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THE LEAST-SQUARES ESTIMATOR OF SINUSOIDAL SIGNAL OF DIFFUSION PROCESS FOR DISCRETE OBSERVATIONS

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Abstract. There is notably paucity of studies on least-squares estimator of diffusion process for discrete observations. This paper discusses sufficient conditions of the least-squares estimator of diffusion process for discrete observations in order to gain an estimator that is strongly consistent of [1]. We assume that the process Y is arranged by a function such as sinusoidal signal $a(\theta, Y_t) = \sin(2\pi t\theta)$, $\theta \in [0, \frac{1}{2}]$ and function $b(\sigma, Y_t)$. For a given a sample $(Y_0, Y_h, \dots, Y_{nh})$, $h \rightarrow 0$, we demonstrate an asymptotic theory of least-squares estimator $\hat{\theta}_n$. The results of the study show that the least-squares estimator is strongly consistent and asymptotic normal, assuming that $nh \rightarrow \infty$ and $n^3h^4 \rightarrow \infty$; θ that represents the frequency of sinusoidal signal of the unity of time which has a rate of convergence, namely $\sqrt{n^3h^4}$.

Keywords: least-squares; diffusion; discrete.

2010 AMS Subject Classification: 62F12.

1. INTRODUCTION

The stochastic differential equation application has been widely used in the field of industry, economics and environment as can be seen in [2], [3], [4], [5], [6], [7]. A part from these examples, studies on nonlinear problems have been investigated by numerous researchers, examples

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of which can be seen in [1] and [8]. [9] exploited the nonlinear model as follows:

$$(1) \quad y_t = \cos(2\pi t \theta_0) + e_t,$$

where $t \in \mathbb{N}$, $\{e_t\}$ are i.i.d normal random variable with mean zero and finite positive variance σ^2 .

The stochastic differential equation (SDE) can be viewed as problems of nonlinear regression model. Few published studies have examined parameter estimation of diffusion process using the least-squares method for discrete observations (see for examples, [10] dan [11]). Hence, this study aimed to investigate the least-squares estimator (LSE) of diffusion process for discrete observations.

Consider $Y = (Y_t)_{t \in \mathbb{R}_+}$ is the solution to the SDE

$$(2) \quad dY_t = a(\theta, Y_t)dt + b(\sigma, Y_t)dw_t, \quad Y_0 = y_0,$$

where

- $a(\theta, Y_t) = \sin(2\pi\theta t)$, $\theta \in [0, \frac{1}{2}]$, a is a measurable function, $a : \mathbb{R} \rightarrow \mathbb{R}$;
- θ is unknown parameter and will be estimated;
- $Y > 0$ and Y_0 is an initial value of Y when $t = 0$;
- $b(\sigma, Y_t) = \sigma$, $\sigma > 0$ is assumed unknown, $b : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$;
- w is a one-dimensional Wiener process.

This study inspired the nonlinear models of [1], [12], [9] and [13]. These studies discussed nonlinear model in general, i.e., $y_t = f_t(\alpha) + e_t$ with $\{e_t\}_{t \in \mathbb{N}}$ which are independent random variables with mean zero and finite variance. In this paper we develop schemes, namely: first we discretize (2), from this discretization model, we define a target function using the least-square approach by minimizing errors of the squares between the process Y and function of $a(\cdot)$. Based on the target function, a verification of the almost sure convergence of the estimator will be performed as suggested by [1] whether it fulfills for discrete observations. With a strong consistency of the estimator θ , we will then discuss how to determine an asymptotic normality for θ and verify an assumption required for asymptotic normal of the estimator.

The paper is structured as follows. The model is presented in Section 2, followed by our theoretical results, namely asymptotic theory of the estimator in Section 3. The study of numeric aiming to simulate the least-squares estimator is set out in Section 4.

