easily by observing the bifurcation diagrams. A chaotic set, an strange attractor, has fractal structure. Correlation dimension gives a *measure of dimensionality* of the chaotic set. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension we use statistical method. It is a very practical and efficient method then other methods, like box counting etc. The procedure to obtain correlation dimension follows the following steps, Martelli (1999):

Consider an orbit $O(\mathbf{x}_1) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots\ldots\}$, of a map $f: U \rightarrow U$, where U is an open bounded set in $\measuredangle^n$. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral, Grassberger and Procaccia (1983),

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H\left( r - \left\| x_i - x_j \right\| \right), \qquad (2.12)$$

where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases},$$

is the unit-step function, (Heaviside function). The summation indicates the the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. C(r) measures the density of pair of distinct vectors $\mathbf{x}_i$ and $\mathbf{x}_j$ that are closer to r.

The correlation dimension $D_c$ of $O(\mathbf{x}_1)$ is defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log r} \qquad (2.13)$$

To obtain $D_c$, log C(r) is plotted against log r and then we find a straight line fitted to this curve. The y- intercept of this straight line provides the value of the correlation dimension $D_c$.

**3.Investigations of Some Discrete Dynamical systems**:

In this section, analysis have been carried out on some discrete one dimensional mathematical models for their regular and chaotic evolution. Maps considered are of interesting character and have extensive applications in various fields. By varying certain control parameter of these models, bifurcation diagrams are obtained for each of them. This provides a clear picture of regular, (periodic), and chaotic motion. In the process, one may observe certain chaotic set, *strange attractor*, which is dense and having fractal properties. For some cases, we observe periodic windows within chaos. Chaotic and non-chaotic nature of evolutions are identified easily by observing the bifurcation and calculating the Lyapunov exponents as well as topological entropy. Results of numerical computation of Lyapunov exponents, topological entropy and correlation dimension are displayed through graphics. Correlation dimension could be obtained as an intercept to y-axis of the straight line fitted by the method of least square in the plot of $\log[c(r)]\ vs\ log\ r$. Detailed explanation and mathematical analysis of the maps considered here are avoided as these can be obtained in suitable books and articles on dynamical systems, (viz. Smale (1967), May (1976), Feigenbaum (1978), Devany (1989 ), Hao Bai-Lin (1984) etc). Therefore, investigations here are confined mainly to their dynamical properties and numerical studies.

In the present work, following one dimensional discrete maps are considered:

**(1) Logistic Map**: $x_{n+1} = \mu x_n (1 - x_n)$ or $f(x) = \mu x (1 - x), 0 < \mu \le 4$ (3.1)

This map has been discussed in numerous articles and has large applications in various fields of studies. It has two fixed points, $x_1^* = 0$ and $x_2^* = (\mu - 1) / \mu$. The fixed point 0 is a fixed point of f for $0 < \mu \le 1$, but when $\mu > 1$ it is unstable while $(\mu - 1) / \mu$ is a fixed point of f for $1 < \mu < 3$. Stability of fixed point changes to different interval for μ during different cycles of evolution and finally we reach to a value of μ, where this evolve chaotically.

Fig.1 shows bifurcation diagrams of this system for two different ranges of values of μ. Appearance of typical periodic windows within chaos are shown in the second diagram.

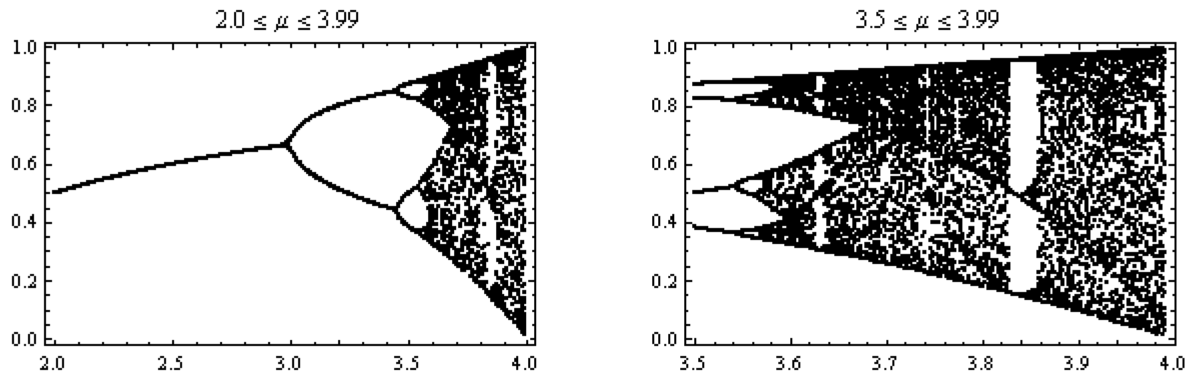


Fig.1: Bifurcations of Logistic map: The second figure clearly shows the appearance of periodic windows within chaos.

