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FIXED POINT THEOREMS FOR $\psi-$CONTRACTIVE MAPPING IN C^*-ALGEBRA VALUED RECTANGULAR B-METRIC SPACES

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Abstract. In this paper, we present a new insight of C^*-algebra valued rectangular $b-$metric spaces in the perspective of the fixed point theory using contractive mapping. Using contractive mapping in the rectangular $b-$metric spaces, we discussed the existence and the uniqueness of the fixed point with mapping satisfying a contractive condition. As a result, we obtained an interesting and important result for the general case of C^*-algebra valued metric spaces. In particular, we study some fixed point theorems in the C^*-algebra valued rectangular $b-$metric spaces using a positive function.

Keywords: C^*-algebra; C^*-algebra valued rectangular b-metric; fixed point; contractive mapping.

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1. INTRODUCTION

C^*-algebra theory is a critical subject in functional analysis and operator theory that plays a central role in fixed point theory and applications.

In this context, several researchers have obtained fixed point results for mapping under multiple contractive conditions in the framework of different types of metric spaces.

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In 1989 Bakhtin [5] introduced the concept of $b-$metric spaces. Later, Czerwik [6] extended the results of the renowned Banach fixed point theorem in the $b-$metric spaces.

In 2000, Branciari [4] introduced the notion of rectangular metric spaces where the triangle inequality of metric spaces was replaced by another inequality, the so-called rectangular inequality. In [8], George et al. established the concept of rectangular $b-$metric space which generalizes the concept of rectangular metric space and $b-$metric space.

Ma et al. [11] introduced the concept of C^*-algebra valued metric space and studied some fixed point theorems. The notion of C^*-algebra valued metric spaces generalized to that of C^*-algebra rectangular $b-$ metric space, where $b-$ is an element of C^*- algebra greater than I, and the triangle inequality is modified into

$$d(x,y) \preceq b[d(x,u)+d(u,v)+d(v.y)].$$

Then, various fixed point theorems are obtained for self-map with contractive condition [9,10].

In this paper, inspired by the work done in [13], we introduce the notion of $\psi-$contractive mapping in C^*-algebra valued rectangular $b-$metric and establish some new fixed point theorems. Moreover, an illustrative example is presented to support the obtained results.

2. Preliminaries

Throughout this paper, we denote $\mathbb{A}$ by an unital (i.e. unity element I) C^*-algebra with linear involution $*$, such that for all $x,y \in \mathbb{A}$,

$$(xy)^* = y^*x^* \text{ and } x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$ if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x.

Using positive element, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta$$

where θ means the zero element in $\mathbb{A}$.

We denote $\mathbb{A}^+ = \{a \in \mathbb{A}, \theta \preceq a\}$ and $\mathbb{A}^{'} = \{a \in \mathbb{A}, ab = ba; \forall b \in \mathbb{A}\}$ and $|x| = (x^*x)^{\frac{1}{2}}$

Remark 2.1. *When $\mathbb{A}$ is a unital C^*-algebra, then for any $x \in \mathbb{A}_+$ we have*

$$x \preceq I \Longleftrightarrow \|x\| \leq 1$$

Definition 2.2. [14] *Let X be a non-empty set and $b \in A$ such that $b \succeq I$. Suppose the mapping $d : X \times X \to \mathbb{A}_+$ satisfies:*

(i) $d(x,y) = \theta$ *if and only if* $x = y$;

(ii) $d(x,y) = d(y,x)$ *for all distinct points* $x,y \in X$;

(iii) $d(x,y) \preceq b[d(x,u) + d(u,v) + d(v,y)]$ *for all* $x,y \in X$ *and for all distinct points* $u,v \in X - \{x,y\}$.

Then $(X,\mathbb{A}_+,d)$ *is called a C^*-algebra valued rectangular $b-$metric space.*

Definition 2.3. [11] *Let $(X,\mathbb{A}_+,d)$ be a C^*-algebra valued rectangular $b-$metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$.*

If for any $\varepsilon > 0$ there is N such that for all $n > N$,$\|d(x_n,x)\| \leq \varepsilon$, then $\{x_n\} \subset X$ is said to be convergent with respect to $\mathbb{A}$ and $\{x_n\} \subset X$ converges to x and x is the limit of $\{x_n\} \subset X$. We denote it by $lim_{n\longrightarrow\infty}x_n = x$.

