Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7892-7905
https://doi.org/10.28919/jmcs/6400
ISSN: 1927-5307

REGIONAL BOUNDARY CONTROLLABILITY OF THE GRADIENT OF SEMI-LINEAR PARABOLIC SYSTEMS

AHAD KHALID NAJI ALSHAMMARI, SID AHMED OULD BEINANE*, MAAWIYA OULD SIDI

RT-M2A Laboratory, Mathematics Department, College of Science, Jouf University, P.O. Box: 2014, Sakaka, Saudi Arabia

Abstract. The purpose of this paper is to explore the concept of regional gradient-controllability for semi linear parabolic distributed systems. We are interested in transferring the state gradient of the system to a desired gradient given on the subregion ω or Γ of the whole domain Ω. We give a definition and delineate some properties of this concept, and we show that under some hypothesis, that the approximate regional controllability for the associated linear system holds, we determine the control ensuring the transfer of the system. The approach developed is based on an extension of the method H.U.M and the Schauder fixed point theorem. The developed method lead to an algorithms for constructing the control ensuring the transfer of the system from a desired gradient and we finish by numerical simulation.

Keywords: partial controllability; semilinear distributed systems; gradient controllability; minimum energy control.

2010 AMS Subject Classification: 93B05, 47B99.

1. INTRODUCTION

The analysis of distributed systems brings together a set of concepts, we will cite the fundamental notions of controllability, stability and stabilization, by duality the notions of observability and

*Corresponding author
E-mail address: sabeinane@ju.edu.sa
Received June 24, 2021

detectability. These various concepts have been the subject of several works and there is very vast literature For an extensive list of publications, see e.g., ([4], [5], [10], [12] and [27]). They are generally approached using two main approaches: the approach variational and the semi-group approach. The development of the first approach, accelerated mainly towards the end of the sixties, is marked among others by the work of [14, 15, 16]. The second approach can be found in the works of ([1], [3], [11], [19] and [20]). This rich literature is generally focused on three types of systems: parabolic, hyperbolic and elliptical.

The regional analysis, introduced in the early 1990s to systems distributed by El Jaï and Zerrik in the works ([7], [8], [9]). One extension which is very important in practical applications is that of the concept of regional controllability. For the theory of analysis of distributed systems, the term regional refers to control problems in which the target which we are interested in is not specified on the whole domain, but only on a ω subregion of the system domain Ω. Then was deepened by Boutoulout's work in ([23] and [25]) for the case where the target region is part of the boundary of Ω note that $\Gamma \subset \partial\Omega$, this is then referred to as boundary regional analysis. Thus, Kamal was interested in the controllability regional gradient of parabolic systems and strategic gradient actuators ([24],[26]) this controllability problems of the gradient is encountered by researchers in the field of industrial engineering. For example, the problem of determining heat exchanges between a plasma jet and a flat target perpendicular to the direction of flow.

The same way Ould Beinane ([13], [17] and [22]) studied the problem of regional controllability of a class of semilinear systems, which constitute an intermediary between linear systems widely treated in the literature and nonlinear ones very close to nature, since real systems are often modeled by derivative equations nonlinear partials.

The objective of the work is to extend the concept of regional controllability of the gradient developed for linear systems to the semi-linear case where the target region Γ is part of the $\partial\Omega$ boundary of Ω. We are interested in transferring the state gradient of the system to a desired gradient given on an boundary Γ part of the domain. We start with definitions of this concept, then we focus on the determination of the explicit expression of such a control. The developed approach is based on an extension of the HUM method and the Schauder fixed point theorem.

Then we lead to algorithms that are successfully implemented numerically and illustrated by examples and simulations.

2. PRELIMINARIES

Let Ω be a regular bounded open set of $I\!R^n$ $(n = 1,2,3)$ with boundary $\partial\Omega$ and $T > 0$.
We denoted by $W = \Omega\times]0,T[$, $\Sigma = \partial\Omega\times]0,T[$ and we consider the system

$$(1)\qquad \begin{cases} \dfrac{\partial z}{\partial t}(x,t) = Qz(x,t) + Nz(x,t) + Bv(t), & W, \\ z(\xi,t) = 0, & \Sigma, \\ z(x,0) = z_0(x), & \Omega, \end{cases}$$

where

- Q is a second-order linear differential operator , and generates a strongly continuous semi-group $(L(t))_{t\geq 0}$ on the Hilbert space $L^2(\Omega)$, the adjoint operator of Q is denoted by Q^*.
- N is a nonlinear operator, $B \in \mathscr{L}(I\!R^p, L^2(\Omega))$ and $v \in V = L^2(0,T,I\!R^p)$.

