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FINITE K- FRAMES IN HILBERT SPACES

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Abstract: *K*-frames were recently introduced by Găvruţa in Hilbert spaces to study atomic systems with respect to a bounded linear operator. Some results on finite k-frames in finite dimensional Hilbert space are studied. The properties of eigen values of k-frame operator are discussed.

Keywords: k-frame; Hilbert spaces; eigen values.
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1. INTRODUCTION

Frames are generalization of bases. D. Han and D.R. Larson [4] have developed a number of basic aspects of operator-theoretic approach to frame theory in Hilbert space. Peter G. Casazza [2] presented a tutorial on frame theory and he suggested the major directions of research in frame theory. The notation of K-frames has been introduced by Gavrutha [3] to study the atomic system with respect to a bounded linear operator K in a Separable Hilbert space H. Fahimeh Arabyani Neyshaburi and Ali Akber Arefijamaal [1] were characterize all duals of a given k-frame and given some approaches for constructing K-frames. Mitra Shamsabadi and Ali Akbar Arefijamaal [5] were study soe properties of K- frames and introduced the K-frame multipliers.

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In [6] A. Rahimi, Sh. Najafzadehnad M. Nouri were developed the concept of controlled K-frames. Some properties of K Dual frames are investigated by Vahid Reza, Mohammad Janfada and Rajabali Kamyabi Gol [7].

In this paper some results on finite k-frames in finite dimensional Hilbert space are studied. The properties of eigen values of k-frame operator are discussed.

2. PRELIMINARIES

Frames are generalizations of orthonormal basis in Hilbert spaces. Here we recall a few basic definitions and results needed in the sequel.

Definition 2.1 A sequence $\{f_j\}_{j\in J}$ of vectors in a Hilbert space H is called a frame if there exists two constants $0 < A \le B < \infty$, such that

$$A\|f\|^2 \le \sum_{j\in J}\left|\langle f, f_j\rangle\right|^2 \le B\|f\|^2 \ \forall f \in H$$

The above inequality is called a frame inequality. The numbers A and B are called the lower and upper frame bounds respectively. If A=B then $\{f_j\}_{j\in J}$ is called tight frame, if A=B=1 then $\{f_j\}_{j\in J}$ is called normalized tight frame. A synthesis operator $T : l_2 \to H$ is defined as $Te_j = f_j$ where $\{e_j\}$ is an orthonormal basis for l_2. The analysis operator $T^* : H \to l_2$ is an adjoint of synthesis operator T and is defined as $T^* f = \sum_{j\in J} < f, f_j > e_j \ \forall f \in H$. A frame operator $S = TT^* : H \to H$ is defined as $Sf = \sum_{j} < f, f_j > f_j \ \forall f \in H$

The following few theorems from [1, 5] are useful in our discussion.

Theorem 2.2. Suppose $\{f_j\}_{j\in J}$ is a frame for H if and only if $AI_{op} \le S \le BI_{op}$ and $\{f_j\}_{j\in J}$ is normalized tight frame for H if and only if $S = I_{op}$, where I_{op} is an identity operator on H.

Theorem 2.3. [5] Let S be a frame operator for the frame $\{f_j\}_{j\in J}$ with frame bounds A and B in the Hilbert space H. Then S^{-1} exists, positive and $B^{-1}I_{op} \le S^{-1} \le A^{-1}I_{op}$.

We consider a sequence of M vectors $\{f_j\}_{j=1}^{M}$ in N-dimensional real or complex Hilbert space H_N, where M and N are positive integers with $M \geq N$. Here l_2 is K^N where K=R or C. For a frame $\{f_j\}_{j=1}^{M} \subset H_N$, the existence of the corresponding synthesis, analysis and frame operators are $T: K^N \to H_N$, $T^*: H_N \to K^N$ and $S = TT^*: H_N \to H_N$ which may be represented as $N \times M$, $M \times N$ and $N \times N$ matrices respectively. With respect to orthonormal basis $\{e_i\}_{i=1}^{N}$ for H_N and the standard basis for K^N, the matrix representations of the synthesis, analysis and frame operators are

$$T = \left[f_1^T \ f_2^T \ . \ . \ . \ f_m^T\right]_{N\,X\,M},\ T^* = \begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ f_m \end{bmatrix}_{M\,x\,N} \quad \text{and } S = TT^* = \left[f_1^T \ \ f_2^T \ \ . \ \ . \ \ f_m^T\right]_{N\,x\,M} \begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ f_m \end{bmatrix}_{M\,x\,N}$$

Consider $\lambda_1, \lambda_2 \ldots \lambda_N$ as the eigen values of S. Here the properties of eigen values of frame operator are discussed.