If for any $\varepsilon > 0$ there is N such that for all $n,m > N$,$\|d(x_n,x_m)\| \leq \varepsilon$,then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

We say $(X,\mathbb{A}_+,d)$ is a complete C^*-algebra valued rectangular $b-$ metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

It is obvious that if X is a Banach space, then $(X,\mathbb{A}_+,d)$ is a complete C^*-algebra valued rectangular $b-$metric space if we set

$$d(x,y) = \|x-y\|I$$

Example 2.4. *Let $X = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \mathbb{N}$. Let $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition, scalar multiplication and multiplication. Define partial ordering on $\mathbb{A}$ as*

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i \text{ for } i = 1,2,3,4$$

For any $A \in \mathbb{A}$ we define its norm as ,$\|A\| = max_{1\leq i\leq 4}|a_i|$

Define $d : X \times X \to \mathbb{A}$ such that $d(x,y) = d(y,x)$ for all $x,y \in X$ and

$$d(x,y)=\begin{cases}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} if x=y\\ \begin{pmatrix}2\alpha & 0\\ 0 & 2\alpha\end{pmatrix}, if x,y\in\{\frac{1}{n};n\in\mathbb{N}^*\}\\ \begin{pmatrix}\frac{\alpha}{2n} & 0\\ 0 & \frac{\alpha}{2n}\end{pmatrix}, if x\in\{\frac{1}{n};n\in\mathbb{N}^*\}\text{ , }y\in\{2,3\}\\ \begin{pmatrix}\alpha & 0\\ 0 & 0\end{pmatrix}, otherwise\end{cases}$$

where $\alpha>0$ is a constant.

Then $(X,\mathbb{A}_+,d)$ is a C^*-algebra valued rectangular $b-$ metric space with coefficient $A=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$ and $\|A\|=2>1$.

Let A be a C^*-algebra and suppose that φ is a linear functional on A. Define

$$\varphi^*(a)=\varphi(a^*)\text{ for all } a\in A.$$

Then φ^* is also a linear function on A .

If $\varphi^*=\varphi$ the function φ is called self- adjoint.

Every linear function on A can be represented in the form $\varphi=\varphi_1+i\varphi_2$ where φ_1,φ_2 are self-adjoint. Specifically $(\varphi_1=\frac{1}{2}(\varphi+\varphi^*);\varphi_2=\frac{1}{2i}(\varphi-\varphi^*))$.

A linear function φ on A is called positive if $\varphi(a^*a)\succeq\theta$ for all $a\in\mathbb{A}$.

We denote the positivity of φ by $\varphi\succeq\theta$. For two self-adjoint linear function φ_1,φ_2, we have $(\varphi_2-\varphi_1\succeq\theta)$ when $\varphi_2\succeq\varphi_1$.

Definition 2.5. [15] *If $\varphi: A\to B$ is a linear mapping in C^*-algebra, it is said to be positive if $\varphi(A^+)\subseteq B^+$. In this case $\varphi(A_h)\subseteq B_h$, and the restriction map $\varphi: A_h\to B_h$ is increasing.*

Proposition 2.6. [15] *Let A be a C^*-algebra with 1 then a positive functional is bounded and $\varphi(1)=\|\varphi\|$.*

Proposition 2.7. [15] *Let A be a C^*-algebra with 1 and let φ be a bounded linear functional on A, such that $\varphi(a)=\|\varphi\|\|a\|$. There exists positive element $a\in A$ such that φ is a positive linear functional.*

Definition 2.8. [12] *Let the function $\psi : A^+ \to A^+$ be positive if having the following constraints:*

(i) *ψ is continous and nondecrasing*

(ii) *$\psi(a) = \theta$ if and only if $a = \theta$*

(iii) *$lim_{n \longrightarrow \infty} \psi^n(a) = \theta$*

Definition 2.9. [12] *Suppose that A and B are C^*-algebra .*

A mapping $\psi : A \to B$ is said to be C^*- homomorphism if :

(i) $\psi(ax+by) = a\psi(x)+b\psi(y)$ *for all $a, b \in \mathbb{C}$ and $x, y \in A$*

(ii) $\psi(xy) = \psi(x)\psi(y)$ *for all $x, y \in A$*

(iii) $\psi(x^*) = \psi(x)^*$ *for all $x \in A$*

(iv) *ψ maps the unit in A to the unit in B.*

Definition 2.10. [12] *Let A and B be C^*-algebra spaces and let $\psi : A \to B$ be a homomorphism, then ψ is called an $*-$ homomorphism if it is one to one $*-$ homomorphism.*

A C^*-algebra A is $*-$isomorphic to a C^*-algebra B if there exists $*-$ isomorphism of A onto B.