2. PRELIMINARIES

Suppose $(Y_t)_{t \in \mathbb{R}_+}$ in (2) is a real-valued process that is

$$(3) \quad Y_t = Y_0 + \int_0^t \sin(2\pi\theta s) ds + \int_0^t \sigma w_s,$$

defined on an underlying complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ where

- the true parameter value is denoted by $\theta_0 \in \Theta$ which does exist with the \mathbb{P}_0 assumed as the true image measure;
- $\theta \in \Theta \subset \mathbb{R}$, where Θ is supposed to be bounded convex domains, and the closure of Θ is denoted by $\bar{\Theta}$ which satisfies

$$\bar{\Theta} \subset \left\{ \theta \in \left[0, \frac{1}{2} \right] \right\};$$

- \mathbb{P}_θ for the image measure of a solution process Y associated with θ .

We assume that Y is observed at discrete sample points $(Y_{t_0}, Y_{t_1}, \dots, Y_{t_n})$ with $0 \equiv t_0 < t_1 < \dots < t_n$, where $t_i^n = t_i = ih, i \leq n$ and $h > 0$ is a non-random sampling discrete of step size such that for $n \rightarrow \infty$ satisfies

$$(4) \quad h \rightarrow 0,$$

$$(5) \quad T_n := nh, T_n \rightarrow \infty,$$

and

$$(6) \quad n^3 h^4 \rightarrow \infty.$$

Several notations which will be used in this paper are: for a function $f(\theta, \cdot)$, $\partial_\theta^j f(\theta, \cdot)$ stands for j th derivative of f with respect to θ , $j = 1, 2$; symbol $\xrightarrow{\mathcal{L}}$ indicates the convergence in law under \mathbb{P}_0 .

3. MAIN RESULTS

Discretization of Y (3) by deploying approximations of Euler-Maruyama we can provide:

$$(7) \quad Y_{t_i} \stackrel{\mathbb{P}_\theta}{=} Y_{t_{i-1}} + \int_{t_{i-1}}^{t_i} (\sin(2\pi\theta s) ds + \sigma dw_s).$$

We also assume

$$(8) \quad \Delta_i Y \stackrel{\mathbb{P}_0}{=} \int_{t_{i-1}}^{t_i} \sin(2\pi\theta_0 s) ds + \sigma \Delta_i w.$$

From here and the next section, we define the increased process ρ_i as $\Delta_i \rho := \rho_i(\cdot) - \rho_{i-1}(\cdot)$.

Next, we define a target function of (2) namely

$$(9) \quad \mathbb{H}_n(\theta) := \sum_{i=1}^n \frac{1}{h} (\Delta_i Y - \sin(2\pi\theta t_{i-1})h)^2.$$

The definition of target function of diffusion process for discrete observations can be seen also in [10] and [11].

From the target function (9), we define the LSE of θ as $\hat{\theta}_n$ namely a measurable function which satisfies

$$(10) \quad \hat{\theta}_n := \underset{\theta \in \bar{\Theta}}{\operatorname{argmin}} \mathbb{H}_n(\theta).$$

Now, let

$$(11) \quad \mathbb{F}_n(\theta) := \frac{1}{nh} [\mathbb{H}_n(\theta) - \mathbb{H}_n(\theta_0)],$$

and based on the function (11), we then shall discuss the consistency of LSE $\hat{\theta}_n$.

Theorem 3.1. *The LSE of diffusion process (2) for discrete observation is strongly consistent under (4) and (5).*

Proof. We shall verify a condition of almost sure convergence of θ from Lemma 1 of [1] whether it can be fulfilled for LSE of diffusion process for discrete observations. Verification

can be done by showing: for $C > 0$, inequality

$$(12) \quad \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \mathbb{F}_n(\theta) > 0 \quad a.s \quad as \quad T_n \rightarrow \infty$$

applies.

