Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 1, 167-176
ISSN: 1927-5307

AN EFFECTIVE NUMERICAL METHOD FOR SOLVING NONLINEAR SECOND-ORDER BOUNDARY VALUE PROBLEMS

BAKER AL-WARDAT[1,*], KHALED ALKHNAIFES[1], AND KAMEL AL−KHALED[2]

[1]Department of Mathematics, University of Damascus, Syria

[2]Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

Abstract. In this paper, we present an efficient numerical algorithm for approximate solutions of nonlinear second-order boundary value problems. We use the Laplace transform decomposition method to develop a new method for computing an approximate solution for nonlinear second-order boundary value problems. The Adomian decomposition method (shortly, ADM) together with the application of Laplace transform integral operator are applied to the differential equation. The new approach provides the solution in the form of a convergence series. An iterative algorithm is constructed for the determination of the infinite series solution. Numerical results are included to demonstrate the reliability and efficiency of the proposed scheme. Comparison between exact and approximate solutions with known results is made.

Keywords: Second order boundary value problems, Adomian decomposition method, Laplace transform, Numerical solutions.

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction

Mathematical modeling of many physical system leads to nonlinear ordinary differential equations. An effective method is required to analyze the mathematical modeling which

*Corresponding author

Received September 21, 2012

provides solutions conforming to physical reality, i.e, the real world of physics. Therefore, we must be able to solve nonlinear ordinary differential equations, in space and time, which maybe strongly nonlinear. Common analytic procedures linearized the system or assume that nonlinearities are relative insignificant. Such procedures change the actual problem to make it attractable by the conventional methods. In short, the physical problem is transformed to a purely mathematical one for which the solution is readily available. This change, sometimes seriously, the solution. Generally the numerical methods such as Rungs Kutta method are based on the discretization techniques, and they only permit us to calculate the approximate solution for some values of time and space variables, which cause us to overlook some important phenomena such as chaos and bifurcations, because generally nonlinear dynamic systems exhibit some delicate structures in very small time and space intervals. Also, the numerical methods require computer-intensive calculations. The ability to solve nonlinear equations by an analytic method is important because linearization change the problem, perturbation is only reasonable when nonlinear effects are very small, and the numerical methods need a substantial amount of computation but only get limited information. Since the beginning of the 1980s, Adomian's has presented and developed a so-called Decomposition Method for solving linear and nonlinear problems such as ordinary differential equations. Adomian's Decomposition Method (ADM) consist of splitting the given solution into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, Identifying the initial and/or boundary conditions and the terms involving the independent variables alone as initial approximation, decomposition the unknown function into series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials. ADM is quantitative rather than quantitative, analytic, requiring neither linearization nor perturbations, and continuous with no resort to discretization and consequent computer-intensive calculations.ADM is are relatively new approach to provide analytical approximation to linear and non-linear problems, and it is particularly valuable as a tool for Scientists and applied mathematicians, because it provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solution to both linear and non-linear differential equations without linearization or discretization. Over the past few years, many new alternative to the use of traditional methods for the numerical solution of differential equations have been proposed. Linear or nonlinear two-point boundary value problems of second order can be readily solved by many methods. In [19] an accurate algorithm for the solution of special fourth-order boundary value problems with two-point boundary conditions is developed. In [10] the authors used Lindstedt-Poincare method, the Krylov-Bogoliubov first approximate method, and the differential transform method to handle second-order differential equations with oscillations. A combined form of the Laplace transform method (LTM) with the differential transform method (DTM) is used to solve non-homogeneous linear partial differential equations (PDEs) [11]. A modified form of the Adomian decomposition method is applied to construct the numerical solutions for such problems. In [13], the author illustrates the use of both, Laplace transform, and the decomposition method to approximate the solution of Bratu's boundary value problem. Numerical solutions for boundary value problems were obtained by the shooting method, and by representing the nonlinear differential equation as an integral equation then applying the decomposition method [8].

In this paper, with the same analysis as in [13], we illustrates how the Laplace transform integral operator and the Adomian's decomposition method (ADM) [1, 2] can be both efficiently manipulated for obtaining explicit and numerical solutions of the nonlinear second-order boundary value problems.

$$\frac{d^2y}{dx^2} = \lambda(x)e^{\mu y}, \quad y(0) = y(1) = 0 \tag{1}$$

where $\mu > 0$ is a constant and the function $\lambda(x) > 0$ may be a polynomial or a rational function, or exponential. Due to the large different possibilities, we will assume $\lambda(x)$ to be any analytic function of x which has a power series expansion. Equation (1) occurs frequently in diffusion theory, Celestial Machines, for example in mechanical problems without dissipation. A good number of research papers which deals with the study of approximate solutions of these kind of equation are available in the literature.