Lemma 2.11. [16] *Let A and B be C^*-algebra spaces and $\psi : A \to B$ is a C^*- homomorphism for all $x \in A$ we have*

$$\sigma(\psi(x)) \subset \sigma(x) \text{ and } \|\psi(x)\| \leq \|\psi\|.$$

Corollary 2.12. [12] *Every C^*- homomorphism is bounded.*

Corollary 2.13. [12]*Suppose that ψ is C^*- isomorphism from A to B, then $\sigma(\psi(x)) = \sigma(x)$ and $\|\psi(x)\| = \|\psi\|$ for all $x \in A$.*

Lemma 2.14. [12] *Every $*-$ homomorphism is positive.*

3. Main Results

In this part, we give some fixed point theorems in C^*-algebra valued rectangular $b-$metric space using a positive function.

Theorem 3.1. *Let $(X,\mathbb{A},d)$ be a complete C^*-algebra valued rectangular $b-$metric space.*

Let $T:X\to X$ satisfy the following condition:

$$d(Tx,Ty)\preceq a^*d(x,y)a-\psi(d(x,y))$$

where ψ is $*-$homomorphism and $lim_{a\longrightarrow\infty}\psi(a)=\infty$ and $\|b\|\|a\|^2<1$.
Then T has a unique fixed point.

Proof. : Let $x_{n+1}=Tx_n$.for each $n\geq 1$, then:

$$\begin{aligned}d(x_{n+1},x_{n+2})&=d(Tx_n,Tx_{n+1})\\&\preceq a^*d(x_n,x_{n+1})a-\psi(d(x_n,x_{n+1}))\\&\preceq a^*d(x_n,x_{n+1})a\\&\preceq ...\\&\preceq a^{*n}d(x_0,x_1)a^n\end{aligned}$$

then

$$\|d(x_{n+1},x_{n+2})\|\preceq\|a\|^n\|d(x_0,x_1)\|$$

Letting $n\to\infty$ we obtain $d(x_{n+1},x_{n+2})\to\theta$
Then for $m\geq 1$ and $p\geq 1$:
$d(x_{m+p},x_m)\preceq b[d(x_{m+p},x_{m+p-1})+d(x_{m+p-1},x_{m+p-2})+d(x_{m+p-2},x_m)]$
$\preceq bd(x_{m+p},x_{m+p-1})+bd(x_{m+p-1},x_{m+p-2})+b[b[d(x_{m+p-2},x_{m+p-3})+d(x_{m+p-3},x_{m+p-4})+$
$d(x_{m+p-4},x_m)]]]$
$=bd(x_{m+p},x_{m+p-1})+bd(x_{m+p-1},x_{m+p-2})+b^2d(x_{m+p-2},x_{m+p-3})+b^2d(x_{m+p-3},x_{m+p-4})+$
$b^2d(x_{m+p-4},x_m)$
$\preceq bd(x_{m+p},x_{m+p-1})+bd(x_{m+p-1},x_{m+p-2})+b^2d(x_{m+p-2},x_{m+p-3})+b^2d(x_{m+p-3},x_{m+p-4})+$
$....+b^{\frac{p-1}{2}}d(x_{m+3},x_{m+2})+b^{\frac{p-1}{2}}d(x_{m+2},x_{m+1})+b^{\frac{p-1}{2}}d(x_{m+1},x_m)$
$\preceq\quad b[(a^{m+p-1})^*d(x_1,x_0)(a^{m+p-1})\quad+\quad(a^{m+p-2})^*)d(x_1,x_0)(a^{m+p-2})]\quad+$
$b^2[(a^{m+p-3})^*)d(x_1,x_0)(a^{m+p-3})\quad+\quad(a^{m+p-4})^*d(x_1,x_0)(a^{m+p-4})]\quad+\quad....\quad+$
$b^{\frac{p-1}{2}}[(a^{m+2})^*)d(x_1,x_0)(a^{m+2})\ +\ (a^{m+1})^*d(x_1,x_0)(a^{m+1})]\ +\ b^{\frac{p-1}{2}}(a^m)^*d(x_1,x_0)(a^m)\ -$
$b[\psi^{m+p-1}(d(x_1,x_0)\ -\ \psi^{m+p-2}(d(x_1,x_0)]\ -\ b^2[\psi^{m+p-3}(d(x_1,x_0)\ -\ \psi^{m+p-4}(d(x_1,x_0)]\ -$
$......-b^{\frac{p-1}{2}}[\psi^{m+2}(d(x_1,x_0)-\psi^{m+1}(d(x_1,x_0)]-b^{\frac{p-1}{2}}\psi^m(d(x_1,x_0)$
$=\ \sum_{k=1}^{\frac{p-1}{2}}b^k(a^*)^{m+p-(2k-1)}d(x_1,x_0)(a^{m+p-(2k-1)})\ +\ \sum_{k=1}^{\frac{p-1}{2}}b^k(a^*)^{m+p-2k}d(x_1,x_0)(a^{m+p-2k})\ +$

