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THE EQUIVALENCE ABOUT THE T-STABILITIES AND THE ALMOST T-STABILITIES BETWEEN MANN AND ISHIKAWA ITERATION

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Abstract: In this paper, we prove that the equivalence about the T-stabilities and the almost T-stabilities of Mann and Ishikawa iteration.

Keywords: a T-stabilities, almost T-stabilities, Mann iteration, Ishikawa iteration.

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1. Introduction

Definition1.1. Let $T:E\to E$ be a mapping, $x_0\in E$ be a given point, $\{\alpha_n\},\{\beta_n\}$, be two sequences in $[0,1]$, and $\{u_n\},\{v_n\},\{r_n\}$, be three bounded sequences in E.

1. the sequence $\{x_n\}\subset E$ defined by:

$$z_{n+1}=(1-\alpha_n)z_n+\alpha_n Tz_n+r_n \tag{1.1}$$

is called the Mann iteration with errors.

2. the sequence $\{x_n\}\subset E$ defined by:

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$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n \tag{1.2}$$

is called the Ishikawa iteration with errors.

3. the sequence $\{x_n\} \subset E$ defined by:

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n T^n z_n + r_n \tag{1.3}$$

is called the modified Mann iteration with errors.

4. the sequence $\{x_n\} \subset E$ defined by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + u_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n + v_n \tag{1.4}$$

is called the modified Ishikawa iteration with errors.

Now we give the stability definition of the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.1) and (1.2)

Definition1.2. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence defined by (1.1), (1.2**)** and such that $x_n \to p \in F(T), F(T)$ denotes the set of fixed point of T. Let $\{s_n\}$ $\{q_n\}$ be any bounded sequence in E by:

(i) $$\delta_n = \left\| q_{n+1} - (1-\alpha_n)q_n - \alpha_n T q_n - r_n \right\|, \quad n \geq 0 \tag{1.5}$$

(ii) $$\varepsilon_n = \left\| s_{n+1} - (1-\alpha_n)s_n - \alpha_n T y_n - u_n \right\|,$$
$$y_n = (1-\beta_n)s_n + \beta_n T s_n + v_n, \quad n \geq 0 \tag{1.6}$$

If $\delta_n \to 0$ implies that $q_n \to p$, then the Mann iterative sequence $\{x_n\}$ is said to be $T-$stable; $\varepsilon_n \to 0$ implies that $s_n \to p$, then the Ishikawa iterative sequence $\{x_n\}$ is said to be $T-$stable; if $\sum_{n=0}^{\infty} \delta_n < +\infty$ implies that $q_n \to p$,then the Mann sequence $\{x_n\}$ is said to be almost $T-$stable; $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$ implies that $s_n \to p$,then the Ishikawa sequence $\{x_n\}$ is said to be almost $T-$stable.

The equivalence between the Ishikawa iteration and Mann iteration for several classes of maps has been studied in [1-5], and proved the equivalence between T-stable of (1.1) and (1.2). In this paper, we shall prove the equivalence between T-stable and almost T-stable of (1.3) and (1.4). Throughout this paper, we shall assume that X is a normed space. T is a map on X with a bounded range and assume that both Mann and Ishikawa iterations with errors converge to a fixed point of T.

2. The equivalence between T-stabilities

Theorem2.1. Let X be a normed space and $T: X \to X$ be a map with a bounded range. For all $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1)$, $\{u_n\}$, $\{r_n\}$, satisfy

$\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0, \|u_n\| \to 0, \|r_n\| \to 0$, as $n \to \infty$

The following are equivalent:

(i) the Ishikawa iteration sequence with errors (1.2) is T-stable.

(ii) the Mann iteration sequence with errors (1.1) is T-stable.

Proof:

We first prove **(ii)**$\Rightarrow$**(i)** In **(1.6)**, let $s_n = q_n$, we have:

$$\begin{aligned} &\|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Tq_n - r_n\| \\ &\quad \le \|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Ty_n - u_n + \alpha_n Ty_n + u_n - \alpha_n Tq_n - r_n\| \\ &\quad \le \|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Ty_n - u_n\| + \|\alpha_n Ty_n - \alpha_n Tq_n\| + \|u_n - r_n\| \\ &\quad \le \|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Ty_n - u_n\| + \alpha_n \|Ty_n - Tq_n\| + \|u_n\| + \|r_n\| \end{aligned} \tag{2.1}$$

from the definition, we can know $\forall \{s_n\} \subset X$ are bounded sequence such that $\{Ts_n\}$ is bounded so we can let

$$M := \max\left\{\sup_{n\in N}\{\|T(y_n)\|\}, \sup_{n\in N}\{\|T(s_n)\|\}, \sup_{n\in N}\{\|T(p_n)\|\},\right\}$$

then $M < +\infty$. Then we obtain:

$$\begin{aligned} &\|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Tq_n - r_n\| \\ &\le \|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Ty_n - u_n\| + 2\alpha_n M + \|u_n\| + \|r_n\| \to 0 \quad (n \to \infty) \end{aligned}$$

We have $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \|q_{n+1} - (1-\alpha_n)q_n - \alpha_n Tq_n - r_n\| = 0 \Rightarrow \lim_{n\to\infty} q_n = p$ by the condition (ii), thus , for $\forall \{q_n\} \subset X$ satisfying

$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\|q_{n+1}-(1-\alpha_n)q_n-\alpha_n Ty_n-u_n\|=0$, we have $\lim_{n\to\infty}q_n = p$ i.e. **(ii)**$\Rightarrow$**(i).**