It is assumed that the system (1) admits a weak solution $z_v(.)$ such that
$z_v(T) \in H^1(\Omega)$.

- The gradient operator

$$\begin{array}{llll} \nabla : & H_0^1(\Omega)\cap H^2(\Omega) & \longrightarrow & (L^2(\Omega))^n \\ & z & \longmapsto & (\dfrac{\partial z}{\partial x_1}, \dots, \dfrac{\partial z}{\partial x_n}). \end{array}$$

- The trace operator

$$\begin{array}{llll} \gamma : & (H^1(\Omega))^n & \longrightarrow & (H^{\frac{1}{2}}(\partial\Omega))^n \\ & z & \longmapsto & \gamma z = (\gamma_0 z_1, \dots, \gamma_0 z_n). \end{array}$$

Where $\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$ the trace operator zero order, continuous linear surjective and γ_0^* (resp. γ^*) be the adjoint operator of γ_0 (resp. γ).

- For $\Gamma \subset \partial\Omega$

$$\begin{array}{llll} \bar{\chi}_\Gamma : & (H^{\frac{1}{2}}(\partial\Omega))^n & \longrightarrow & (H^{\frac{1}{2}}(\Gamma))^n \\ & z & \longmapsto & \bar{\chi}_\Gamma z = z_{|\Gamma}. \end{array}$$

The restriction operator to Γ and $\bar{\chi}_\Gamma^*$ denotes the adjoint operator of $\bar{\chi}_\Gamma$.

Definition 2.1.

The system (1) is exactly (resp. approximate) gradient controllable in Γ if for all g_d a desired gradient, there exist a control $v \in V$ such that $\bar{\chi}_\Gamma(\gamma\nabla z_v(T) = g_d$
(resp. $\|\bar{\chi}_\Gamma(\gamma\nabla z_v(T)) - g_d\| \leq \varepsilon$).

Problem

The problem of the controllability of the gradient on a boundary region $\Gamma \subset \partial\Omega$ for a semi-linear parabolic system is formulated as follows:

$$\text{(2)} \qquad \begin{cases} \text{For } g_d \text{ a desired gradient , there exist a control} \\ \\ v \in V \text{ such that } \bar{\chi}_\Gamma(\gamma\nabla z_v(T) = g_d \ ? \end{cases}$$

3. Main Results

Consider the system (1) which is energized by a zone actuator (D, f) where D is the actuator support and $f \in L^2(D)$ defined its spatial distribution. Under the hypothesis that the system (1) is approximate gradient controllable in the internal region containing Γ, a characterization of the control solution of the problem is determined (2).

Let $p > 0$ an integer, $F_p = \bigsqcup_y B(y, \dfrac{1}{p})$ and $\omega_p = F_p \cap \Omega$ where $B(y, \frac{1}{p})$ is the open ball of the radius $\frac{1}{p}$ and the center y. Consider

$$\Theta = \left\{ g \in (L^2(\Omega))^n \mid g = 0 \text{ in } \Omega \setminus \omega_p \right\} \cap \{\nabla g \mid g \in H_0^1(\Omega)\}$$

Let $\varphi_0 \in H_0^1(\Omega)$, we consider the system:

$$\text{(3)} \qquad \begin{cases} \dfrac{\partial \varphi}{\partial t}(x,t) & = -Q^*\varphi(x,t) & W \\ \\ \varphi\ (\xi, t) & = 0 & \Sigma \\ \\ \varphi(x, T) & = \varphi_0(x) & \Omega \end{cases}$$

this system admits a unique solution $\varphi \in L^2(0,T;H_0^2(\Omega)) \cap C^3(\Omega\times]0,T[)$.

For $\varphi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, for $\tilde{\varphi}_0 \in \Theta$ such that $\tilde{\varphi}_0 = \nabla\varphi_0$, we consider the system (3).

We denoted by Θ the completion of the set Θ and we consider the system

(4)
$$\begin{cases} \dfrac{\partial \Upsilon}{\partial t}(x,t) & = Q\Upsilon(x,t) + N\Upsilon(x,t) + \sum_{i=1}^{n} \langle f, \dfrac{\partial \varphi}{\partial x_i} \rangle_{L^2(D)} \chi_D f(x), \quad W, \\ \Upsilon(\xi,t) & = 0, \quad \Sigma, \\ \Upsilon(x,0) & = \Upsilon_0(x), \quad \Omega, \end{cases}$$