By re-calling (9) and (8), we have

$$(13) \quad \mathbb{F}_n(\theta) = \frac{1}{n} \sum_{i=1}^n [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})]^2 - \frac{2\sigma}{nh} \sum_{i=1}^n \Delta_i w [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})].$$

Let us observe the first part of the right hand-side of an equation (13),

$$\begin{aligned} & \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \frac{1}{n} \sum_{i=1}^n [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})]^2 \\ &= \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \frac{1}{n} \sum_{i=1}^n \sin^2(2\pi\theta t_{i-1}) \\ &+ \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \frac{1}{n} \sum_{i=1}^n \sin^2(2\pi\theta_0 t_{i-1}) \\ &- \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \left[\frac{1}{n} \sum_{i=1}^n \sin(2\pi\theta t_{i-1}) \sin(2\pi\theta_0 t_{i-1}) \right]. \end{aligned}$$

Since we apply the following results,

$$\frac{1}{n} \sum_{i=1}^n \sin^2(2\pi\theta t_i) = \frac{1}{2} + o(1),$$

$$\frac{1}{n} \sum_{i=1}^n \sin(2\pi\theta t_i) \cos(2\pi\theta t_i) = o(1),$$

then we have

$$(14) \quad \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \frac{1}{n} \sum_{i=1}^n [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})]^2 = \frac{1}{4} + o(1) > 0.$$

Next, we need to know that the last part of the right hand-side of an equation (13). Since

$$\begin{aligned} & \liminf_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \frac{2\sigma}{nh} \sum_{i=1}^n \Delta_i w [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})] \\ & \leq \limsup_{T_n \rightarrow \infty} \sup_{|\theta - \theta_0| \geq C} \frac{2\sigma}{nh} \sum_{i=1}^n \Delta_i w \sin(2\pi\theta t_{i-1}) - \limsup_{T_n \rightarrow \infty} \sup_{|\theta - \theta_0| \geq C} \frac{2\sigma}{nh} \sum_{i=1}^n \Delta_i w \sin(2\pi\theta_0 t_{i-1}) \end{aligned}$$

We can complete above by adapting Lemma 4.2 of [13], namely showing that $\{\sigma \Delta_i w\}_{i \leq n}$ satisfies Assumption 3.1 and 3.2 of [13].

Now we focus on $\{\sigma\Delta_i w\}_{i \leq n}$, where $\{\Delta_i w = w_i - w_{i-1}\}$ is the i th increment of the Wiener process w with $E[\sigma\Delta_i w] = 0$ and $Var(\sigma\Delta_i w) = \sigma^2 h < \infty, h \rightarrow 0$. Now, we follow Theorem 2.2.1 of [12]: assume there are the sequence of $\{\sigma_1, \dots, \sigma_i\}_{i \leq n}$ and $\{\sigma_i := \sigma; \sigma > 0\}_{i \leq n}$, because the sequence of random variables $\{\Delta_i w\}_{i \leq n}$ satisfy $\sum_{i=1}^n |\sigma_i^2| < \infty$ and $E[\Delta_i^2 w] = h < \infty$ then there exists a sequence of $\{\sigma\Delta_i w\}_{i \leq n}$ such that

$$\varepsilon_i = \sum_{i=1}^n \sigma\Delta_i w \quad a.s \quad as \quad T_n \rightarrow \infty,$$

$$\lim_{T_n \rightarrow \infty} \mathbb{E} \left| \varepsilon_i - \sum_{i=1}^n \sigma\Delta_i w \right|^2 = 0 \quad and, \mathbb{E} |\varepsilon_i|^2 < \infty.$$

Therefore, $\{\sigma\Delta_i w\}_{i \leq n}$ can be seen as $\{\varepsilon_i\}_{i \leq n}$ that fulfill Assumption 3.1 and 3.2 of [13], hence, we can state the following Lemma.