Lyapunov exponents for map (3.1) are calculated and drown for two ranges of values of μ, 2.5 ≤ μ ≤ 4.0 and 3.5 ≤ μ ≤ 4.0 and represented by Fig. 2. The negative value of Lyapunov exponents in the figure,Fig.2 (b), shows clearly the existence of periodic orbits within chaos. The same is indicated in the topological entropy plots, Fig. 3. Fig. 3(c), provides similarity in results obtained by Lyapunov exponents and topological entropy.

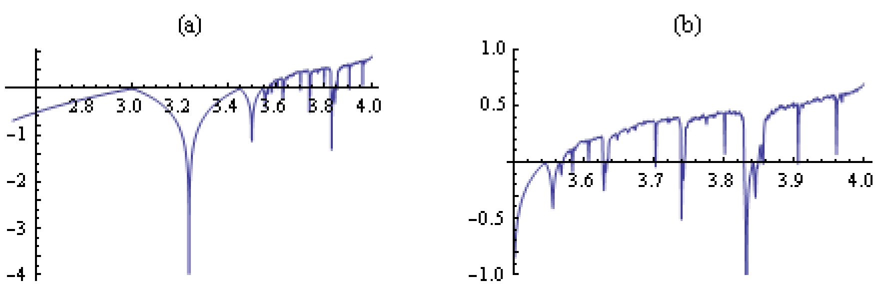


Fig. 2: Lyapunov exponents of the logistic map respectively for ranges 2.5 ≤ μ ≤ 4.0 and 3.5 ≤ μ ≤ 4.0