We decomposed the $\Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3$

Where Υ_1 is solution of the system

(5)
$$\begin{cases} \dfrac{\partial \Upsilon_1}{\partial t}(x,t) & = Q\Upsilon_1(x,t) \quad W \\ \Upsilon_1(\xi,t) & = 0 \quad \Sigma \\ \Upsilon_1(x,0) & = z_0(x) \quad \Omega \end{cases}$$

Υ_2 is solution of

(6)
$$\begin{cases} \dfrac{\partial \Upsilon_2}{\partial t}(x,t) & = Q\Upsilon_2(x,t) + \sum_{i=1}^{n} \langle f, \dfrac{\partial \varphi}{\partial x_i} \rangle_{L^2(D)} \chi_D f(x) \quad W \\ \Upsilon_2(\xi,t) & = 0 \quad \Sigma \\ \Upsilon_2(x,0) & = 0 \quad \Omega \end{cases}$$

and Υ_3 is the solution of the system

(7)
$$\begin{cases} \dfrac{\partial \Upsilon_3}{\partial t}(x,t) & = Q\Upsilon_3(x,t) + N(\Upsilon_1 + \Upsilon_2 + \Upsilon_3) \quad W \\ \Upsilon_3(\xi,t) & = 0 \quad \Sigma \\ \Upsilon_3(x,0) & = 0, \quad \Omega \end{cases}$$

We set the nonlinear operator $\tilde{\mu}$ defined by:

$$\begin{aligned} \tilde{\mu} \ : \quad \Theta \longrightarrow & \quad \Theta^* \\ \tilde{\varphi}_0 \longrightarrow & \quad \tilde{\mu}(\tilde{\varphi}_0) = \tilde{\mathscr{P}}(\nabla\Upsilon_2(T)) + \tilde{\mathscr{P}}(\nabla\Upsilon_3(T)) \end{aligned}$$

where $\tilde{\mathscr{P}}$ be the projection operator and Θ^* is the dual of Θ.

If $\wedge \ : \ \Theta \ \longrightarrow \ \Theta^*$ and $K : \Theta \ \longrightarrow \ \Theta^*$ operators defined by

$$\wedge(\tilde{\varphi}_0) = \tilde{\mathscr{P}}(\nabla\Upsilon_2(T))$$

and

$$K(\tilde{\varphi}_0) = \tilde{\mathscr{P}}(\nabla\Upsilon_3(T)).$$

Hence the problem of regional controllability of the gradient (2) comes to solving the equation

$$\tilde{\mu}(\tilde{\varphi}_0) = \bar{\chi}^*_{\omega_p} g_d - \tilde{\mathscr{P}}(\nabla\Upsilon_1(T)).$$

Which implies

$$\wedge(\tilde{\varphi}_0) = \bar{\chi}^*_{\omega_p} g_d - K(\tilde{\varphi}_0) - \tilde{\mathscr{P}}(\nabla\Upsilon_1(T)).$$

Let the linear system be associated with (4)

$$(8) \qquad \begin{cases} \dfrac{\partial \Upsilon}{\partial t}(x,t) & = Q\Upsilon(x,t) + \displaystyle\sum_{i=1}^{n} \langle f, \frac{\partial \varphi}{\partial x_i} \rangle_{L^2(D)} \chi_D f(x), \quad W, \\ \Upsilon(\xi,t) & = 0, \qquad \Sigma, \\ \Upsilon(x,0) & = \Upsilon_0(x), \qquad \Omega, \end{cases}$$

we suppose that the linear system (8) is approximate gradient controllable in the region ω_p, then the operator $\wedge$ is invertible and therefore we have

$$\wedge^{-1}(\bar{\chi}^*_{\omega_p} g_d) - \wedge^{-1} K(\tilde{\varphi}_0) - \wedge^{-1} \tilde{\mathscr{P}}(\Upsilon_1(T)) = \tilde{\varphi}_0$$

Now, we define the nonlinear operator $\tilde{K} : \Theta \ \longrightarrow \ \Theta^*$ by

$$(9) \qquad \tilde{K}(\tilde{\varphi}_0) = \wedge^{-1}(\bar{\chi}^*_{\omega_p} g_d) - \wedge^{-1} K(\tilde{\varphi}_0) - \wedge^{-1} \tilde{\mathscr{P}}(\nabla\Upsilon_1(T))$$

Proposition 3.1.

If the linear system (8) is approximate gradient controllable in ω_p, then the equation (9) admits a fixed point $\tilde{\varphi}_0 \in \Theta$ and the control $v_p^*(t) = \langle F, \nabla \varphi(t) \rangle_{(L^2(D))^n}$ steers the system (4) has a desired gradient g_d in the boundary region Γ, where $F = (f, ..., f)$, φ is the solution of the system (3).

Proof.