In recent years a lot of attention has been developed to the study of the decomposition method to investigate various scientific models [14, 15, 18, 6, 7].

2. Laplace transform algorithm

In this section, the Laplace transform decomposition method is introduced, and the scheme is implemented for the solution of special nonlinear second-order boundary value problem of the form

$$\frac{d^2y}{dx^2} = \lambda(x)e^{\mu y}, \quad 0 < x < 1 \tag{2}$$

with boundary conditions

$$y(0) = y(1) = 0 \tag{3}$$

The exact solution of (2)-(3) is given by [12]

$$y(x) = \frac{2}{\mu}\ln\Big[\frac{\sqrt{2}(\sqrt{\lambda_1} \mp \sqrt{\lambda_0})}{\sqrt{\mu\lambda(x)(\sqrt{\lambda_1}(1-x) \pm \sqrt{\lambda_0}x)}}\Big] \tag{4}$$

provided that

$$\frac{(\sqrt{\lambda_1} \mp \sqrt{\lambda_0})^2}{\lambda_1\lambda_0} = \frac{\mu}{2}$$

. The technique consists of applying Laplace transform integral operator (denoted by ℓ) to both sides of equation (2). Hence

$$\ell[\frac{d^2y}{dx^2}] = \ell[\lambda(x)e^{\mu y}]$$

Applying the formulas on Laplace transform, we obtain

$$s^2\ell[y] - sy(0) - y'(0) = \ell[\lambda(x)e^{\mu y}] \tag{5}$$

Using the initial condition $y(0) = 0$ in equation (3), and setting $A = y'(0)$, it follows that

$$s^2\ell[y] - A = \ell[\lambda(x)e^{\mu y}] \tag{6}$$

The constant A will be determined later by using the boundary condition at $x = 1$. Rewrite equation (6) as

$$\ell[y] = \frac{A}{s^2} + \frac{1}{s^2}\ell[\lambda(x)e^{\mu y}] \tag{7}$$

The Laplace transform decomposition introduces the solution $y(x)$ by a decomposition series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{8}$$

and the nonlinear function $e^{\mu y}$ is decomposed as

$$e^{\mu y} = f(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, ..., y_n) \tag{9}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recursively, and $A_n(y_0, y_1, ..., y_n)$ are the Adomian polynomials that can be constructed for various classes of nonlinearity according to specific algorithms set by Adomian [1, 2] and Wazwaz [18].

$$A_0 = f(y_0),$$

$$A_1 = y_1 f'(y_0),$$

$$A_2 = f''(y_0)\tfrac{y_1^2}{2!} + f'(y_0)y_2,$$

The first few polynomials are given by

$$A_3 = \tfrac{1}{3!}f'''(y_0)y_1^3 + f''(y_0)y_1y_2 + f'(y_0)y_3$$

$$A_4 = \tfrac{1}{4!}f^{(4)}(y_0)y_1^4 + \tfrac{1}{2}f^{(3)}(y_0)y_1^2y_2 + \tfrac{1}{2}f''(y_0)y_2^2 + f'(y_0)y_4$$

$$\vdots$$

Substituting equations (8) and (9) into equation (7) yields

$$\ell[\sum_{n=0}^{\infty} y_n] = \frac{A}{s^2} + \frac{1}{s^2}\ell[\sum_{n=0}^{\infty} \lambda(x)A_n] \tag{10}$$