$b^{\frac{p-1}{2}}(a^*)^m d(x_1,x_0) - \sum_{k=1}^{\frac{p-1}{2}} b^k \psi^{m+p-(2k-1)}(d(x_1,x_0)) - \sum_{k=1}^{\frac{p-1}{2}} b^k \psi^{m+p-2k}(d(x_1,x_0)) - b^{\frac{p-1}{2}} \psi^m(d(x_1,x_0))$
$= \sum_{k=1}^{\frac{p-1}{2}}(d(x_1,x_0)^{\frac{1}{2}} b^{\frac{k}{2}} a^{m+p-(2k-1)})^*(d(x_0,x_1)^{\frac{1}{2}} b^{\frac{k}{2}} a^{m+p-(2k-1)}) +$
$\sum_{k=1}^{\frac{p-1}{2}}(d(x_1,x_0)^{\frac{1}{2}} b^{\frac{k}{2}} a^{m+p-2k})^*(d(x_0,x_1)^{\frac{1}{2}} b^{\frac{k}{2}} a^{m+p-2k}) + (d(x_0,x_1)^{\frac{1}{2}} b^{\frac{p-1}{4}} a^m)^*(d(x_0,x_1)^{\frac{1}{2}} b^{\frac{p-1}{4}} a^m) -$
$\sum_{k=1}^{\frac{p-1}{2}} b^k \psi^{m+p-(2k-1)}(d(x_1,x_0)) - \sum_{k=1}^{\frac{p-1}{2}} b^k \psi^{m+p-2k}(d(x_1,x_0)) - b^{\frac{p-1}{2}} \psi^m(d(x_1,x_0))$
$\preceq \sum_{k=1}^{\frac{p-1}{2}} \|b^{\frac{k}{2}}(a^*)^{m+p-(2k-1)} d(x_1,x_0)^{\frac{1}{2}}\|^2 I + \sum_{k=1}^{\frac{p-1}{2}} \|b^{\frac{k}{2}}(a^*)^{m+p-2k} d(x_1,x_0)^{\frac{1}{2}}\|^2 I +$
$\| (d(x_0,x_1)^{\frac{1}{2}} b^{\frac{p-1}{4}} a^m \|^2 I \preceq \| d(x_0,x_1) \| \sum_{k=1}^{\frac{p-1}{2}} \| b \|^{\frac{k}{2}} \| a \|^{2(m+p-(2k-1))} I +$
$\| d(x_0,x_1) \| \sum_{k=1}^{\frac{p-1}{2}} \| a \|^{2(m+p-2k)} I + \| d(x_0,x_1) \| \| b \|^{\frac{p-1}{2}} \| a \|^{2m} I$
$= \| d(x_0,x_1) \| \| b \| \| a \|^{2(m+p-1)} [\frac{\| b \|^{\frac{p-1}{2}} \| a \|^{2(-p+1)} - 1}{\| b \| \| a \|^{-4} - 1}] I \| d(x_0,x_1) \|$
$\| b \| \| a \|^{2(m+p-2)} [\frac{\| b \|^{\frac{p-1}{2}} \| a \|^{2(-p+1)} - 1}{\| b \| \| a \|^{-4} - 1}] I + \| d(x_0,x_1) \| \| b \|^{\frac{p-1}{2}} \| a \|^{2m} I$
$\preceq \frac{\| d(x_0,x_1) \| \| b \|^{\frac{p+1}{2}} \| a \|^{2(m+2)}}{\| b \| - \| a \|^4} + \frac{\| d(x_0,x_1) \| \| b \|^{\frac{p+1}{2}} \| a \|^{2(m+1)}}{\| b \| - \| a \|^4} +$
$\| d(x_0,x_1) \| \| b \|^{\frac{p-1}{2}} \| a \|^{2m} I \to \theta (m \to \infty)$
$\to \theta (m \to \infty)$

Therefore x_n is a Cauchy sequence with respect to $\mathbb{A}$.By the completness of $(X,\mathbb{A},d)$ there exists an $x \in X$ such that

$lim_{n\to\infty} x_n = lim_{n\to\infty} T x_{n-1} = x = Tx.$

Let y be another fixed point of T where:

$$\theta \preceq d(x,y) = d(Tx,Ty) \preceq a^* d(x,y) a - \psi(d(x,y)) \preceq a^* d(x,y) a$$

we have

$$\begin{aligned} 0 \leq \| d(x,y) \| &= \| d(Tx,Ty) \| \\ &\leq \| a^* d(x,y) a \| \\ &\leq \| a^* \| \| d(x,y) \| \| a \| \\ &= \|a\|^2 \|d(x,y)\| \\ &< \|d(x,y)\| . \end{aligned}$$

wich is a contradiction Hence $d(x,y) = \theta$ and $x = y$,wich implies that the fixed point is unique.