First, we prove that $\tilde{K}$ is compact and we apply Schauder's fixed point theorem, to prove that the operator $\tilde{K}$ has a fixed point.

Step 1. We prove that the operator $\tilde{K}$ is compact.

Let $r > 0$, consider the closed ball $B_r = B(0, r)$ of $(L^2(\Omega))^n$, we consider

$$\tilde{K}(B_r) = \left\{ \tilde{\mathscr{P}}(\nabla \Upsilon_3(t)) \mid \tilde{\varphi}_0 \in B_r, t \in [0, T] \right\}.$$

Let us show that $\tilde{K}(B_r)$ is relatively compact, in fact:

The solution $\Upsilon_3(.)$ of the system (7) is written ([18]).

$$\Upsilon_3(t) = \int_0^t L(t-\tau) N(\Upsilon_1(\tau) + \Upsilon_2(\tau) + \Upsilon_3(\tau)) d\tau \tag{10}$$

$\Upsilon_3 \in C(0, T; L^2(\Omega))$ (see [18]), $\tilde{\mathscr{P}}$ and ∇ are two linear operators, then there exists $c_1 > 0$ such that

$$\|\tilde{\mathscr{P}}(\nabla \Upsilon_3(t))\|_{\Theta^*} \leq c_1 \|\Upsilon_3(t)\|_{L^2(\Omega)}.$$

Since $L(.)$ is a strongly continuous semigroup in $[0, T]$, then there exist $\pi > 0$ such that

$$\|L(t)\|_{\mathscr{L}(L^2(\Omega))} \leq \pi, \ \forall t \in [0, T]$$

by (10), we have

$$\begin{aligned} \|\Upsilon_3(t)\|_{L^2(\Omega)} &\leq \int_0^t \|L(t-\tau)(N(\Upsilon_1(\tau) + \Upsilon_2(\tau) + \Upsilon_3(\tau))\|_{L^2(\Omega)} d\tau \\ &\leq \pi c \left(\int_0^t \|\Upsilon_1(\tau)\|_{L^2(\Omega)} d\tau + \int_0^t \|\Upsilon_2(\tau)\|_{L^2(\Omega)} d\tau + \int_0^t \|\Upsilon_3(\tau)\|_{L^2(\Omega)} d\tau \right). \end{aligned}$$

As Υ_1 is solution of the system (5), with $\Upsilon_1(\tau) = L(\tau) z_0$ and we have

$$\int_0^t \|\Upsilon_1(\tau)\|_{L^2(\Omega)} d\tau \ \leq \ \pi T \|z_0\|_{L^2(\Omega)} \tag{11}$$

and Υ_2 is solution of the system (6), then we have

$$\Upsilon_2(\tau) = \int_0^{\tau} L(\tau - s) \sum_{j=1}^{n} \langle f, \frac{\partial \varphi}{\partial x_j}(s) \rangle_{L^2(D)} \chi_D f ds$$

and

$$\begin{aligned} \|\Upsilon_2(\tau)\|_{L^2(\Omega)} &= \| \int_0^{\tau} L(\tau - s) \sum_{j=1}^{n} \langle f, \frac{\partial \varphi}{\partial x_j}(s) \rangle_{L^2(D)} \chi_D f ds \| \\ &\leq \pi n \|f\|^2_{L^2(D)} \int_0^{\tau} \|\nabla \varphi(s)\|_{(L^2(\Omega))^n} ds \end{aligned}$$

by the continuity of the gradient there exists c_2 such that

$$\|\Upsilon_2(\tau)\|_{L^2(\Omega)} \leq c_2 \pi n \|f\|^2_{L^2(D)} \int_0^{\tau} \|\varphi(s)\|_{L^2(\Omega)} ds$$

where $\varphi(s) = L(s)\varphi_0$ is solution of the system (3).

We have

$$\|\Upsilon_2(\tau)\|_{L^2(\Omega)} \leq c_2 n T \pi^2 \|f\|^2 \|\varphi_0\|_{H_0^1(\Omega)} \tag{12}$$