The decomposition method [18] identifies the zeroth component $y_0(x)$ by all terms that arise from the boundary conditions at $x = 0$ and from the Laplace transform of the source term (if any). Based on this identification, matching both sides of equation (10) yields the following iterative algorithm:

$$\ell[y_0(x)] = \frac{A}{s^2} \tag{11}$$

$$\ell[y_1(x)] = \frac{1}{s^2}\ell[\lambda(x)A_0] \tag{12}$$

$$\ell[y_2(x)] = \frac{1}{s^2}\ell[\lambda(x)A_1] \tag{13}$$

In general,

$$\ell[y_{k+1}(x)] = \frac{1}{s^2}\ell[\lambda(x)A_k], \quad k \geq 0 \tag{14}$$

For the determination of the components $y_n(x)$ of $y(x)$. Applying the inverse Laplace transform to equation (11) we obtain

$$y_0(x) = Ax \tag{15}$$

Next, substituting this value of $y_0(x)$ and that of A_0 into equation (12) gives

$$\ell[y_1(x)] = \frac{1}{s^2}\ell[\lambda(x)f(y_0)] \tag{16}$$

Applying the inverse Laplace transform to equation (16) we obtain the value of $y_1(x)$. Substituting the values of y_0, y_1 and A_1 into equation (13), and then applying the Laplace inverse yields the value of $y_2(x)$. Continue this way and note that the efficiency of this approach can be dramatically enhanced by determining more components of $y(x)$ as far as we wish. To determine the constant A, we define the $n-$term approximate solution by

$$\phi_n = \sum_{i=0}^{n-1} y_i(x;A) = y_0(x;A) + y_1(x;A) + y_2(x;A) + ...$$

This solution has yet to satisfy the remaining boundary condition at $x = 1$ in equation (3). Setting $\phi_n(1) = 0$, then solving the resulting algebraic equation in A, will determine the unknown constant A and eventually the numerical solution of our nonlinear second-order boundary value problem. Finally the solution $y(x)$ is given by $y(x) = \lim_{n\to\infty} \phi_n$. The existence and uniqueness of the solution is guaranteed by a result in [12]. To give a clear overview of the content of this work, in the next section, an illustrative example has been selected to demonstrate the efficiency of the method.

3. Numerical Results

In this section we provide a numerical example which verify the convergence of the Laplace decomposition algorithm, described in section 2.

The example included here illustrate various features of the method, demonstrating the ease of implementation and assembly of the iterative technique in using the ADM.

Example 3.1. *Consider the nonlinear boundary value problem*

$$y''(x) = \pi e^{y(x)}, \quad 0 < x < 1 \tag{17}$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 0, \tag{18}$$

The exact solution for this problem is $y(x) = -\ln[1 + \sin(\pi x)]$. In this example the nonlinear operator $f(y) = \pi e^{y(x)}$, and the first three Adomian polynomials are given by

$$A_0 = -6e^{-4y_0},$$

$$A_1 = 24y_1 e^{-4y_0},$$

$$A_2 = -48e^{-4y_0}y_1^2 + 24e^{-4y_0}y_2,$$

$$A_3 = 64e^{-4y_0}y_1^3 - 96e^{-4y_0}y_1y_2 + 24e^{-4y_0}y_3,$$

$$\vdots$$

and so on for the other polynomials. In order to avoid evaluating the Laplace transform of some difficult terms, in matching both sides of equation (10) and upon using the modified decomposition method [19] we introduce the recursive relation

$$y_0(x) = x,$$

$$y_1 = x + \frac{A}{2}x^2 + \ell^{-1}\left[\frac{1}{s^2}\ell(g(x))\right] + \ell^{-1}\left[\frac{1}{s^2}\ell(f(y_0))\right],$$

$$y_{k+1}(x) = \ell^{-1}\left[\frac{1}{s^2}\ell(A_k)\right], \quad k \geq 1$$

where ℓ^{-1} is to denote the inverse Laplace transform. The first four components of the solution $y(x)$ are given by

$$y_0(x) = Ax,$$

$$y_1(x) = \frac{3(1-e^{-4Ax}-4Ax)}{8A^2},$$

$$y_2(x) = \frac{-9e^{-8Ax}(1+4e^{4Ax}(1+4Ax)+e^{8Ax}(-5+8Ax))}{64A^4}$$

The approximate solution is $\phi_4 = \sum_{i=0}^{3} y_i(x;A)$, which has yet to satisfy the remaining two boundary conditions. Imposing the boundary conditions $y(1) = 0,$, on the 4−term approximation ϕ_4 gives an algebraic system for the unknown A. Upon solving this system (using *Mathematica*) we get $A = -0.136357$, and eventually the numerical solution is obtained. Table 1 show results for the two example. The error between the numerical approximation y_i and the true solutions $y(x_i)$ is determined and reported as $|y_i - y(x_i)|$. All results are tabulated in the form $a.aa - E\gamma$ which represent $a.aa \times 10^{-\gamma}$. The error of our method is displayed in the third column of each Table. For comparison the error of the method taken from [19] are recorded in the second column, this comparison indicated that Laplace decomposition method is better than the modified decomposition method.