Theorem 2.4[8]. Let $\{f_j\}_{j=1}^{M}$ be a frame for H_N,then sum of the eigen values of frame operator S is equal to the sum of lengths of the frame vectors.

Proof: Let $\{f_j\}_{j=1}^{M}$ be a frame for H_N. Suppose $\lambda_1, \lambda_2 \ldots \lambda_n$ be the eigen values of the frame operator S.

Consider $\sum_{j=1}^{N} \lambda_j = \text{Tr(S)} = \text{Tr}(T\,T^*) = \sum_{j=1}^{M} f_j\ f_j^* = \sum_{j=1}^{M} \|f_j\|^2$

Example: The set of vectors {(1, 0), (0, 1), (1, -1)} is a frame for R^2. Here the dimension of space is 2 where as the number of frame vectors are 3. They are f_1= (1, 0); f_2 = (0,1); f_3 = (1,-1).

Let $T = \left[f_1^T \ f_2^T \ f_3^T\right]_{2x3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}_{2x3}$ and $T^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

Therefore $S = T\,T^* = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{2x2}$, eigen values of the frame operator S are 1, 3

$\sum_{j=1}^{2}\lambda_j$ = $\lambda_1 + \lambda_2$ = 4; and $\sum_{j=1}^{3}\|f_j\|^2$ = 1 + 1 + 2 = 4.

Therefore we have $\sum_{j=1}^{2}\lambda_j = \sum_{j=1}^{3}\|f_j\|^2$.

3. K-FRAMES

Definition3.1. Let $K \in B(H)$. A sequence $\{f_j\}_{j\in J}$ in Hilbert space H is said to be a K-frame for H if there exist two constants $0 < A \le B < \infty$, such that

$$A\|K^* f\|^2 \le \sum_{j\in J}|< f, f_j >|^2 \le B\|f\|^2, \quad \forall f \in H .$$

Where A and B are called lower and upper frame bounds for k-frame respectively. If K=I, then K-frames are just ordinary frames.

Definition 3.2: Let $\{f_j\}_{j\in J}$ is a K- frame for H. Obviously it is a Bessel sequence, so we can define the following operator $T: l^2 \to H$ by $T(c_j) = \sum_{j\in J} c_j f_j \ \forall \{c_j\} \in l^2$ is called Synthesis operator for K- frame $\{f_j\}_{j\in J}$. Also we have

$T^*: H \to l^2$ by $T^* f = \{< f, f_j >\}_{j\in J}$ $for\ all\ f \in H$ is called Analysis operator for K- frame $\{f_j\}_{j\in J}$.

The frame operator is given by S^k : H $\to$H is defined as S^k f = $\sum_{j\in J}\langle f, f_j\rangle f_j$, for all f$\in$ H.

K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K Because of the higher generality of K-frames many properties for ordinary frames may not hold for K-frames such as the corresponding synthesis operator for K-frames is not surjective.

The different types of k-frames are given below.

S.No	**Description**	Type of frame $\{f_j\}$
1	$A\|K^* f\|^2 \le \sum_{j\in J} \|< f, f_j >\|^2 \le B\|f\|^2, \ \forall f \in H$	K-frame for H
2	If $A\|K^* f\|^2 = \sum_{j\in J} \|< f, f_j >\|^2 \ \forall f \in H$	A-tight K- frame for H
3	If $\|K^* f\|^2 = \sum_{j\in J} \|< f, f_j >\|^2 \ \forall f \in H$	normalized K- frame or Parseval K-frame for H.
4	If $A\|K^* f\|^2 = \sum_{j\in J} \|< f, f_j >\|^2 \ \forall f \in H$ and $\|f_j\| = a \ for \ all \ j$	equal norm A-tight K- frame for H.
5	If $\|K^* f\|^2 = \sum_{j\in J} \|< f, f_j >\|^2 \ \forall f \in H$ and $\|f_j\| = aforallj$	equal norm normalized K- frame for H.
6	If $A\|K^* f\|^2 = \sum_{j\in J} \|< f, f_j >\|^2 \ \forall f \in H$ and $\|f_j\| = 1 \ forallj$	unit norm A-tight K- frame for H.
7	If $\|K^* f\|^2 = \sum_{j\in J} \|< f, f_j >\|^2 \ \forall f \in H$ and $\|f_j\| = 1 \ forallj$	unit norm normalized K- frame for H.