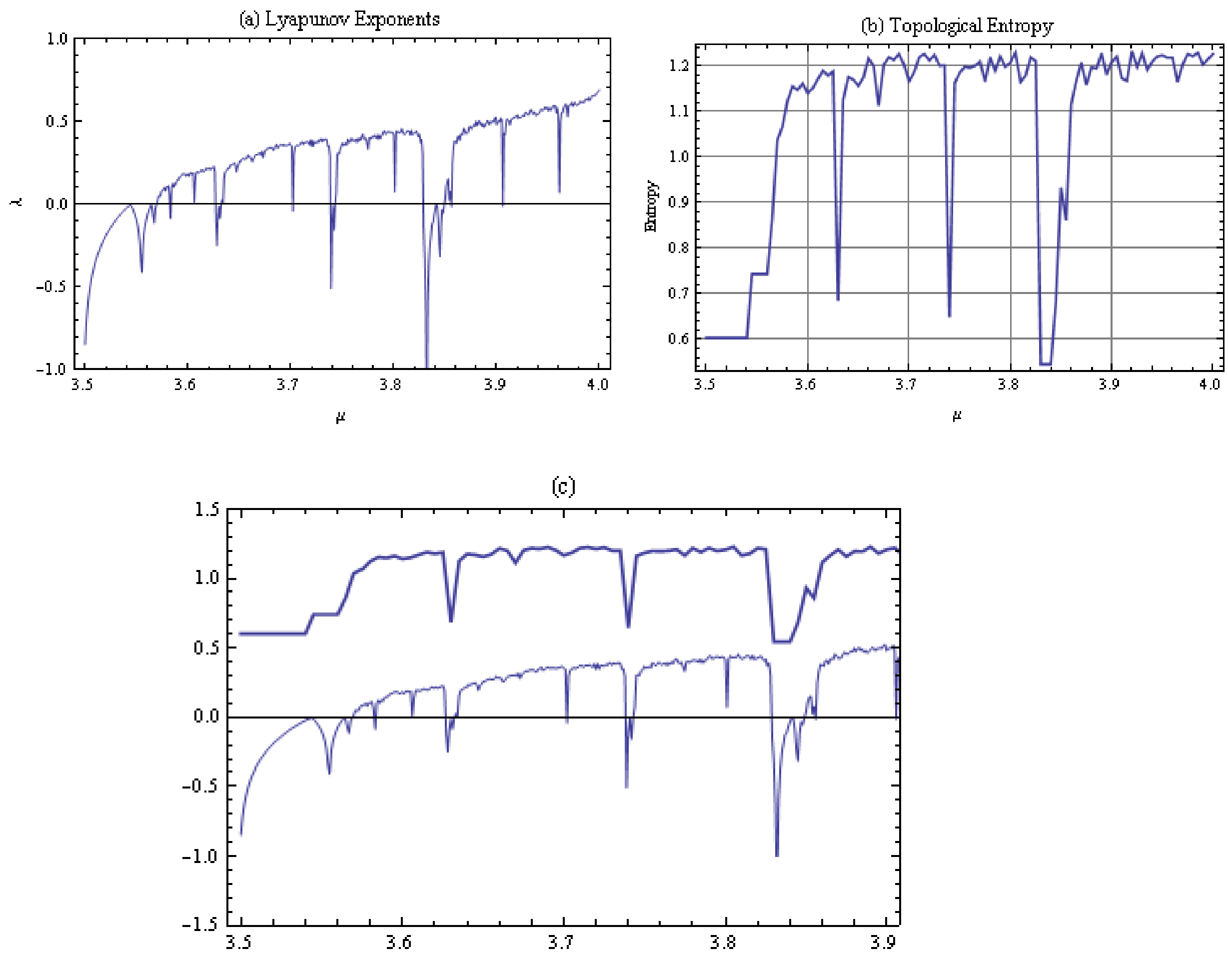


Fig.3: Plots for chaotic in logistic map: (a) Lyapunov exponents, (b)topological Entropy and (c) Comparison of these two, upper curve is for topological entropy.

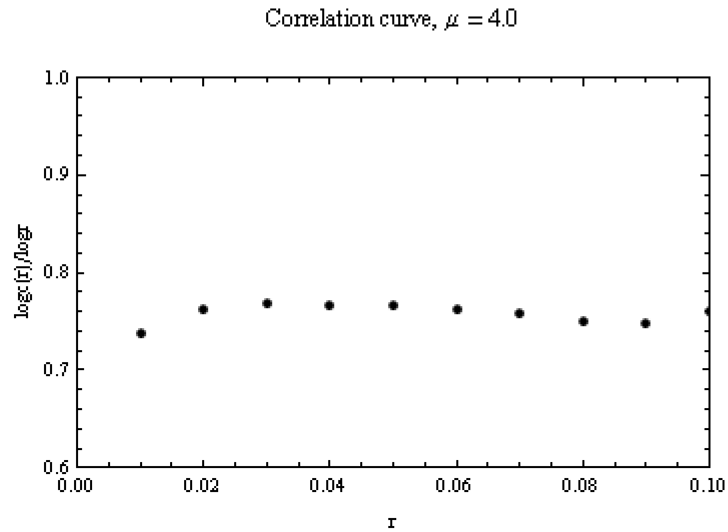


Fig.4: Plot of Correlation integrals curve for chaotic logistic map when $\mu = 4.0$

In figure, Fig. 4, we have plotted a curve, called correlation integrals curve for log C(r)/log r versus r, and then by applied the least square fit method to obtain the equation of the straight line

$$Y = 0.758645 - 0.00767199\, x$$

The y intercept of this st. line, $0.758645 \approx 0.76$, is then the correlation dimension of the fractal set obtained from the logistic map.

**(2) Salmon Map**: $x_{n+1} = x_n\, e^{\mu(1-x_n)}$ or $f(x) = x\, e^{\mu(1-x)}$ (3.2)

This map again evolve into chaos through period doubling bifurcation like that of logistic map and also, periodic windows appear within its chaotic zone.