$\varphi_0 \in H_0^1(\Omega)$ and Ω is a regular open, we apply the Poincare inequality in the equation (12), then there exist $c_3 > 0$ such that

$$\int_0^t \|\Upsilon_2(\tau)\|_{L^2(\Omega)} d\tau \leq c_2 c_3 n T^2 \pi^2 \|f\|^2 \|\nabla \varphi_0\|_{(L^2(\Omega))^n}$$

where $\tilde{\varphi}_0 = \nabla \varphi_0$, then we have

$$\int_0^t \|\Upsilon_2(\tau)\|_{L^2(\Omega)} d\tau \leq c_2 c_3 n T^2 \pi^2 \|f\|^2 \|\tilde{\varphi}_0\|_{\Theta} \tag{13}$$

according to (11) and (13), we have

$$\|\Upsilon_3(t)\|_{L^2(\Omega)} \leq \pi c \left(\pi T \|z_0\|_{L^2(\Omega)} + c_2 c_3 n T^2 \pi^2 \|f\|^2 \|\tilde{\varphi}_0\|_{\Theta} + \int_0^t \|\Upsilon_3(\tau)\|_{L^2(\Omega)} d\tau \right)$$

applying the Gronwall Lemma, we get

$$\|\Upsilon_3(t)\|_{L^2(\Omega)} \leq \left(\pi^2 c T \|z_0\|_{L^2(\Omega)} + c_2 c_3 n T^2 \pi^3 c \|f\|^2 \|\tilde{\varphi}_0\|_{\Theta} \right) e^{\pi c T}.$$

From

$$\|\tilde{\mathscr{P}}(\nabla \Upsilon_3(t))\|_{\Theta} \leq c_1 \left(\pi^2 c T \|z_0\|_{L^2(\Omega)} + c_2 c_3 n T^2 \pi^3 c \|f\|^2 r \right) e^{\pi c T}, \quad \forall\, \tilde{\varphi}_0 \in B_r. \tag{14}$$

Then, $\tilde{K}(B_r)$ is bounded.

On the other hand, for $0 < t_1 < T$ and $h > 0$, we have

$$\begin{aligned}\Upsilon_3(t_1+h)-\Upsilon_3(t_1) &= \int_0^{t_1}(L(t_1+h-\tau)-L(t_1-\tau))N(\Upsilon_1(\tau)+\Upsilon_2(\tau)+\Upsilon_3(\tau))d\tau \\ &+ \int_{t_1}^{t_1+h}L(t_1+h-\tau)N(\Upsilon_1(\tau)+\Upsilon_2(\tau)+\Upsilon_3(\tau))d\tau \\ &= \mathscr{J}_1+\mathscr{J}_2\end{aligned}$$

where

$$\mathscr{J}_1 = \int_0^{t_1}(L(t_1+h-\tau)-L(t_1-\tau))N(\Upsilon_1(\tau)+\Upsilon_2(\tau)+\Upsilon_3(\tau))d\tau$$

and

$$\mathscr{J}_2 = \int_{t_1}^{t_1+h}L(t_1+h-\tau)N(\Upsilon_1(\tau)+\Upsilon_2(\tau)+\Upsilon_3(\tau))d\tau.$$

For all $\varepsilon_1 > 0$, there exist $\alpha > 0$ such that $|h| < \alpha$, we have

$$\|L(t_1+h-\tau)-L(t_1-\tau)\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} \leq \varepsilon_1,\ \forall\tau\in[0,T].$$

Which give

$$\begin{aligned}\|\mathscr{J}_1\|_{L^2(\Omega)} &\leq \int_0^{t_1}\|L(t_1+h-\tau)-L(t_1-\tau)\|_{L^2(\Omega)}\|N(\Upsilon_1(\tau)+\Upsilon_2(\tau)+\Upsilon_3(\tau))\|_{L^2(\Omega)}d\tau \\ &\leq \varepsilon_1 c\left(\int_0^{t_1}\|\Upsilon_1(\tau)\|_{L^2(\Omega)}d\tau+\int_0^{t_1}\|\Upsilon_2(\tau)\|_{L^2(\Omega)}d\tau+\int_0^{t_1}\|\Upsilon_3(\tau)\|_{L^2(\Omega)}d\tau\right).\end{aligned}$$

According to (11), (13) and (14), we obtain

$$\|\mathscr{J}_1\|_{L^2(\Omega)} \leq \varepsilon_1 c\left[(\pi T\|z_0\|_{L^2(\Omega)}+c_2c_3nT^2\pi^2\|f\|^2_{L^2(D)}\|\tilde{\varphi}_0\|_\Theta)(1+\pi cTe^{\pi cT})\right]$$

and

$$\|\mathscr{J}_2\|_{L^2(\Omega)} \leq \alpha\pi c\left[(\pi\|z_0\|_{L^2(\Omega)}+c_2c_3nT^2\pi^2\|f\|^2_{L^2(D)}\|\tilde{\varphi}_0\|_\Theta)(1+\pi cTe^{\pi cT})\right]$$