The fallowing examples illustrates that a sequence $\{f_j\}$ is a K-frame.

Example 3.3: Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H and define $K \in B(H)$ as fallows

$$Ke_{2n} = e_{2n} + e_{2n-1}, Ke_{2n-1} = 0 \ n = 1,2, \ldots$$

Then for each $f \in H$ we have

$$Kf = K(\sum_{n=1}^{\infty} < f, e_n > e_n) = K(\sum_{n=1}^{\infty} < f, e_{2n} > e_{2n}) + \sum_{n=1}^{\infty} < f, e_{2n-1} > e_{2n-1})$$

$$= \sum_{n=1}^{\infty} < f, e_{2n} > (e_{2n} + e_{2n-1})$$

It is easy to check that the adjoint operator $K^*: H \to H$ is given by

$$K^* f = \sum_{n=1}^{\infty} < f, e_{2n} + e_{2n-1} > e_{2n} \ \forall f \in H$$

But since, for all $f \in H$

$$\|K^*f\|^2 = \left\|\sum_{n=1}^{\infty} < f, e_{2n} + e_{2n-1} > e_{2n} \; \forall f \in H \right\|^2$$

$$= \sum_{n=1}^{\infty} |< f, e_{2n} + e_{2n-1} >|^2$$

$$\leq 2\sum_{n=1}^{\infty} |< f, e_{2n} >|^2 + 2\sum_{n=1}^{\infty} |< f, e_{2n-1} >|^2 \leq 4\|f\|^2$$

$$\Longrightarrow \|K^*f\|^2 = \sum_{n=1}^{\infty} |< f, e_{2n} + e_{2n-1} >|^2 \leq 4\|f\|^2$$

It follows that $\{f_n\}_{n=1}^{\infty} = \{e_{2n} + e_{2n-1}\}_{n=1}^{\infty}$ is a K-frame for H. here frame bounds A=1 and B= 4

Example 3.4: Let $H = \mathbb{C}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H. Define $K: H \to H$ by $Ke_1 = e_1, Ke_2 = e_1, Ke_3 = e_2$. Then $\{f_i\}_{i=1}^{3} = \{Ke_1, Ke_2, Ke_3\}_{i=1}^{3}$ is a K-frame for H.

And $S^K = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a K- frame operator for H.

Also let T= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $f = e_3 \in H$. then $\sum_{i=1}^{3} |< f, f_i >|^2 = 0$ and $\|T^*f\|^2 = 4$. Hence $\{f_i\}_{i=1}^{3}$ is not a T-frame for H

Example 3.5: Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in l_2. Define T and K on $l_2$by $Te_i = e_{i-1} \, for \; i > 1$

And $Te_1 = 0$. And $Ke_i = e_{i+1} \;\; for \; all \; i$ respectively then it is clear that $\{Ke_i\}_{i=1}^{\infty}$ is K-frame for l_2 but it not a T-frame for l_2.

Example 3.6: $F = \left\{\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$ in $H = \mathbb{C}^2$ and K be an orthogonal projection onto the subspace spanned be e_1 where $\{e_1, e_2\}$ is the orthonormal basis of $\mathbb{C}^2$. Now for all $f = (a, b) \in \mathbb{C}^2$ we obtain

$$\|K^*f\|^2 \leq \sum_{i \in I} |< f, f_i >|^2 = \frac{3}{2}(a^2 + b^2) - ab \leq 2\|f\|^2$$

Then F is a K-frame for $H = \mathbb{C}^2$.

Theorem 3.7[6]: Suppose $\{f_j\}$ is a K- frame for H iff $AKK^* \leq S^K \leq BI$ and Suppose $\{f_j\}$ is called normalized K- frame for H iff $S^K = KK^*I$.