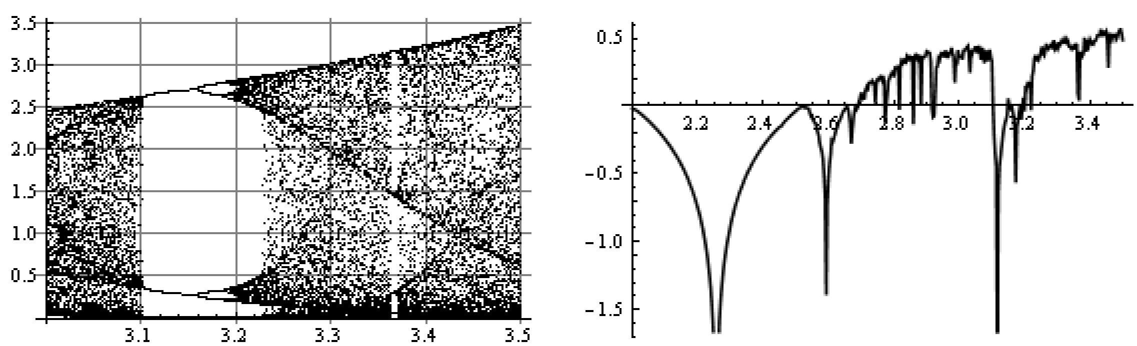


Fig.5 : Bifurcations (left fig.) and Lyapunov exponents (right fig) in the Salmon map.

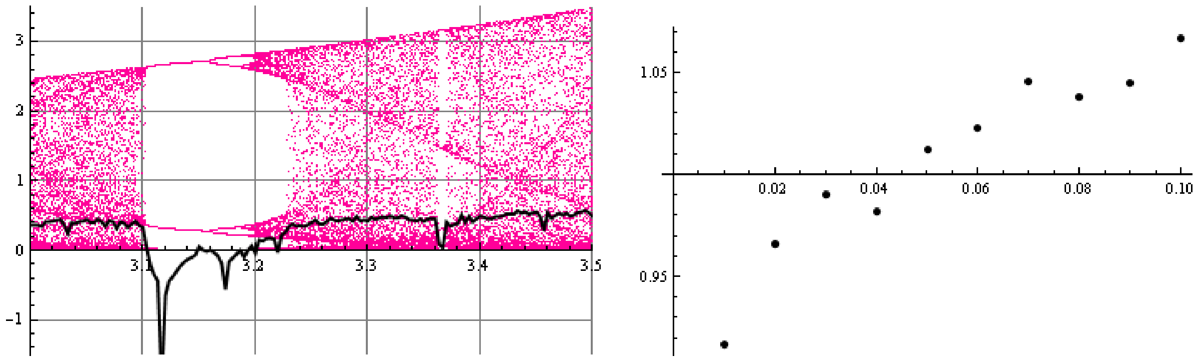


Fig.6: Left figure shows the negativity of Lyapunov exponents where window of periodic orbits exist. The right plot is for correlation integrals.

By least square linearize method, the straight line fitting this curve is

$$Y = 0.93023 + 1.42204\, x$$

The y intercept of this line provides the correlation dimension of the chaotic attractor which is 0.93.

**(3) Gauss Map**: $x_{n+1} = \exp(-a\, x_n^2) + b$ (3.3)

One can observe a very special type of bifurcation in Gauss map shown in Fig.7 by varying b from -1 to +1. The system started evolving chaotically when a exceeds the value 4.65. At values of a higher than 8.0 the system becomes highly chaotic.

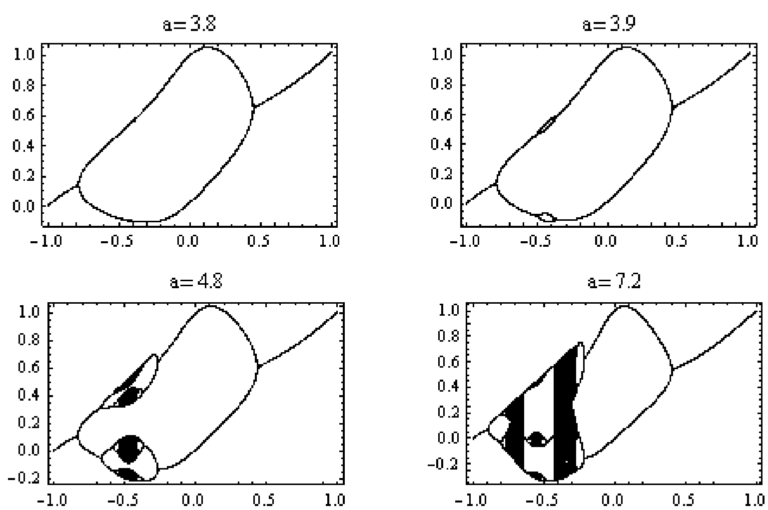


Fig.7: Bifurcation diagrams of Gaus map for different values of a.