then

$$\begin{aligned}\|\tilde{\mathscr{P}}(\nabla\Upsilon_3(t_1+h))-\tilde{\mathscr{P}}(\nabla\Upsilon_3(t_1))\|_{\Theta^*} &\leq c_1\|(\Upsilon_3(t_1+h))-(\Upsilon_3(t_1))\|_{L^2(\Omega)} \\ &\leq \varepsilon_1\mathscr{J}_3+\alpha\mathscr{J}_4\end{aligned}$$

where

$$\mathscr{J}_3 = c_1 c \left[(\pi T \|z_0\|_{L^2(\Omega)} + c_2 c_3 n T^2 \pi^2 \|f\|^2_{L^2(D)} \|\tilde{\varphi}_0\|_\Theta)(1 + \pi c T e^{\pi c T}) \right]$$

and

$$\mathscr{J}_4 = c_1 \pi c \left[(\pi \|z_0\|_{L^2(\Omega)} + c_2 c_3 n T^2 \pi^2 \|f\|^2_{L^2(D)} \|\tilde{\varphi}_0\|_\Theta)(1 + \pi c T e^{\pi c T}) \right].$$

For $\varepsilon_1 \leq \dfrac{\varepsilon}{2\mathscr{J}_3}$ and $\alpha \leq \text{Inf}\left(dist(B_r, C\Omega); \dfrac{\varepsilon}{2\mathscr{J}_4} \right)$, we obtain

$$\|\tilde{\mathscr{P}}(\nabla \Upsilon_3(t_1 + h)) - \tilde{\mathscr{P}}(\nabla \Upsilon_3(t_1))\|_\Theta \leq \varepsilon.$$

Then, according to (14) $\tilde{K}(B_r)$ is bounded and $\forall \varepsilon > 0$, $\exists\, \alpha > 0$ and $h > 0$ such that $|h| < \alpha$

$$\|\tilde{\mathscr{P}}(\nabla \Upsilon_3(t_1 + h)) - \tilde{\mathscr{P}}(\nabla \Upsilon_3(t_1))\|_\Theta \leq \varepsilon.$$

Finally, by the Kolmogorov-Riesz-Freshet theorem [2, 21], $\tilde{K} : \Theta \longrightarrow \Theta^*$ is a compact operator.

Step 2. According to (9) and (14), we have

$$\begin{aligned} \|\tilde{K}(\tilde{\varphi}_0)\|_\Theta &\leq \| \wedge^{-1} (\bar{\chi}^*_\omega g_d) - \wedge^{-1} \tilde{\mathscr{P}}(\nabla \Upsilon_1(T))\|_\Theta + \| \wedge^{-1} K(\tilde{\varphi}_0)\|_\Theta \\ &\leq \mathscr{J}_5 + \| \wedge^{-1} \|_{\mathscr{L}(\Theta^*, \Theta)} c_1 c_2 c_3 n \pi^3 c T^2 \|f\|^2_{L^2(D)} e^{\pi c T} \|\tilde{\varphi}_0\|_\Theta \end{aligned}$$

where

$$\mathscr{J}_5 = \| \wedge^{-1} (\bar{\chi}^*_\omega g_d) - \wedge^{-1} \tilde{\mathscr{P}}(\nabla \Upsilon_1(T))\|_\Theta + \| \wedge^{-1} \|_{\mathscr{L}(\Theta^*, \Theta)} c_1 \pi^2 c T \|z_0\|_{L^2(\Omega)} e^{\pi c T}$$

and c a constant, such that

$$\| \wedge^{-1} \|_{\mathscr{L}(\Theta^*, \Theta)} c_1 c_2 c_3 n \pi^3 c T^2 \|f\|^2_{L^2(D)} e^{\pi c T} \leq \frac{1}{2}.$$

Let $0 < s \leq r$, such that $s \geq 2\mathscr{J}_5$, then we have

$$\|\tilde{K}(\tilde{\varphi}_0)\|_\Theta \leq s \quad \forall \tilde{\varphi}_0 \in \Theta \text{ such that } \|\tilde{\varphi}_0\|_\Theta \leq s$$

and by application of Schauder's fixed point theorem [21], the operator $\tilde{K}$ admits a fixed point.

□

4. Example

Consider the two-dimensional diffusion system

(15)
$$\begin{cases} \frac{\partial z}{\partial t}(x,y,t) &= \beta(\frac{\partial^2 z}{\partial x^2}(x,y,t)+\frac{\partial^2 z}{\partial y^2}(x,y,t))+\sum_{i,j=1}^{\infty}|\langle z,\varphi_{ij}\rangle|\langle z,\varphi_{ij}\rangle\varphi_{ij}+Bv(t) \quad \Omega\times]0,T[\\ z(\xi,\eta,t) &= 0 \quad \partial\Omega\times]0,T[\\ z(x,y,0) &= 0 \quad \Omega \end{cases}$$

Where

- **•:** $T=2$ and the actuator $b=(b_1,b_2)$ with $b_1=0.68, b_2=0.42$.
- **•:** $\omega =]0,0.25[\times]0,1[$: the internal region.
- **•:** $\Gamma=\{0\}\times]0,1[$: the boundary region.
- **•:** $g_d=(y(y-1),0)$: the desired gradient on Γ.
- **•:** $\bar{g}_d=[(2x-1)y(y-1),(2y-1)x(x-1)]$: the extension of the desired gradient g_d on ω.