Proof: By using the definition of K-frame operator we have

$$< S^K f, f > = < \sum_{j\in J} < f, f_j > f_j , f > = \sum_{j\in J} < f, f_j >< f_j, f > = \sum_{j\in J} \left|< f, f_j >\right|^2 \forall f \in H$$

$$\Rightarrow < S^K f, f >= \sum_{j\in J} \left|< f, f_j >\right|^2 \forall f \in H \qquad \ldots (1)$$

Consider $A\|K^*f\|^2 = A < K^*f, K^*f > = < AKK^*f, f >$. ... (2)

Suppose $\{f_j\}$ is a K- frame for H $\Leftrightarrow A\|K^*f\|^2 \leq \sum_{j\in J}\left|< f, f_j >\right|^2 \leq B\|f\|^2 \forall f \in H$

$$\Leftrightarrow A\|K^*f\|^2 \leq \quad < S^K f, f > \quad \leq B\|f\|^2 \quad \forall f \in H \text{ by } \quad (1)$$

$$\Leftrightarrow < AKK^*f, f > \leq < S^K f, f > \leq < BIf, f > \forall f \in H$$

$$\Leftrightarrow AKK^* \leq S^K \leq BI \qquad \ldots\ldots..(3)$$

$\Rightarrow \{f_j\}$ is a K- frame for H $\quad \Leftrightarrow AKK^* \leq S^K \leq BI$.

Suppose $\{f_j\}$ is called normalized K- frame for H

$$\Leftrightarrow \|K^*f\|^2 = \sum_{j\in J} \left|< f, f_j >\right|^2 \forall f \in H$$

$$\Leftrightarrow < S^K f, f > = < KK^*f, f > \forall f \in H \quad \ldots\ldots \text{by } (1)$$

$$\Leftrightarrow S^K = KK^*I.$$

$\therefore \{f_j\}$ is called normalized K- frame for H $\quad \Leftrightarrow S^K = KK^*I$.

4. FINITE K-FRAMES IN FINITE DIMENSIONAL HILBERT SPACES

Definition4.1: Let $K \in B(H)$. A sequence $\{f_j\}$in Hilbert space H is said to be a finite K-frame forH_N if there exist two constants $0 < A \leq B < \propto$, such that

$A\|K^*f\|^2 \leq \sum_{j=1}^{m}\left|< f, f_j >\right|^2 \leq B\|f\|^2 \forall f \in H_N$. Here H_N is an N-dimensional Hilbert space.

In the fallowing propositions, we express inequalities for different types of K-frames.

Proposition4.2: If $\{f_j\}_{j=1}^{m}$ is a A- tight K-frame for Hilbert space H_N then for every j=1,2,…m we have $\|f_j\| \leq \sqrt{\|AKK^*\|}$.

Proof: We have $\{f_j\}_{j=1}^{m}$ is a A- tight K-frame for Hilbert space H_N then

$$A\|K^*f\|^2 = \sum_{j=1}^{m}\left|< f, f_j >\right|^2$$

Now for any$1 \leq i \leq m$, $A\|K^*f_i\|^2 = \sum_{j=1}^{m}\left|< f_i, f_j >\right|^2$

$$= \|f_i\|^4 + \sum_{i \neq j=1}^{m}\left|< f_i, f_j >\right|^2$$

$$\Longrightarrow \|f_i\|^4 - A\|K^*f_i\|^2 = -\sum_{i \neq j=1}^{m}\left|< f_i, f_j >\right|^2 \leq 0$$

$$\Longrightarrow \|f_i\|^4 - A\|K^*f_i\|^2 \leq 0$$

$$\Longrightarrow \|f_i\|^4 \leq A\|K^*f_i\|^2$$

$$\Longrightarrow \|f_i\|^4 I - AKK^*\|f_i\|^2 \leq 0$$

$\Longrightarrow I\|f_i\|^2 - AKK^* \leq 0 \Longrightarrow \|f_i\| \leq \sqrt{\|AKK^*\|}$ For i=1,2…m

Corollary 4.3: If $\{f_j\}_{j=1}^{m}$ is a normalised K-frame for Hilbert space H_N then for every j=1,2,…m we have $\|f_j\| \leq \sqrt{\|KK^*\|}$.

Proof: From the Proposition 4.2 we have $\|f_i\| \leq \sqrt{\|AKK^*\|}$ when $\{f_j\}_{j=1}^{m}$ is a A- tight K-frame for Hilbert space H_N. Given that $\{f_j\}_{j=1}^{m}$ is a normalised K-frame for Hilbert space H_N. Then

A=1

Hence $\|f_j\| \leq \sqrt{\|KK^*\|}$.