□

Lemma 3.2. [13] *Let $(X,\mathbb{A},d)$ be a C^*-algebra valued rectangular $b-$metric space such that $d(x,y)\in\mathbb{A}^+$, for all $x,y\in X$ where $x\neq y$.*

Let $\phi:\mathbb{A}^+\longrightarrow\mathbb{A}^+$ be a function with the following propreties:

(i) $\phi(a)=\theta$ *if* $a=\theta$

(ii) $\phi(a)\prec a$ *for* $a\in\mathbb{A}^+$

(iii) *Either* $\phi(a)\preceq d(x,y)$ *or* $d(x,y)\preceq\phi(a)$ *where* $a\in\mathbb{A}^+$ *and* $x,y\in X$.

Theorem 3.3. *Let $(X,\mathbb{A},d)$ be a complete C^*- algebra valued rectangular $b-$metric space. Let $T:X\longrightarrow X$ be a mapping function:*

$$\psi(d(Tx,Ty))\preceq\phi(d(x,y))$$

where ψ is a $*-$homomorphism and $\phi:\mathbb{A}^+\longrightarrow\mathbb{A}^+$ is a continuous function with the constraint $\psi(a)\prec\phi(a)$. Then, T has a fixed point.

Proof. Let $x_0\in X$, we define :

$x_1=Tx_0,x_2=Tx_1,......,x_n=Tx_{n-1}$.

$$\psi(d(x_{n+1},x_n))=\psi(d(Tx_n,Tx_{n-1}))\preceq\phi(d(x_n,x_{n-1}))$$

We have $d(x_{n+1},x_n)\preceq d(x_n,x_{n-1})$ then $\|d(x_{n+1},x_n)\|\preceq\|d(x_n,x_{n-1})\|$

Hence, the sequence $d(x_{n+1},x_n)$ is norm decreasing,from the condition of the condition of the theorem we have $d(x_{n+1},x_n)\longrightarrow\theta$ this implies $\|d(x_{n+1},x_n)\|\rightarrow 0$

Then for $m\geq 1$ and $p\geq 1$:

$d(x_{m+p},x_m)\preceq b[d(x_{m+p},x_{m+p-1})+d(x_{m+p-1},x_{m+p-2})+d(x_{m+p-2},x_m)]$

$\preceq bd(x_{m+p},x_{m+p-1})+bd(x_{m+p-1},x_{m+p-2})+b[b[d(x_{m+p-2},x_{m+p-3})+d(x_{m+p-3},x_{m+p-4})+d(x_{m+p-4},x_m)]]]$

$=bd(x_{m+p},x_{m+p-1})+bd(x_{m+p-1},x_{m+p-2})+b^2d(x_{m+p-2},x_{m+p-3})+b^2d(x_{m+p-3},x_{m+p-4})+b^2d(x_{m+p-4},x_m)$

$\preceq bd(x_{m+p},x_{m+p-1})+bd(x_{m+p-1},x_{m+p-2})+b^2d(x_{m+p-2},x_{m+p-3})+b^2d(x_{m+p-3},x_{m+p-4})+ ....+b^{\frac{p-1}{2}}d(x_{m+3},x_{m+2})+b^{\frac{p-1}{2}}d(x_{m+2},x_{m+1})+b^{\frac{p-1}{2}}d(x_{m+1},x_m)$

$\psi(d(x_{m+p},x_m))\preceq\psi(bd(x_{m+p},x_{m+p-1}))+\psi(bd(x_{m+p-1},x_{m+p-2}))+\psi(b^2d(x_{m+p-2},x_{m+p-3}))+...+\psi(b^{\frac{p-1}{2}}d(x_{m+1},x_m))$

$= \psi(b)\psi(d(x_{m+p},x_{m+p-1})) + \psi(b)\psi(d(x_{m+p-1},x_{m+p-2})) + ... + \psi(b^{\frac{p-1}{2}})\psi(d(x_{m+1},x_m))$

$|| \psi(d(x_{m+p},x_m)) || \preceq || \phi |||| b |||| d(x_{m+p},x_{m+p-1}) || + || \phi |||| b |||| d(x_{m+p-1},x_{m+p-2}) ||$
$+ || \phi |||| b^2 |||| d(x_{m+p-2},x_{m+p-3}) || + ... +$
$|| \phi |||| b^{\frac{p-1}{2}} |||| d(x_{m+1},x_m) || \to 0\,(m \to \infty)$