We obtain the following figures:

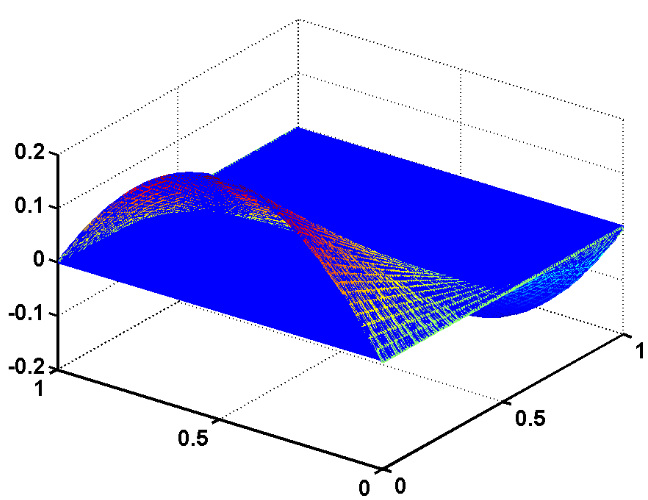

Fig. 1 Desired gradient on ω_r.

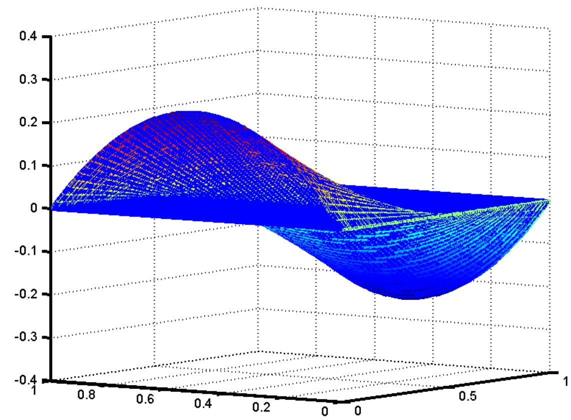

Fig. 2 Final gradient on ω_r.

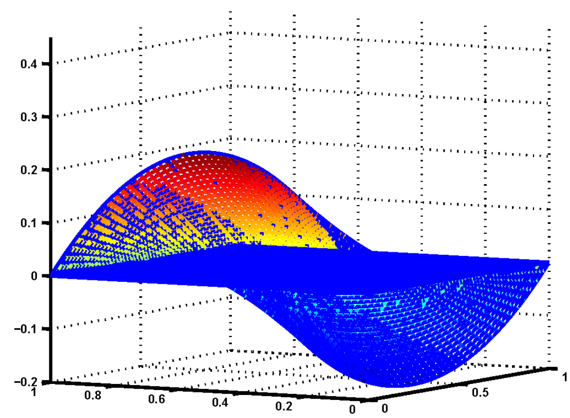

Fig. 3 Desired and final gradient on ω_r.

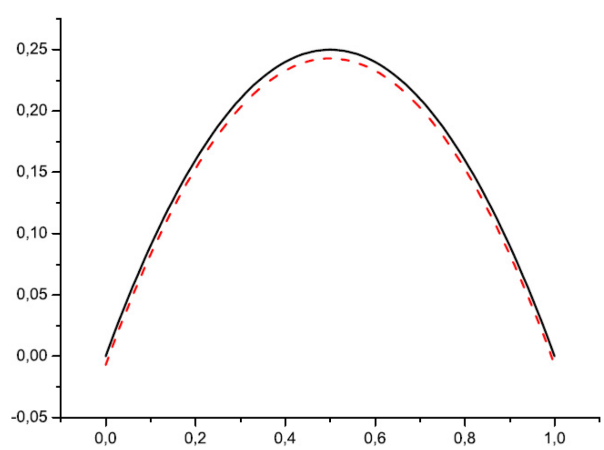

Fig. 4 The trace on Γ.

The desired gradient g_d is reached with an error
$\|\bar{\chi}_\Gamma \gamma \nabla z_{u^*}(T) - g_d\| = 6.89 * 10^{-3}$ and a transfer cost $\|u^*\| = 5.37 * 10^{-4}$.