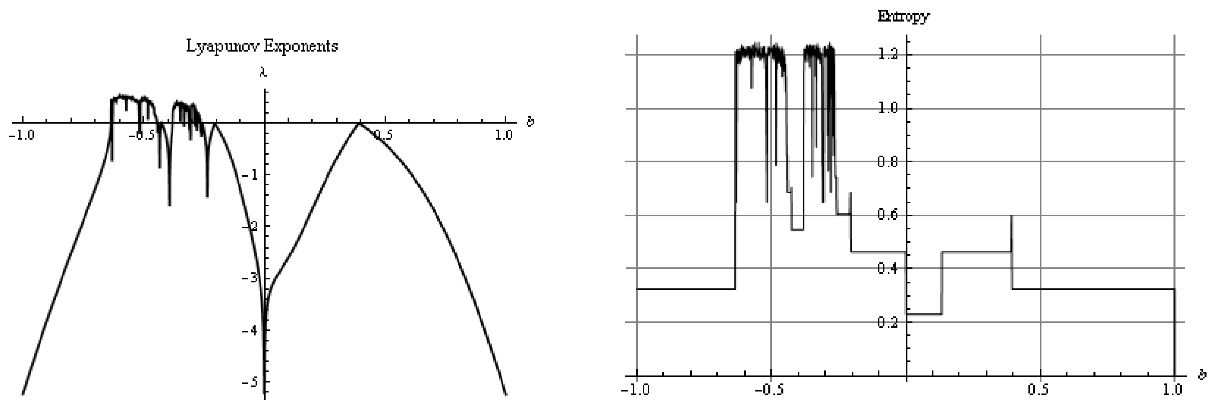


Fig.8: Plots of Lyapunov Exponents (left fig.) and Topological entropy of Gauss map (right fig.).

The correlation integrals curve for chaotic set of Gauss map when a = 8.0, b = -0.6 is shown in Fig. 9.

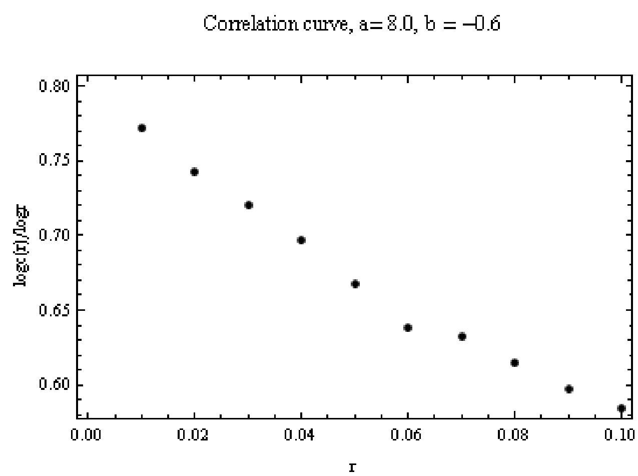


Fig. 9: Correlation integrals curve of chaotic Gauss map.

And by linear least square fit we obtain the appropriate straight line fitting the above curve is

$$Y = 0.781956 - 2.09579\, x$$

The y intercept of this curve, 0.781596 ≈ 0.782, is the correlation dimension of the chaotic set of the Gauss map for above mentioned parameter value.