4. Conclusions

In this paper, the Laplace transform decomposition method produced a reliable computational method for handling boundary value problems. The results in this paper indicates that our procedure can be used to obtain accurate numerical solutions of nonlinear boundary value problems with very little computational effort.

x_i	Wazwaz Error [19]	Our Error
0	0.000000	0.000000
0.1	6.9E-09	8.88E-16
0.2	1.3E-08	1.33E-15
0.3	1.6E-08	1.55E-15
0.4	2.3E-08	0.000000
0.5	2.5E-08	4.44E-16
0.6	2.4E-08	2.22E-15
0.7	2.2E-08	3.10E-15
0.8	1.7E-08	1.77E-15
0.9	1.2E-08	1.59E-14
1.0	2.0E-09	5.41E-14

Table 1. Numerical Values for Example 3.1

References

[1] G. Adomian, A Review of the Decomposition Method in Applied Mathematics. *J. Math. Anal. Appl.*, 135(1988), 501-544.

[2] G. Adomian, (1994) *Solving Frontier Problems of Physics: The decomposition method*. Kluwer Academic Publishers, Boston.

[3] R. P. Agarwal, (1986) *Boundary Value Problems for High Ordinary Differential Equations,* World Scientific, Singapore.

[4] E. A. Al-Said, M. A. Noor, Computational method for fourth order obstacle boundary value problems, *Comm. Appl. Nonlinear Analy.* 2(3), (1995) 73-83.

[5] E. A. Al-Said, M. A. Noor, Numerical solutions of a system of fourth order boundary value problems, *Internt J. Comput. Math.* 71 (1998) 347-355.

[6] Y. Cherrualt, Convergence of Adomian's method, *Kybernetes* 18(1989), 31-38.

[7] Y. Cherrualt; G. Adomian, Decomposition methods: a new proof of convergence, *Math. Comput. Model.* 18(1993) 103-106.

[8] E. Deeba, S. A. Khuri, S. Xie, An algorithm for solving boundary value problems, *J. Comput. Phy.* 159 (2000) 125-138.

[9] M. El-Gamel; S. H. Behiry; H. Hashish; Numerical method for the solution of special nonlinear fourth-order boundary value problems. *Appl. Math. Comput.*, 145 (2003) 717-734.

[10] Marwan Alquran; Kamel Al-Khaled; Effective approximate methods for strongly nonlinear differential equations with oscillations. *Mathematical Sciences*, 6(32) (2012) doi:10.1186/2251-7456-6-32.

[11] Marwan Alquran; Kamel Al-Khaled; Mohammed Ali; A. Ta'any; The combined Laplace transform-differential transform method for solving linear non-homogeneous PDEs. *Journal of Mathematical and Computational Science*, 3(2) (2012) 690-701.

[12] V. Goyal; On a second order nonlinear boundary value problem. *Appl. Math. Letters*, In press.

[13] S. A. Khuri, A new approach to Btatu's problem, *Appl. Math. Comput.* 147 (2004) 131-136.

[14] D. Kaya; I. E. Inan, Exact and numerical traveling wave solutions for nonlinear coupled equations using symbolic computation. *Appl. Math. Comp.* In press.

[15] D. Kaya; A numerical simulation of solitary-wave solutions of the generalized regualized long-wave equation. *Appl. Math. Comput.* 149 (2004) 833-841.

[16] A. C. Morelet, Convergence of the sinc method for a fourth-order ordinary differential equation with an application. *SIAM J. Numer. Analy.* Vol. 32, No. 5 (1995) 1475-1503.

[17] Ralph C. Smith, Gary A. Bogar, Kenneth L. Bowers, and John Lund, The Sinc-Galerkin method for fourth-order differential equations. *SIAM J. Numer. Analy.* Vol. 28, No. 3 (1991) 760-788.

[18] A. M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators. *Appl. Math. Comput.*, 111 (2000) 53-69.

[19] A. M. Wazwaz, The numerical solution of special fourth-order boundary value problems by the modified decomposition method, *Internt. J. computer Math.*, Vol. 79(3)2002, 345-356.