Acknowledgment

The authors would like to thanks the Deanship of Graduate Studies at Jouf University for funding and supporting this research through the initiative of DGS, Graduate Students Research Support (GSR) at Jouf University, Saudi Arabia.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] A. V. Balakrishnan. Applied functional analysis. Springer, 1976.

[2] H. Brezis, Analyse fonctionnelle: théorie et application. Masson. 1983.

[3] R.F. Curtain, H. Zwart, An introduction to infinite dimensional linear systems theory. Texts in Applied Mathematics, Springer Verlag, 21, 1995.

[4] J.A.M.F. De Souza, A.J. Pritchard, Control of semi-linear distributed parameter systems. Telecomunication and Control, INPE Press, Sao Josè dos Campos, Brasil, 1985, pp. 202-207.

[5] J.A.M.F. De Souza, Control of nonlinear distributed parameter systems. Proc, VIII Congreso Chileno De Ingenieria Electrica, Chile, 1989, pp. 370-375.

[6] A. El Jai. Etude dálgorithmes pour la commande optimale de systemes a parametres repartis de type parabolique. These etat, Universite de Paul Sabatier, Toulouse, 1978.

[7] A. El Jai, A.J. Pritchard, Capteurs et actionneurs dans l'analyse des systèmes distribués. Masson. RMA 3. Paris. 1986.

[8] A. El Jai, A.J. Pritchard, Sensors and actuators in distributed systems analysis. Ellis Horwood series in Applied Mathematics (Wiley). 1988.

[9] A. El Jai, A.J. Pritchard, M.C. Simon, E. Zerrik, Regional controllability of distributed systems. Int. J. Control, 62 (1995), 1351-1365.

[10] C. Fabre, J.-P. Puel, E. Zuazua, Approximate controllability of the semilinear heat equation, Proc. R. Soc. Edinburgh: Sect. A. Math. 125 (1995), 31-61.

[11] H.O. Fattorini. Boundary control systems. SIAM J. Cont. 6 (1978), 349-388.

[12] E. Fernandez-Cara, Null controllability of the semilinear heat equation, ESAIM: COCV. 2 (1997), 87-103.

[13] A. Kamal, A. Boutoulout, S. Beinane, Regional Controllability of Semi-Linear Distributed Parabolic Systems: Theory and Simulation. Intell. Control Autom. 03(2012), 146-158.

[14] J.L. Lions. Controle optimal des systemes gouvernes par des equations aux derivees partielles. Dunod, Paris, 1968.

[15] J.L. Lions. Controlabilite exacte. Perturbations et stabilisation des systemes distribues, volume Tome 1, Controlabilite Exacte. Masson, Paris, 1988.

[16] J.L. Lions, E. Magenes. Problemes aux limites non homogenes et application, volume Vol. 1. Dunod, Paris, 1968.

[17] S.A. Ould Beinane, A. Kamal, A. Boutoulout, Regional Gradient controllability of semi-linear parabolic systems. Int. Rev. Autom. Control. 6(2013), 641-653.

[18] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer-Verlag, New-York, 1983.

[19] D.L. Russell. Controllability and stabilizability theory for linear partial differential equations. recent progress and open. SIAM Rev. 20 (1978), 639-739.

[20] R. Triggiani. Controllability and observability in banach spaces with bounded operators. SIAM J. Cont. 13 (1975), 462-491.

[21] E. Zeidler, Applied functional analysis : Applications to mathematical physics, Vol. 108. Springer-Verlag, New-York, 1995.

[22] E. Zerrik, A. Kamal, Output controllability for semi-linear distributed parabolic systems. J. Dyn. Control Syst. 13 (2007), 289-306.

[23] E. Zerrik , A. Boutoulout, A. El Jai, Actuators and regional boundary controllability of parabolic systems. Int. J. Syst. Sci. 31 (2000), 73-82.

[24] E. Zerrik, A. Kamal, Flux target: actuators and simulations, Sensors and Actuators A: Physical. 121 (2005), 22?30.

[25] E. Zerrik, A. Kamal, A. Boutoulout, Regional flux target with minimum energy. IEE Proceedings-Control Theory and Applications. 149 (2002), 349-356.

[26] E. Zerrik, A. Kamal, A. Boutoulout, Regional gradient controllability and actuators. Int, J, Syst, Sci, 33 (2002), 239-246.

[27] E. Zuazua, Contrôlabilité exacte d'un modèle de plaques vibrantes en un temps arbitrairement petit. C.R.A.S, Paris. Série I Math, 1987.