**(4) Epidemic Model** : $f(x) = 1 - a\,x^2$, $x_{n+1} = 1 - a\,x_n^2$ (3.4 )

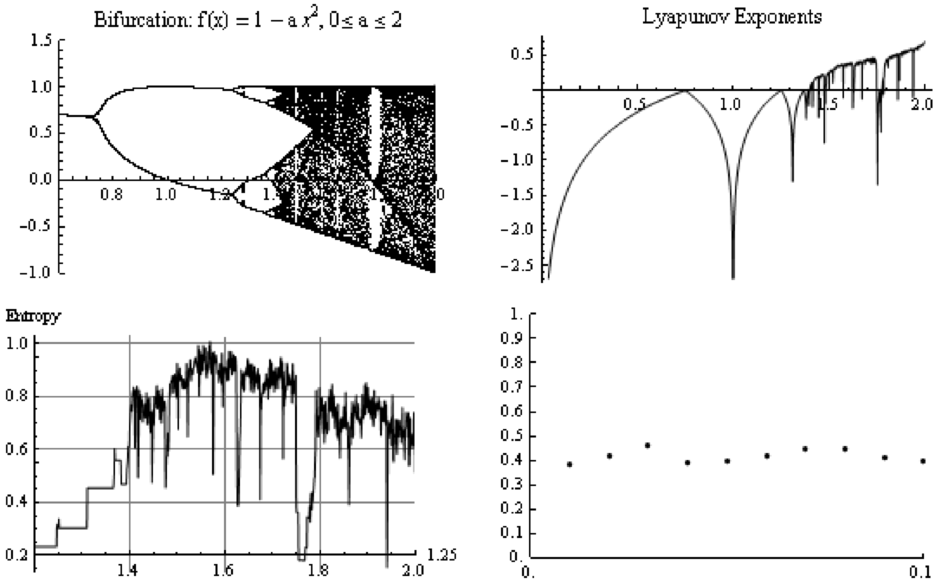


Fig.10: Plots of bifurcations, Lyapunof exponents, topological entropy and correlation integrals curve of map (3.4).

The correlation integrals data fitted to the st. line

$$y = 0.410138 + 0.133082\,x$$

provides the correlation dimension as 0.410318 ≈ 0.41.

**(5) Sine circle map:**

$$f(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x) \quad \text{or} \quad x_{n+1} = x_n + \Omega - \frac{K}{2\pi}\sin(2\pi x_n) \qquad (3.5)$$

where $0 \le x \le 1$ and $\Omega = 0.65$.

Due to presence of sine function the system is nonlinear and that nonlinearity is controlled by parameter $K$. For $K \le 1$, the map shows periodicity and regular. However, for $K > 1$ the map can exhibit chaos. The figure shown below, Fig.11 , represent the bifurcation of this map. The bifurcation scenario display very interesting feature, within chaos appearance of certain periodic windows are located.

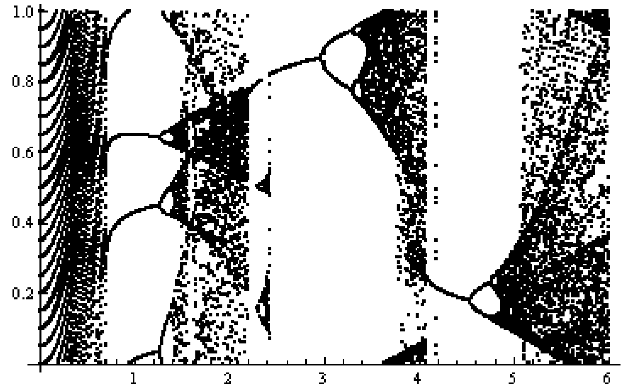


Fig.11 : Bifurcation diagram of sine circle map by varying K .

Lyapunov exponents plot of sine circle map, shown in Fig.12 left, shows an interesting characteristic called *Devil's Staircase* for $0.25 \leq K \leq 0.75$.

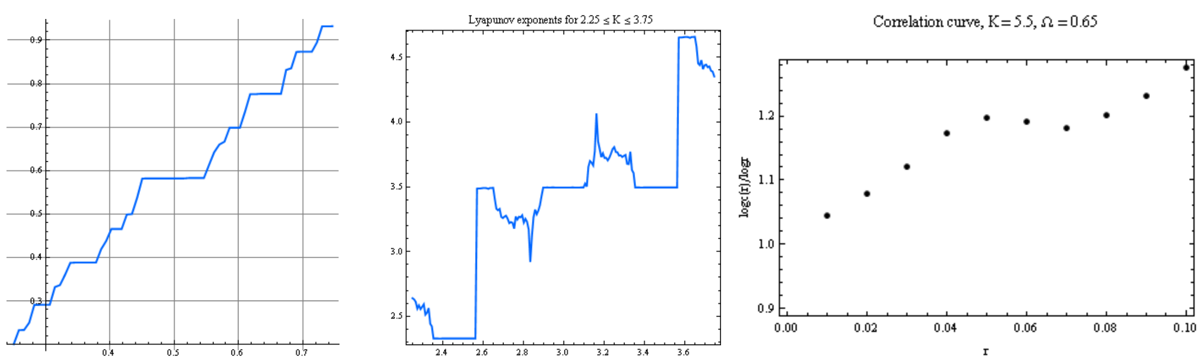


Fig. 12: Plots for Lyapunov exponents, topological entropies and correlation integrals curve.