Then $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there exists $x \in X$ such that $lim_{n\to\infty}x_n = x$. Due to the continuity of T,

$$lim_{n\to\infty}x_n = x = lim_{n\to\infty}Tx_{n-1} = Tx.$$

□

Example 3.4. *Let $X = [0,2]$ and $\mathbb{A} = \mathbb{C}$ with a norm $|| z || = | z |$*
be a C^*- algebra.
We define $\mathbb{C}^+ = \{z = (x,y) \in \mathbb{C}; x = Re(z) \geq 0, y = Im(z) \geq 0\}$.
The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ i s the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.
Let $d : X \times X \to \mathbb{C}$
Suppose that $d(x,y) = (2 \mid x-y \mid, 2 \mid x-y \mid)$ for $x,y \in X$. Then, $(X,\mathbb{C},d)$ is a C^*- algebra valued rectangular $b-$ metric space where $b = 1$ with the required propreties of theorem 3.3.
Let $\psi,\phi : \mathbb{C}^+ \to \mathbb{C}^+$ such that they can defined as follows: for $t = (x,y) \in \mathbb{C}^+$,

$$\psi(t) = \begin{cases} (x,y) \; if x \leq 2 \; and \; y \leq 2 \\ (x^2,y) \; if x > 2, y \leq 2 \\ (x,y^2) \; if x \leq 2 \; and \; y > 2 \\ (x^2,y^2) \; if x > 2 \; and \; y > 2 \end{cases}$$

and for $s = (s_1,s_2) \in \mathbb{C}^+$ with $v = min\{s_1,s_2\}$,

$$\phi = \begin{cases} (\frac{v^2}{4},\frac{v^2}{4}) \; if v \leq 2 \\ \\ (\frac{1}{4},\frac{1}{4}) \; if v > 2 \end{cases}$$

Then ψ and ϕ have the propreties mentioned in definitions 2.8 and 2.9.

Let $T : X \to X$ be defined as follows : $T(x) = \begin{cases} 0 \; if 0 \leq x \leq 1 \\ \\ \frac{1}{8} \; if 1 < x \leq 2 \end{cases}$

Then ,T has the required properties montioned in theorem 3.3 we show that 0 is a fixed point of T

Theorem 3.5. *Let $(X, \mathbb{A}, d)$ be a complete C^*- algebra valued rectangular $b-$metric space. Let $T : X \longrightarrow X$ be a mapping function and :*

$$\psi(d(Tx,Ty)) \preceq \psi(M(x,y)) - \phi(d(x,y))$$

and

$$M(x,y) = a_1 d(x,y) + a_2[d(Tx,y) + d(Ty,x)] + a_3[d(Tx,x) + d(Ty,y)]$$

where $b \in \mathbb{A}'_+$, $a_1, a_2, a_3 \geq 0$, $a_1 + 2a_2 b + (2+b)a_3 \leq 1$,

ψ and ϕ are $*-$ homomorphisms and with the constraint $\psi(a) < \phi(a)$.

Then, T has a fixed point.

Proof. Let $x_0 \in X$ and define

$$x_1 = Tx_0, x_2 = Tx_1, ..., x_n = Tx_{n-1}$$

We have

$$\psi(d(x_{n+2}, x_{n+1})) = \psi(d(Tx_{n+1}, Tx_n))$$

$$\preceq \psi(M(x_{n+1}, x_n)) - \phi(d(x_{n+1}, x_n))$$

$$= \psi(a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]) - \phi(d(x_{n+1}, x_n)).$$

Using a proprety of ϕ, we have

$\psi(d(x_{n+2}, x_{n+1})) \preceq \psi(a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]$

Using the strongly monotone proprety of ψ, we have

$d(x_{n+2}, x_{n+1}) \preceq a_1 d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]$

that is $(1-a_2b-a_3b)d(Tx_{n+1},Tx_n) \preceq (a_1+a_2b+a_3(2+b))d(x_{n+1},x_n)$

Therefore

$$d(x_{n+2},x_{n+1}) \preceq \frac{a_1+a_2b+a_3(2+b)}{1-a_2b-a_3b}d(x_{n+1},x_n)$$

wich implies that

$$d(x_{n+2},x_{n+1}) \preceq d(x_{n+1},x_n)$$

since

$$\frac{a_1+a_2b+a_3(2+b)}{1-a_2b-a_3b} < 1$$

Therefore $\{d(x_{n+1},x_n)\}$ is monotone decreasing sequence.