Then we prove **(i)**$\Rightarrow$**(ii)**. In (1.5), Let $q_n = s_n$, we have:

$$\begin{aligned}
&\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ty_n-u_n\| \\
&\quad\le \|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n-\alpha_n Ty_n-u_n+\alpha_n Ts_n+r_n\| \\
&\quad\le \|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n\|+\|\alpha_n Ty_n-\alpha_n Ts_n\|+\|u_n-r_n\| \\
&\quad\le \|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n\|+\alpha_n\|Ty_n-Ts_n\|+\|u_n\|+\|r_n\| \\
&\quad\le \|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n\|+2\alpha_n M+\|u_n\|+\|r_n\|\to 0 \quad (n\to\infty)
\end{aligned} \tag{2.2}$$

We have $\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ty_n-u_n\|=0 \Rightarrow \lim_{n\to\infty}s_n = p$ by the condition (i), thus, for $\forall\{s_n\}\subset X$ satisfying $\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n\|=0$, we have $\lim_{n\to\infty}s_n = p$ i.e.**(i)**$\Rightarrow$**(ii)**.

3. Main results

Theorem3.1. Let X be a normed space and $T: X\to X$ be a map with a bounded range. For all $\{\alpha_n\}\subset(0,1),\{\beta_n\}\subset(0,1)$ $\{u_n\}$, $\{r_n\}$, satisfy

$$\sum_{n=1}^{\infty}\alpha_n<+\infty,\sum_{n=1}^{\infty}\beta_n<+\infty,\sum_{n=1}^{\infty}u_n<+\infty,\sum_{n=1}^{\infty}r_n<+\infty,$$

The following are equivalent:

(i) the Ishikawa iteration with errors (1.2)is almost T-stable.

(ii) the Mann iteration with errors (1.1) is almost T-stable.

Proof: in (2.1), we have

$$\begin{aligned}
&\|q_{n+1}-(1-\alpha_n)q_n-\alpha_n Tq_n-r_n\| \\
&\quad\le \|q_{n+1}-(1-\alpha_n)q_n-\alpha_n Ty_n-u_n\|+2\alpha_n M+\|u_n\|+\|r_n\| \\
&\quad\le \varepsilon_n+2\alpha_n M+\|u_n\|+\|r_n\|
\end{aligned} \tag{3.1}$$

Because $\sum_{n=1}^{\infty}\alpha_n<+\infty, M<+\infty$, $\sum_{n=1}^{\infty}u_n<+\infty,\sum_{n=1}^{\infty}r_n<+\infty$ and we have

We have $\sum_{n=0}^{\infty}\delta_n = \sum_{n=0}^{\infty}\|q_{n+1}-(1-\alpha_n)q_n-\alpha_n Tq_n-r_n\|<+\infty \Rightarrow \lim_{n\to\infty}q_n = p$ by the condition (ii), thus, for $\forall\{q_n\}\subset X$ satisfying

$\sum_{n=0}^{\infty}\varepsilon_n=\sum_{n=0}^{\infty}\|q_{n+1}-(1-\alpha_n)q_n-\alpha_n Ty_n-u_n\|<+\infty$, we have $\lim_{n\to\infty}q_n=p$ i.e. **(ii)$\Rightarrow$(i).**

Conversely, In **(2.2)** we have:

$$\begin{aligned}&\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ty_n-u_n\|\\&\quad\le\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n\|+2\alpha_n M+\|u_n\|+\|r_n\|\end{aligned}\tag{3.2}$$

Because $\sum_{n=1}^{\infty}\alpha_n<+\infty, M<+\infty$, $\sum_{n=1}^{\infty}u_n<+\infty, \sum_{n=1}^{\infty}r_n<+\infty$ and we have

We have $\sum_{n=0}^{\infty}\varepsilon_n=\sum_{n=0}^{\infty}\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ty_n-u_n\|<+\infty\Rightarrow\lim_{n\to\infty}s_n=p$ by the

condition(i),thus, for $\forall\{s_n\}\subset X$ satisfying

$$\sum_{n=0}^{\infty}\delta_n=\sum_{n=0}^{\infty}\|s_{n+1}-(1-\alpha_n)s_n-\alpha_n Ts_n-r_n\|<+\infty,$$

We have $\lim_{n\to\infty}s_n=p$ i.e. **(i)$\Rightarrow$(ii)**.

In (1.1) and **(**1.2), set $T:=T^n$, we can obtain the modified Mann and Modified Ishikawa iteration with errors (1.3) (1.4). We also suppose that the modified Mann and Modified Ishikawa iteration with errors converge to a fixed point of T. Note that Definition1.2 and Theorem 2.1 and Theorem 3.1 hold in this case too.

Theorem3.2. Let X be a normed space and $T:X\to X$ be a map with a bounded range. For all $\{\alpha_n\}\subset(0,1),\{\beta_n\}\subset(0,1)$, $\{u_n\},\{v_n\}$, satisfy

$\lim_{n\to\infty}\alpha_n=0,\lim_{n\to\infty}\beta_n=0,\|u_n\|\to 0,\|r_n\|\to 0$ as $n\to\infty$ The following are equivalent:

(i) the modified Ishikawa iteration with errors is T-stable.

(ii) the modified Mann iteration with errors is T-stable.

Theorem3.3. Let X be a normed space and $T:X\to X$ be a map with a bounded range. .for all $\{\alpha_n\}\subset(0,1),\{\beta_n\}\subset(0,1)$ $\{r_n\}$, $\{u_n\}$, satisfy

$$\sum_{n=1}^{\infty}\alpha_n < +\infty, \sum_{n=1}^{\infty}\beta_n < +\infty, \sum_{n=1}^{\infty}u_n < +\infty, \sum_{n=1}^{\infty}r_n < +\infty$$

The following are equivalent:

(i) the modified Ishikawa iteration with errors is almost T-stable.

(ii). the modified Mann iteration with errors is almost T-stable.

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