**(6) Biological Model:** $x_{n+1} = b\, x_n^r \exp[- s\, x_n] + (1 - a)\, x_n$ (3.6)

This is a very interesting biological model. Its evolutionary behavior shows some characteristic feature similar to the Gauss map discussed above. The bifurcation diagram reflects many significant motion of this model and can be seen from the diagram, Fig.13 , when $b = 1.1 \times 10^6$, $r = 8$ and $s = 16$, by varying a. Its plots for Lyapunov exponents and topological entropy provides many important information.

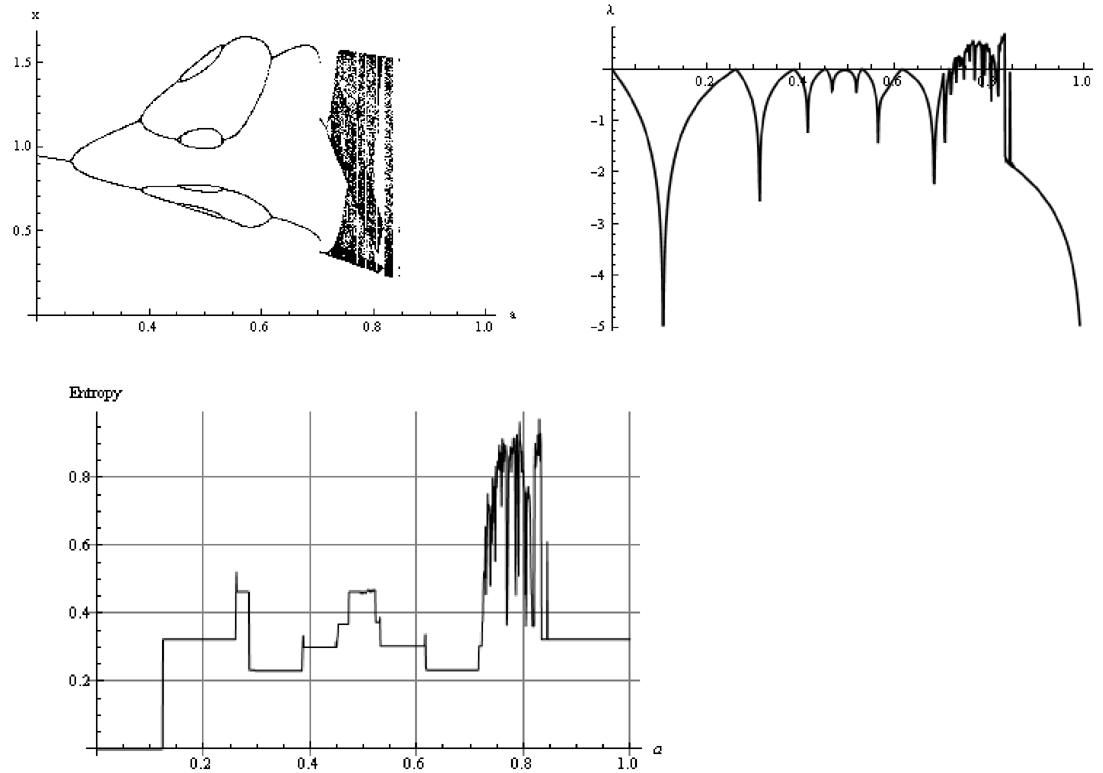


Fig.13. Shows plots bifurcations, Lyapunov exponents and topological entropies.

The correlation dimension of its chaotic attractor at a = 0.78 is obtained as 1.264
From the least square linear fit we obtain the st. line fitting the correlation integrals data as

$$Y = 1.26422 + 2.17595\, x$$

**4**.**Discussions**:

The discrete one dimensional maps used in this article have many applications in different areas of science. The chaotic sets emerging during evolution, *strange attractors*, have fractal structure and so self similar property. The dimensions of such sets are non-integers as shown by the computed correlation dimension data for each case. The studies made in this work provide explanations for numerous fractal structures observed in our universe.

Higher dimensional systems may reveal more significant results. Such studies will be communicated in our near future work.

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