Hence by lemma 3.2 there exists $u \in \mathbb{A}^+$ such that $d(x_{n+1},x_n) \to u$ as $n \to \infty$.

Taking $n \to \infty$ in

$$\psi(d(x_{n+2},x_{n+1})) = \psi(a_1d(x_{n+1},x_n)+a_2[d(x_{n+2},x_n)+d(x_{n+1},x_{n+1})]+a_3[d(x_{n+2},x_{n+1})+ d(x_{n+1},x_n)])-\phi(d(x_{n+1},x_n))$$

Using the continuities of ψ and ϕ, we have

$\psi(u) \preceq \psi((a_1+2a_2+2a_3)u)-\phi(u)$ wich implies that $\psi(u) \preceq \psi(u)-\phi(u)$ (since $a_1+2a_2+2a_3 \leq 1$ and ψ is strongly monotonic increasing) which is a contradiction unless $u=\theta$. Hence $d(x_{n+1},x_n) \to \theta$ as $n \to \infty$ (1).

Next we show that $\{x_n\}$ is a Cauchy sequence .If $\{x_n\}$ is not a Cauchy sequence then by lemma 3.2 ,there exists $c \in \mathbb{A}$ such that $\forall n_0 \in \mathbb{N}$, $\exists n,m \in \mathbb{N}$ with $n > m \geq n_0$ $\phi(c) \preceq d(x_n,x_m)$. Therefore there exists sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ such that for all positive integers k, $n_k > m_k > k$ and $d(x_{n_{(k)}},x_{m_{(k)}}) \succeq \phi(c)$ and $d(x_{n_{(k)-1}},x_{m_{(k)}} \preceq \phi(c)$

then

$$\phi(c) \preceq d(x_{n_{(k)}},x_{m_{(k)}}) \preceq b[d(x_{n_{(k)}},x_{n_{(k)-1}})+d(x_{n_{(k)-1}},x_{n_{(k)-2}})+d(x_{n_{(k)-2}},x_{m_{(k)}})]$$

that is

$$\phi(c) \preceq d(x_{n_{(k)}},x_{m_{(k)}}) \preceq b[d(x_{n_{(k)}},x_{n_{(k)-1}})+d(x_{n_{(k)-1}},x_{n_{(k)-2}})+\phi(c)]$$

letting $k \to \infty$ we have

$$lim_{k\to\infty}d(x_{n_{(k)}},x_{m_{(k)}}) = b\phi(c)\,(2)$$

again

$$d(x_{n_{(k)}},x_{m_{(k)}}) \preceq b[d(x_{n_{(k)}},x_{n_{(k)+1}})+d(x_{n_{(k)+1}},x_{m_{(k)+1}})+d(x_{m_{(k)+1}},x_{m_{(k)}})]$$

and

$$d(x_{n_{(k)+1}},x_{m_{(k)+1}}) \preceq b[d(x_{n_{(k)+1}},x_{n_{(k)}})+d(x_{n_{(k)}},x_{m_{(k)}})+d(x_{m_{(k)}},x_{m_{(k)+1}})]$$

letting $k \to \infty$ in above inequalities , we have

$$lim_{k\to\infty}d(x_{n_{(k)+1}},x_{m_{(k)+1}}) = b\phi(c)\,(3)$$

Again

$$d(x_{n_{(k)}},x_{m_{(k)+1}}) \preceq b[d(x_{n_{(k)}},x_{n_{(k)+1}})+d(x_{n_{(k)+1}},x_{m_{(k)}})+d(x_{m_{(k)}},x_{m_{(k)+1}})]$$

and

$$d(x_{n_{(k)+1}},x_{m_{(k)}}) \preceq b[d(x_{n_{(k)+1}},x_{n_{(k)}})+d(x_{n_{(k)}},x_{m_{(k)+1}})+d(x_{m_{(k)+1}},x_{m_{(k)}})]$$

Further,

$$d(x_{n_{(k)+1}},x_{m_{(k)}}) \preceq b[d(x_{n_{(k)+1}},x_{n_{(k)}})+d(x_{n_{(k)}},x_{m_{(k)+1}})+d(x_{m_{(k)+1}},x_{m_{(k)}})]$$

and

$$d(x_{n_{(k)}},x_{m_{(k)}}) \preceq b[d(x_{n_{(k)}},x_{n_{(k)+1}})+d(x_{n_{(k)+1}},x_{m_{(k)+1}})+d(x_{m_{(k)+1}},x_{m_{(k)}})]$$