Lemma 3.2.

$$\sup_{\theta} \left| \frac{\sigma}{T_n} \sum_{i=1}^n \Delta_i w \sin(2\pi\theta t_i) \right| \rightarrow 0 \quad a.s \quad as \quad T_n \rightarrow \infty.$$

Corollary 3.3.

$$\sup_{\theta} \left| \frac{\sigma}{T_n^{m+1}} \sum_{i=1}^n t_i^m \sigma\Delta_i w \sin(2\pi\theta t_i) \right| \rightarrow 0 \quad a.s \quad as \quad T_n \rightarrow \infty,$$

where $m \in \mathbb{N}_0$.

Lemma 3.2 and Corollary 3.3 are true for cosine function. Proof of 3.2 is similar to the one provided by [13] in Lemma 4.2 in view of discrete observations.

Therefore, using Lemma 3.2 we may know that (12) is fulfilled because

$$(15) \quad \sup_{|\theta - \theta_0| \geq C} \left| \frac{2\sigma}{nh} \sum_{i=1}^n \Delta_i w [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})] \right| \rightarrow 0 \quad a.s \quad as \quad T_n \rightarrow \infty,$$

so that

$$(16) \quad \lim_{T_n \rightarrow \infty} \inf_{|\theta - \theta_0| \geq C} \frac{2\sigma}{nh} \sum_{i=1}^n \Delta_i w [\sin(2\pi\theta t_{i-1}) - \sin(2\pi\theta_0 t_{i-1})] = 0 \quad a.s \quad T_n \rightarrow \infty,$$

and since we have (14) and (16), the verification has been completed. \square

Next, we shall discuss the asymptotic normality of $\hat{\theta}_n$.

Theorem 3.4.

$$(17) \quad \sqrt{n^3 h^4} (\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2\sigma^2 \Sigma_2^{-1})$$

where $\Sigma_2 = \frac{4\pi^2}{3}$.

Proof. Note that, from the equation (9), we have the relation

$$(18) \quad 0 = \sum_{i=1}^n v_{i-1}(\theta_0) j_{i-1}(\theta_0) + (\hat{\theta}_n - \theta_0) \sum_{i=1}^n [v_{i-1}(\theta_0) q_{i-1}(\theta_0) - h[j_{i-1}]^2]$$

where

$$\begin{aligned} v_{i-1}(\theta_0) &= \Delta_i Y - \sin(2\pi\theta_0 t_{i-1})h, \\ j_{i-1}(\theta_0) &= 2\pi t_{i-1} \cos(2\pi t_{i-1} \theta_0), \\ q_{i-1}(\theta_0) &= -4\pi^2 t_{i-1}^2 \sin(2\pi t_{i-1} \theta_0). \end{aligned}$$

To solve the equation (18), we need the following Lemma.

Lemma 3.5. *Under (5) and (6), we have*

$$\frac{1}{\sqrt{n^3 h^4}} \sum_{i=1}^n v_{i-1}(\theta_0) q_{i-1}(\theta_0) \xrightarrow{P} 0.$$

Proof. Re-calling v and q , we get

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{\sqrt{n^3 h^4}} \sum_{i=1}^n v_{i-1}(\theta_0) q_{i-1}(\theta_0) \right] \\ &= \mathbb{E} \left[\frac{1}{\sqrt{n^3 h^4}} \sum_{i=1}^n [\Delta_i Y - \sin(2\pi t_{i-1} \theta_0)] [-4\pi^2 t_{i-1}^2 \sin(2\pi t_{i-1} \theta_0)] \right] \\ &\leq \frac{1}{\sqrt{n^3 h^4}} \sum_{i=1}^n \left\{ \mathbb{E} \left| [\Delta_i Y - \sin(2\pi t_{i-1} \theta_0)] [-4\pi^2 t_{i-1}^2 \sin(2\pi t_{i-1} \theta_0)] \right|^2 \right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{n^3 h^4}} \sum_{i=1}^n \left\{ \mathbb{E} |\sigma \Delta_i w|^2 \right\}^{\frac{1}{2}} \left\{ \mathbb{E} |-4\pi^2 t_{i-1}^2 \sin(2\pi t_{i-1} \theta_0)|^2 \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{1}{n^3 h^4} o(1) \right\}^{\frac{1}{2}} = \frac{1}{\sqrt{n^3 h^4}}. \end{aligned}$$

□

Remark 3.6. $n^3 h^4 \rightarrow \infty$ has to be assumed in the model setting of (2). In paper [10], he assumes that $n^{\frac{1}{2}} h \rightarrow 0$ for LSE of diffusion process, meanwhile in the maximum likelihood method, the assumption of $nh^3 \rightarrow 0$ given by [14], whereas [15] assumes $nh^p, p \in \mathbb{N}$.