Proposition 4.4: Let $\{e_i\}_{i=1}^{N}$be an orthonormal basis forH_N. If $\{f_j\}_{j=1}^{m}$ be a K-frame for H_N with frame bounds A and B then $AKK^*N \leq \sum_{j=1}^{m}\|f_j\|^2 \leq BN$

Proof: Suppose $\{e_i\}_{i=1}^{N}$be an orthonormal basis for H_N. By the parsevals identy we have

$$\|f\|^2 \leq \sum_{i=1}^{n}|< f, e_i >|^2 \forall f \in H_N$$

$$\Rightarrow \|f_j\|^2 \leq \sum_{i=1}^{n}|< f_j, e_i >|^2 \forall f \in H_N\ \forall \mathrm{j} = 1,2, \dots \mathrm{m}$$

$$\Rightarrow \sum_{j=1}^{m}\|f_j\|^2 \leq \sum_{j=1}^{m}\sum_{i=1}^{n}|< f_j, e_i >|^2$$

… (3)

Given that $\{f_j\}_{j=1}^{m}$ is a K-frame for H_N with frame bounds A and B then we have

$$A\|K^*f\|^2 \leq \sum_{j=1}^{m}|< f, f_j >|^2 \leq B\|f\|^2 \forall f \in H_N$$

Replacing f by e_i in above

$$A\|K^*e_i\|^2 \leq \sum_{j=1}^{m}|< e_i, f_j >|^2 \leq B\|e_i\|^2 \forall i = 1,2, \dots n.$$

$$AKK^*\sum_{i=1}^{N}\|e_i\|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{m}|< e_i, f_j >|^2 \leq B\sum_{i=1}^{N}\|e_i\|^2$$

$AKK^*N \leq \sum_{j=1}^{m}\|f_j\|^2 \leq BN$by (3)

In the fallowing propositions, we express two equalities for K-frames by using eigen values of frame operator.

Proposition 4.5: Suppose that $\{f_j\}_{j=1}^{m}$ is a K-frame for $R(K) \subseteq H_N$ and $\{\lambda_j\}_{j=1}^{N}$denote the

eigen

values for frame operator S^K then $\sum_{j=1}^{N}\lambda_j = \sum_{j=1}^{m}\|f_j\|^2$.

Proof: Consider

$$\sum_{j=1}^{N}\lambda_j = Tr\left(S^K\right) = Tr(TT^*) = \sum_{j=1}^{m} f_j f_j^{\,*} = \sum_{j=1}^{m}\left\|f_j\right\|^2$$

Proposition 4.6: Let $\{f_j\}_{j=1}^{m}$ be a normalized K-frame forH_N.Then $\sum_{j=1}^{N}\lambda_j = NTr(K)^2$ and

$$NTr(K)^2 = \sum_{j=1}^{N}\left\|f_J\right\|^2.$$

Proof: $\sum_{j=1}^{N}\lambda_j = Tr\left(S^K\right) = Tr(KK^*I) = Tr(K)Tr(K^*)Tr(I) = Tr(K)^2 N.$

Also, we have $Tr\left(S^K\right) = \sum_{j=1}^{N}\left\|f_j\right\|^2$, which implies $\sum_{j=1}^{N}\lambda_j = NTr(K)^2 = \sum_{j=1}^{N}\left\|f_j\right\|^2$.

Proposition 4.7[7]: Let $0 \neq K \in B(H_N)$. Let $\emptyset = \{f_j\}_{j=1}^{m}$ be a K-frame for R(K) witj K-frame operator S^K with eigen values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N > 0$ then λ_1 is the optimal upper K-frame bound and if $\lambda_N \neq 0\ then\,\frac{\lambda_N}{\|K\|^2}$ is the optimal lower K-frame bound.