Letting $k \to \infty$ in the above four inequalities we have

$$lim_{k\to\infty}d(x_{n_{(k)}},x_{m_{(k)+1}}) = b\phi(c)\,(4)$$

$$lim_{k\to\infty}d(x_{n_{(k)+1}},x_{m_{(k)}}) = b\phi(c)\,(5)$$

Using (1), (2), (4), $and\,(5)$ we have

$$lim_{k\to\infty}M(x_{n_{(k)}},x_{m_{(k)}}) = lim_{k\to\infty}a_1d(x_{n_{(k)}},x_{m_{(k)}})+a_2[d(x_{n_{(k)+1}},x_{m_{(k)}})+d(x_{m_{(k)+1}},x_{n_{(k)}})]+$$
$$a_3[d(x_{n_{(k)+1}},x_{n_{(k)}})+d(x_{m_{(k)+1}},x_{m_{(k)}})]$$
$$=(a_1+2a_2)b\phi(c)\;(6)$$

Clearly $x_{m_k} \preceq x_{n_k}$.Putting $x = x_{n_{(k)}}$,$y = x_{m_{(k)}}$

$$\psi(d(x_{n_{(k)+1}},x_{m_{(k)+1}})) = \psi(d(Tx_{n_{(k)}},Tx_{m_{(k)}}))$$
$$\preceq \psi(M(x_{n_{(k)}},x_{m_{(k)}}))-\phi(x_{n_{(k)}},x_{m_{(k)}})$$

Letting $k \to \infty$ in the above inequality using (2), (3) and(6) and the continuities of ψ and ϕ we have

$$\psi(b\phi(c)) \preceq \psi((a_1+2a_2)b\phi(c)) - \phi(b\phi(c))$$

that is

$\psi(b\phi(c)) \preceq \psi(\phi(c)) - \phi(\phi(c))$,(since $(a_1+2a_2)b < 1$) and ψ is strongly monotonic increasing. Which a contradiction by virtue of a proprety of ϕ. Hence $\{x_n\}$is a Cauchy sequence. From the completness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is continous and $Tx_n \to Tz$ as $n \to \infty$ that is $lim_{n\to\infty}x_{n+1} = Tz$, that is $z = Tz$. Hence z is a fixed point of T. $\square$

Example 3.6. *Let $X = [0,1]$ and $\mathbb{A} = \mathbb{C}$ with a norm $||\ z\ || = |\ z\ |$ be a C^*- algebra.*
We define $\mathbb{C}^+ = \{z = (x,y) \in \mathbb{C}; x = Re(z) \geq 0, y = Im(z) \geq 0\}$.
The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.
Let $d : X \times X \to \mathbb{C}$
Suppose that $d(x,y) = (|\ x-y\ |, |\ x-y\ |)$ for $x,y \in X$.
Then, $(X,\mathbb{C},d)$ is a C^*- algebra valued rectangular $b-$ metric space where $b = 1$ with the required propreties of theorem 3.5.
Let $\psi,\phi : \mathbb{C}^+ \to \mathbb{C}^+$ such that they can defined as follows:
for $t = (x,y) \in \mathbb{C}^+$,

$$\psi(t) = \begin{cases} (x,y)\ if x \leq 1\ and\ y \leq 1 \\ (x^2,y)\ if x > 1, y \leq 1 \\ (x,y^2)\ if x \leq 1\ and\ y > 1 \\ (x^2,y^2)\ if x > 1\ and\ y > 1 \end{cases}$$

and for $s = (s_1,s_2) \in \mathbb{C}^+$ with $v = min\{s_1,s_2\}$,

$$\phi = \begin{cases} (\frac{v^2}{2},\frac{v^2}{2})\ if v \leq 1 \\ \\ (\frac{1}{2},\frac{1}{2})\ if v > 1 \end{cases}$$

Then ψ and ϕ have the propreties mentioned in definitions 2.8 and 2.9.

Let $T : X \to X$ *be defined as follows :* $T(x) = \begin{cases} 0 \; if 0 \leq x \leq \dfrac{1}{2} \\ \\ \dfrac{1}{16} \; if \dfrac{1}{2} < x \leq 1 \end{cases}$

Then ,T has the required properties mentioned in theorem 3.5.

Let $a_1 = \dfrac{1}{2}, a_2 = \dfrac{1}{8}$ *and* $a_3 = \dfrac{1}{8}$. *It can be verified that*

$$\psi(d(Tx,Ty)) \preceq \psi(M(x,y)) - \phi(d(x,y)) \text{ for all } x,y \in X \text{ with } y \preceq x$$

the conditions of theorem 3.5 are satisfied .Here it is seen that 0 is a fixed point of T

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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