By applying Lemma 3.5, we can rewrite (18) as follows

$$-\sum_{i=1}^n v_{i-1}(\theta_0) j_{i-1}(\theta_0) = \left[-h \sum_{i=1}^n [j_{i-1}]^2 \right] (\hat{\theta}_n - \theta_0),$$

or we can write it as

$$-\partial \mathbb{H}_n(\theta_0) = \partial^2 \mathbb{H}_n(\theta_0) (\hat{\theta}_n - \theta_0).$$

By applying Central Limit Theorem (CLT), we shall determine Σ_1 such that

$$-\frac{1}{\sqrt{n^3 h^4}} \partial \mathbb{H}_n(\theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma_1),$$

where Σ_1 is a variance of $\left[-\frac{1}{\sqrt{n^3 h^4}} \partial \mathbb{H}_n(\theta_0) \right]$. From (9), we can obtain:

$$(19) \quad \partial \mathbb{H}_n(\theta_0) = -4\pi \sum_{i=1}^n t_{i-1} [\Delta_i Y - \sin(2\pi t_{i-1} \theta_0) h] \cos(2\pi t_{i-1} \theta_0).$$

Use (8) and (19) for expectation and variance of $\left[-\frac{1}{\sqrt{n^3 h^4}} \partial \mathbb{H}_n(\theta_0) \right]$. Note that,

$$\begin{aligned} \mathbb{E} \left[-\frac{1}{\sqrt{n^3 h^4}} \partial \mathbb{H}_n(\theta_0) \right] &= \mathbb{E} \left[\frac{4\pi}{\sqrt{n^3 h^4}} \sum_{i=1}^n t_{i-1} [\Delta_i Y - \sin(2\pi t_{i-1} \theta_0) h] \cos(2\pi t_{i-1} \theta_0) \right] \\ &= 4\pi^2 \sqrt{n} \mathbb{E} \left[\frac{1}{T_n^2} \sum_{i=1}^n t_{i-1} \sigma \Delta_i w \cos(2\pi t_{i-1} \theta_0) \right], \end{aligned}$$

by using Corrolary 3.3 it is obvious that

$$\mathbb{E} \left[-\frac{1}{\sqrt{n^3 h^4}} \partial \mathbb{H}_n(\theta_0) \right] = 0.$$

and

$$\begin{aligned} \Sigma_1 &= \frac{16\pi^2 \sigma^2}{n^3 h^4} \mathbb{E} \left[\sum_{i=1}^n t_{i-1} \Delta_i w \cos(2\pi t_{i-1} \theta_0) \right]^2 \\ &= 16\pi^2 \sigma^2 \mathbb{E} \left[\frac{1}{T_n^3} \sum_{i=1}^n t_{i-1}^2 \cos^2(2\pi t_{i-1} \theta_0) \right], \end{aligned}$$

by using the result of trigonometry of identity:

$$\frac{1}{T_n^3} \sum_{i=1}^n t_i^2 \cos^2(2\pi t_i \theta) = \frac{1}{6} + o(1),$$

we obtain $\Sigma_1 = \frac{8\pi^2\sigma^2}{3} + o(1)$. So we can make a claim that

$$(20) \quad -\frac{1}{\sqrt{n^3h^4}}\partial\mathbb{H}_n(\theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{8\pi^2\sigma^2}{3}\right)$$