Proof: Suppose that $\{e_i\}_{i=1}^{N}$ is an orthonormal eigen basis of the frame operator S^K with associated eigenvalues $\{\lambda_i\}_{i=1}^{N}$ given in decreasing order. We can write

$$f = \sum_{j=1}^{N} < f, e_j > e_j \text{ for all } f \epsilon H_N$$

We have for fϵH_N

$$S^K x = S^K\left(\sum_{j=1}^{N} < f, e_j > e_j\right) = \sum_{j=1}^{N} < f, e_j > S^K e_j$$

$$= \sum_{j=1}^{N} < f, e_j > \lambda_j e_j$$

$$= \sum_{j=1}^{N}\lambda_j < f, e_j > e_j \ldots\ldots. (4)$$

Now

$$\sum_{j=1}^{N}\left|< f,\mathrm{f_j} >\right|^2 =< S^K f,f >= < \sum_{j=1}^{N}\lambda_{\mathrm{j}} < f,\mathrm{e_j} > \mathrm{e_j},\mathrm{f} >$$

$$= \sum_{j=1}^{N}\lambda_{\mathrm{j}}\left|< f,\mathrm{e_j} >\right|^2$$

$$\leq \textstyle\sum_{j=1}^{N}\lambda_1\left|< f,\mathrm{e_j} >\right|^2 \quad \text{since } \lambda_{\mathrm{j}} \leq \lambda_1 \text{ for j} = 1,2,\dots \text{n.}$$

$$= \lambda_1 \textstyle\sum_{j=1}^{N}\left|< f,\mathrm{e_j} >\right|^2 = \lambda_1\|f\|^2 \dots.(5)$$

Now

$$\frac{\lambda_N}{\|K\|^2}\|Kf\|^2 \leq \lambda_N\|f\|^2 = \lambda_N\sum_{j=1}^{N}\left|< f,\mathrm{e_j} >\right|^2$$

$$\leq \lambda_{\mathrm{j}}\sum_{j=1}^{N}\left|< f,\mathrm{e_j} >\right|^2 = \sum_{j=1}^{N}\lambda_{\mathrm{j}}\left|< f,\mathrm{e_j} >\right|^2 =< S^K f,f >$$

$$\dots\dots(6)$$

From (5) and (6) we have

$$\frac{\lambda_N}{\|K\|^2} \leq< S^K f,f >\leq \lambda_1\|f\|^2$$

Proposition 4.8: If $f = \{f_j\}_{j=1}^{m}$ is an A-tight K-frame for H_N then $\underset{j=1,2,\dots m}{\mathrm{Max}}\left\|f_j\right\|^2 \leq A\|K\|^2$.

Proof: For any j=1, 2,…. m we have

$$\left\|f_j\right\|^4 \leq \left|< f_j,f_j >\right|^2 \leq \sum_{j=1}^{m}\left|< f_j,f_j >\right|^2 = A\left\|K^* f_j\right\|^2$$

$$\leq A\|K^*\|^2\left\|f_j\right\|^2 \leq A\|K\|^2\left\|f_j\right\|^2 \text{since}\|K^*\| = \|K\|$$

$$\Longrightarrow \left\|f_j\right\|^4 \leq A\|K\|^2\left\|f_j\right\|^2$$

Hence, we have

$$\left\|x_j\right\|^2 \leq A\|K\|^2 forallj = 1,2,\dots m$$

$$\Longrightarrow \underset{j=1,2,\dots m}{\mathrm{Max}}\left\|f_j\right\|^2 \leq A\|K\|^2.$$

Proposition 4.9: If $\emptyset = \{f_j\}_{j=1}^{m}$ is a unit norm A-tight K-frame for H_N, then $A\|K\|^2 N \geq m$.

Proof: Since $\{e_j\}_{j=1}^{N}$ is an orthonormal basis for H_N then for any f we have

$$f = \sum_{i=1}^{N} < f, e_i > e_i \Rightarrow \|f\|^2 = \sum_{i=1}^{N} |< f, e_i >|^2$$

$$\Longrightarrow \|f_j\|^2 = \sum_{i=1}^{N} \left|< f_j, e_i >\right|^2$$

Now

$$m = \sum_{j=1}^{m} \|f_j\|^2 = \sum_{j=1}^{m} \sum_{i=1}^{N} \left|< e_i, f_j >\right|^2$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{m} \left|< e_i, f_j >\right|^2 = \sum_{j=1}^{N} A\|K^* e_j\|^2$$

since $\{f_j\}_{j=1}^{m}$ is a unit norm A-tight K-frame for H_N

$$\leq \sum_{j=1}^{N} A\|K^*\|^2 \|e_j\|^2 = \sum_{j=1}^{N} A\|K\|^2 \|e_j\|^2 = A\|K\|^2 \sum_{j=1}^{N} \|e_j\|^2$$

$$= A\|K\|^2 N$$

Hence $m \leq A\|K\|^2 N$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests

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