Next, using Law of Large Number (LLN), we find Σ_2 such that

$$(21) \quad \frac{1}{\sqrt{n^3h^4}}\partial^2\mathbb{H}_n(\theta_0) \frac{1}{\sqrt{n^3h^4}} \xrightarrow{P} \Sigma_2.$$

Note that

$$\frac{1}{n^3h^4}\partial^2\mathbb{H}_n(\theta_0) = \frac{8\pi^2}{n^3h^4} \sum_{i=1}^n t_{i-1}^2 [\sigma\Delta_i w \sin(2\pi t_{i-1}\theta_0) - h \cos^2(2\pi t_{i-1}\theta_0)],$$

and we get

$$\Sigma_2 = 8\pi^2 \left(\mathbb{E} \left[\frac{1}{n^3h^4} \sum_{i=1}^n t_{i-1}^2 \sigma\Delta_i w \sin(2\pi t_{i-1}\theta_0) \right] + \mathbb{E} \left[\frac{1}{T_n^3} \sum_{i=1}^n t_{i-1}^2 \cos^2(2\pi t_{i-1}\theta_0) \right] \right).$$

By using Corollary 3.3 and the result of trigonometry identity, we obtain $\Sigma_2 = \frac{4\pi^2}{3}$.

Now let us say $\frac{1}{\sqrt{n^3h^4}}\partial^2\mathbb{H}_n(\theta_0) \frac{1}{\sqrt{n^3h^4}}$ as $\mathcal{H}_n(\theta_0)$,

$$\begin{aligned} \mathbb{E}[\mathcal{H}_n(\theta_0)]^2 &= \mathbb{E} \left[\frac{8\pi^2}{n^3h^4} \sum_{i=1}^n (t_{i-1}^2 \sigma\Delta_i w \sin(2\pi t_{i-1}\theta_0) + h \cos^2(2\pi t_{i-1}\theta_0)) \right]^2 \\ &= 64\pi^4 \mathbb{E} \left[\frac{1}{h^2} \left[\frac{1}{T_n^3} \sum_{i=1}^n t_{i-1}^2 \sigma\Delta_i w \sin(2\pi t_{i-1}\theta_0) \right]^2 \right] \\ &+ 64\pi^4 \mathbb{E} \left[\frac{1}{T_n^4} \left[\frac{1}{T_n} \sum_{i=1}^n \cos^2(2\pi t_{i-1}\theta_0) \right]^2 \right] \\ &+ 64\pi^4 \mathbb{E} \left[\frac{1}{n^3h^4} \frac{1}{T_n^3} \sum_{i=1}^n t_{i-1}^2 \sigma\Delta_i w \sin(2\pi t_{i-1}\theta_0) \right], \end{aligned}$$

By using Corollary 3.3, we obtain the first and third parts of the right hand-side of an equation seen above as 0. The second part of the right hand-side of an equation also generate 0 because $T_n^4 \rightarrow \infty$. Hence, we can conclude that

$$\mathbb{E}[\mathcal{H}_n(\theta_0)]^2 = 0.$$

□

4. SIMULATION

The simulation is done through the use of software R. In the first simulation, we choose $\theta_0 = 0.25$, with replications 10000 and h remains 0.08. Estimator θ_n at different N (indicated by the parentheses) obtained through: 0.19 (7500); 0.191 (8500); 0.216 (9000); and 0.2532 (10000). In the second simulation, we choose $\theta_0 = 0.35$, with the amount of N remains 10000 at different h (indicated by the parentheses). Estimator θ_n is obtained in the second simulation, i.e.,: 0.428 (0.905); 0.4264 (0.805); 0.436 (0.705); 0.383 (0.605); and 0.346 (0.505).

5. CONCLUSION

Based on the above discussion, we can conclude that: the LSE $\hat{\theta}_n$ of diffusion process (2) at discrete observations is strongly consistent according to Wu [1] under (4), (5) and (6) with rate of convergence is $\sqrt{n^3 h^4}$.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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